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∞–OPEN–MULTICOMMUTATIVITY IN THE CATEGORY COMP

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In this paper the notion of ∞-open-multicommutativity of functors in the category of compact Hausdorff spaces is considered. This property is a generalization of the open-multicommutativity on the case of infinite diagrams. It is proved that every open-multicommutative functor is ∞-open-multicommutative.

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Рассматривается понятие ∞-открытой мультикоммутативности нормальных функторов, которое является обобщением открытой мультикоммутативности на бесконченые диаграммы. Доказано, что функтор вероятностных мер является ∞-открыто-мультикоммутативным.

Introduction. We continue to study the properties of openness and bicommutativity of functors in the category Comp of compact Hausdorff spaces. The functor in the category Comp is open if it preserves the class of open and surjective maps. The question whether the notions of openness and bicommutativity are equivalent for normal functors has been set by Shchepin [2] till now it is solved only in the case of finite power functor by Zarichnyi [3]. But in general it is still open.

Due to this problem Kozhan and Zarichnyi [1] introduce the notion of open-multi commutativity of normal functors. This property is a generalization of the bicommutativity extended from square diagrams to more complicated finite diagrams. Simultaneously this notion includes also the property of openness of the functor which is a necessary condition of the open–multicommutativity. Kozhan and Zarichnyi [1] have proved that the functor of probability measures is open–multicommutative.

There is a natural generalization of this property on infinite diagrams. Construction of ∞–open–multicommutativity, which is introduced in this note, is analogous to open–multicommutativity for the class of infinite diagram. The main contribution of the paper is criterium of the ∞–open–multicommutativity for functors in category Comp.

In Section “Definitions and Preliminaries” we give necessary definitions and notions concerning to openness and bicommutativity of functors. Section “∞–Open–Multicommutativity” provides the main result of the paper.

Definitions and Preliminaries. Suppose that $\mathcal{G}$ is a finite partially ordered set and we also regard it as a finite directed graph. Denote by $\mathcal{VG}$ the class of all vertices of the graph $\mathcal{G}$ and by $\mathcal{EG}$ the set of its edges. A map $\mathcal{O}\colon \mathcal{G} \to$ Comp is called a *diagram*.

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The set of morphisms

$$(X \overset{g_A}{\to} \mathcal{O}(A))_{A\in\mathcal{VG}} \tag{1}$$

is said to be a *cone over the diagram* $\mathcal{O}$ if and only if for every objects $A, B \in \mathcal{VG}$ and for every edge $\varphi\colon A \to B$ in $\mathcal{EG}$ the diagram

$$\begin{array}{ccc} & X & \\ {}^{g_A}\swarrow & & \searrow^{g_B} \\ \mathcal{O}(A) & \xrightarrow[\mathcal{O}(\varphi)]{} & \mathcal{O}(B) \end{array}$$

is commutative.

The cone (1) is called a *limit of the diagram* $\mathcal{O}$ if the following condition is satisfied: for each cone $C' = (X' \overset{g'_A}{\to} \mathcal{O}(A))_{A\in\mathcal{VG}}$ there exists a unique morphism $\chi_{C'}\colon X' \to X$ such that $g'_A = g_A \circ \chi_{C'}$ for every $A \in \mathcal{VG}$.

Further we denote this cone by $\lim(\mathcal{O})$. The map $\chi_{C'}$ is called the *characteristic map* of C'.

The cone $C' = (X' \overset{g'_A}{\to} \mathcal{O}(A))_{A\in\mathcal{VG}}$ is called *open–multicommutative* if the characteristic map $\chi_{C'}$ is open and surjective.

The cone is called ∞–*open–multicommutative* if it is open–multicommutative over an infinite graph $\mathcal{G}$.

Let F be a covariant functor in the category Comp. Define the diagram $F(\mathcal{O})\colon \mathcal{G} \to \text{Comp}$ in the following way: for every $A \in \mathcal{VG}$ let $F(\mathcal{O})(A) = F(\mathcal{O}(A))$ and for every edge $\varphi \in \mathcal{EG}$ we set $F(\mathcal{O})(\varphi) = F(\mathcal{O}(\varphi))$.

The functor F is called *open–multicommutative* if it preserves the open–multicommutative cones, i.e. the cone

$$F(C') = (F(X') \overset{Fg'_A}{\to} F(\mathcal{O}(A)))_{A\in\mathcal{VG}}$$

over the diagram $F(\mathcal{O})$ is open–multicommutative.

The functor is called ∞–*open–multicommutative* if it is open–multicommutative over an infinite graph $\mathcal{G}$.

∞–Open–Multicommutativity. We assume that the graph $\mathcal{G}$ is infinite, i.e. the set $\mathcal{VG}$ is infinite. Let us denote by $(Y \overset{\mathrm{pr}_A}{\to} F(\mathcal{O}(A)))_{A\in\mathcal{VG}}$ the limit of the diagram $F(\mathcal{O})$. By definition given the cone $(F(X) \overset{Fg_A}{\to} F(\mathcal{O}(A)))_{A\in\mathcal{VG}}$ there exists the unique characteristic map $\chi\colon F(X) \to Y$.

Let E be a set of all finite subsets of $\mathcal{VG}$. Let us define for every $\alpha \in E$ a finite graph $\mathcal{G}_\alpha$ in the following way: $\mathcal{VG}_\alpha = \alpha$ and φ, which connects vertices A and B, belongs to $\mathcal{EG}_\alpha$ if and only if $A, B \in \mathcal{VG}_\alpha$ and $\varphi \in \mathcal{EG}$. Let for every $\alpha \in E$ the cone $(X_\alpha \overset{g^\alpha_A}{\to} \mathcal{O}(A))_{A\in\mathcal{VG}_\alpha}$ be the limit of the diagram $\mathcal{O}$ over the graph $\mathcal{G}_\alpha$. Define a partial order relation on the set E as $\alpha \le \beta \Leftrightarrow \alpha \subseteq \beta$. We denote by $S = \{X_\alpha, \mathrm{pr}^\beta_\alpha, E\}$ the inverse system.

Lemma 1. $X = \lim S$.

Proof. Let us show that the set of morphisms $(X \overset{\mathrm{pr}_\alpha}{\to} X_\alpha)_{\alpha\in E}$ is a limit of the inverse system S. For arbitrary point $(x_A)_{A\in\mathcal{VG}} \in X \subseteq \prod_{A\in\mathcal{VG}} X_A$ its image $\mathrm{pr}_\alpha((x_A)_{A\in\mathcal{VG}}) = (x_A)_{A\in\alpha} \in X_\alpha$ for each $\alpha \in E$. Indeed, for all $A, B \in \alpha$ and $\varphi \in \mathcal{VG}_\alpha(A, B)$ we have

$\mathrm{pr}_B = \mathcal{O}(\varphi) \circ \mathrm{pr}_A$ because $A, B \in \mathcal{VG}$ and $\varphi \in \mathcal{VG}(A, B)$. This implies that $(X \overset{\mathrm{pr}_\alpha}{\to} \mathcal{O}(A))_{A\in\alpha}$ is a cone. Let us show that this cone is a limit of S. For every cone $(X' \overset{h_\alpha}{\to} \mathcal{O}(A))_{A\in\alpha}$ we define a map $\zeta\colon X' \to X$ such that $\zeta(x') = (h_{\{A\}}(x'))_{A\in\mathcal{VG}}$. Since $h_\alpha(x') \in X_\alpha \subseteq \prod_{A\in\alpha} \mathcal{O}(A)$ for all $\alpha \in E$, we obtain $h_\alpha(x') = (h_{\{A\}}(x'))_{A\in\alpha}$ and therefore $h_\alpha = \mathrm{pr}_\alpha \circ \zeta$. The uniqueness of the map ζ proves that the cone $(X \overset{\mathrm{pr}_\alpha}{\to} \mathcal{O}(A))_{A\in\alpha}$ is the limit of the inverse system S. □

Let us consider the cone $(F(X_\alpha) \overset{F\mathrm{pr}_A}{\to} F(\mathcal{O}(A)))_{A\in\alpha}$ over the diagram $F(\mathcal{O})$ and finite graph $\mathcal{G}_\alpha$. We denote by $(Y_\alpha \overset{\mathrm{pr}_A}{\to} F(\mathcal{O}(A)))_{A\in\alpha}$ the limit of this diagram over $\mathcal{G}_\alpha$, $\alpha \in E$. We construct the inverse systems

$$S_1 = \{F(X_\alpha), F(\mathrm{pr}^\beta_\alpha), E\} \text{ and } S_2 = \{Y_\alpha, \mathrm{pr}^\beta_\alpha, E\}.$$

Let $\chi\colon F(X) \to Y$ be a limit map of the morphisms $(\chi_\alpha)_{\alpha\in E}$, where $\chi_\alpha\colon F(X_\alpha) \to Y_\alpha$ is a characteristic map of the cone

$$(F(X_\alpha) \overset{F\mathrm{pr}_A}{\to} F(\mathcal{O}(A)))_{A\in\alpha}.$$

Lemma 1 implies that $Y = \lim\{Y_\alpha, \mathrm{pr}^\beta_\alpha, E\}$. From the uniqueness of the characteristic maps χ_α it follows that χ is a characteristic map of the cone $(F(X) \overset{F\mathrm{pr}_A}{\to} F(\mathcal{O}(A)))_{A\in\mathcal{VG}}$.

Lemma 2. *Let F be a bicommutative functor. For all $\alpha, \beta \in E$ such that $\alpha \le \beta$ the diagram*

$$\begin{array}{ccc} F(X_\beta) & \xrightarrow{\chi_\beta} & Y_\beta \\ {\scriptstyle F\mathrm{pr}^\beta_\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}^\beta_\alpha} \\ F(X_\alpha) & \xrightarrow[\chi_\alpha]{} & Y_\alpha \end{array} \tag{2}$$

is bicommutative.

Proof. Without loss of generality we assume that $\mathrm{Card}(\beta) = \mathrm{Card}(\alpha)+1$ and let $C \in \beta\backslash\alpha$. A sufficient condition for the bicommutativity of diagram (2) is the fact that given $\tau_0 \in F(X_\alpha)$ and $\mu = (\mu_A)_{A\in\beta} \in Y_\beta$ such that

$$\chi_\alpha(\tau_0) = \mathrm{pr}^\beta_\alpha(\mu) = (\mu_A)_{A\in\alpha} \tag{3}$$

there exists an element $\tau \in F(X_\beta)$ which satisfies

$$\chi_\beta(\tau) = \mu \text{ and } F\mathrm{pr}^\beta_\alpha(\tau) = \tau_0.$$

Consider the diagram

$$\begin{array}{ccc} F(X_\beta) & \xrightarrow{F\mathrm{pr}_C} & F(C) \\ {\scriptstyle F\mathrm{pr}^\beta_\alpha}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_{F(C)}} \\ F(X_\alpha) & \xrightarrow[\mathbf{1}_{F(X_\alpha)}]{} & F(*) \end{array}$$

It is obvious that this diagram is bicommutative, since there exists an element $\tau \in F(X_\beta)$ such that $F\mathrm{pr}^\beta_\alpha(\tau) = \tau_0$ and $F\mathrm{pr}_C(\tau) = \mu_C$. Due to (3) we see that

$$\begin{aligned} \chi_\beta(\tau) &= \prod_{A\in\beta} F\mathrm{pr}_A(\tau) = \prod_{A\in\alpha} F\mathrm{pr}_A(\tau) \times F\mathrm{pr}_C(\tau) = \\ &= \left(\prod_{A\in\alpha} F\mathrm{pr}_A \circ F\mathrm{pr}_\alpha\right) \times F\mathrm{pr}_C = (\mu_A)_{A\in\beta}. \end{aligned}$$

□

Theorem 1. *Every open–multicommutative functor F is ∞–open–multicommutative.*

Proof. It is known (see [1]) that for all $\alpha \in E$ the map χ_α is open. Lemma 2 (open–multicommutativity of F implies its bicommutativity) and Lemma 2.1 of Zarichnyi [4] imply that the limit map χ is also open. The surjectivity of the characteristic map is followed from the surjectivity of morphisms χ_α, $\alpha \in E$. This implies that F is ∞–open–multicommutative. □

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