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TWO EXAMPLES OF RINGS WITH CERTAIN PROPERTIES OF LATTICES OF I–RADICALS

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Two examples of rings with specific properties of lattices of I–radicals are given.

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Построены два примера колец со специфическими свойствами решёток I–радикалов.

Throughout the whole text, all rings are considered to be associative with unit $1 \neq 0$ and all modules are left unitary ([1, 2, 3]). A subset of R is called an ideal if it is a twosided one. Let R be a ring. The category of left R–modules will be denoted by R–Mod. We shall write $N \leq M$ if N is a submodule of M. Recall some definitions. R is semisimple if $R = \mathrm{Soc}({}_RR)$. R is simple if there exist exactly two ideals of R. A subset I of R is right T-nilpotent in case for every sequence $a_1, a_2, \dots$ in I there is an n such that $a_1a_2\dots a_n = 0$.

Let S be an ideal of R. Then we define r_S in the following way:

$$r_S(M) = \sum \{N|\ N \leq M, SN = n\} \text{ for every left } R\text{–module } M \ ([1, 4, 5, 6]).$$

It is easy to see that r_S is an idempotent radical of R–Mod ([1]). A preradical r is called an I-radical if $r = r_S$ for some ideal S of R ([1]). The set of all I–radicals of R–Mod will be denoted by $\mathrm{Ir}(l, R)$. Let r, t be I–radicals. We shall write $r \leq t$ if $r(M) \leq t(M)$ for every left R–module M. It was proved that the poset$(\mathrm{Ir}(l, R), \leq)$ is a lattice and $r_S \wedge r_V = r_{SV}, r_S \vee r_V = r_{S+V}$ for all ideals S, V of R, where $x \wedge y = \inf\{x, y\}$, $x \vee y = \sup\{x, y\}$ in the poset $(\mathrm{Ir}(l, R), \leq)$ ([5]). It is well–known that this lattice is complete for many types of rings ([5, 6]). For example for all noetherian rings and for all left perfect rings.

Proposition 1. *The lattice* $(Ir(l, \mathbb{R}[x_1, x_2, \dots, x_i, \dots]), \leq)$ *is not complete.*

Proof. Let $R := \mathbb{R}[x_1, x_2, \dots, x_i, \dots], A_i := Rx_i (i \in \mathbb{N}), V := \{r_{A_i}|i \in \mathbb{N}\}$ We shall show that V has no supremum. Let us assume that r_B is a supremum of V for some ideal B of R. Then $r_{A_i} = r_{A_iB}$ for every $i \in \mathbb{N}$ ([5, p.45]). Taking into account proposition 1 ([5]), theorem 3 ([5]), we have that $A_i/(A_iB)$ is a right T–nilpotent ideal of $R/(A_iB)$. Since $x_i \in A_i$, there exist some $n_i \in \mathbb{N}$ such that $x_i^{n_i} \in A_iB \subseteq B$. Let $L := \sum\{Rx_i^{n_i}|i \in \mathbb{N}\}$. It is obvious that $L \subseteq B$. Since $r_{A_i} = r_{A_i^{n_i}}$, r_L is a supremum of V. Similarly we obtain that r_N is also a supremum of V, where $M = \sum\{Rx_i^{n_i+1}|i \in \mathbb{N}\}$. Then $r_L = r_M, M \subseteq L$. Again taking into

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account Proposition 1 ([5]) and Theorem 3 ([5]) it is clear that L/M is right T–nilpotent. Since $\{x_1^{n_1}, x_2^{n_2}, \ldots, x_i^{n_i}, \ldots\} \subseteq L$, there exists $k \in \mathbb{N}$ such that $x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k} \in M$. Hence $x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k} = x_1^{n_1+1}f_1 + x_2^{n_2+1}f_2 + \ldots + x_u^{n_u+1}f_u$, where $f_1, f_2, \ldots, f_u \in R, k \le u$. Hence $x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k} = x_1^{n_1+1}g_1 + x_2^{n_2+1}g_2 + \ldots + x_k^{n_k+1}g_k$, where $g_1, g_2, \ldots, g_k \in \mathbb{R}[x_1, x_2, \ldots, x_k]$.

Define $D_i\colon R \to R$ in the following way $D_i(f) = \left[\dfrac{\partial^{n_1} f}{\partial x_i^{n_i}}\right]_{x_i=0}$, $n_1!n_2!\ldots n_{k-1}!x_k^{n_k} = (D_{k-1}\ldots D_2D_1)(x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k}) = (D_{k-1}\ldots D_2D_1)(x_1^{n_1+1}g_1 + x_2^{n_2+1}g_2 + \ldots + x_k^{n_k+1}g_k) = x_k^{n_k+1}g$, where $g \in \mathbb{R}[x_k]$. Hence $n_1!n_2!\ldots n_{k-1}! = x_kg$. And now we have a contradiction. □

Let P be a ring and let $R := \begin{pmatrix} P & P & P \\ 0 & P & P \\ 0 & P & P \end{pmatrix}$ denote the set of all matrices $\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ 0 & p_{22} & p_{23} \\ 0 & p_{32} & p_{33} \end{pmatrix}$ with $p_{ij} \in P$ ($\forall\, i, j$). In R, define addition and multiplication as in ordinary matrices. Then R is a ring.

It is well–known that there exists a simple ring which is not semisimple ([3]).

Proposition 2. *Let P is a simple ring, which is not semisimple. Then $(Ir(l, R), \le)$ is Boolean, R is not left (right) perfect and it is indecomposable as a ring. Moreover*

$$\mathrm{Card}(\mathrm{Ir}(l, R)) = 4.$$

Proof. Let $e = \begin{pmatrix} \alpha & \beta_1 & \beta_2 \\ 0 & \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{21} & \gamma_{22} \end{pmatrix}$ be a central idempotent of R. Then

$$0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e - e \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i. e. $\beta_1 = \beta_2 = 0$. Since $e^2 = e, \alpha^2 = \alpha$, $\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}^2 = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$. Let a, c_{ij} $(i, j \in \{1, 2\})$ be arbitrary elements of P. Then

$$0 = \begin{pmatrix} a & 0 & 0 \\ 0 & c_{11} & c_{12} \\ 0 & c_{21} & c_{22} \end{pmatrix} e - e \begin{pmatrix} a & 0 & 0 \\ 0 & c_{11} & c_{12} \\ 0 & c_{21} & c_{22} \end{pmatrix}.$$

Hence $a\alpha = \alpha a$, $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$. It is clear that

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},\ \alpha \in \{0, 1\}$$

because P is a simple ring. Therefore

$$e \in \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let $u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. But $us - su = s$, $(1-u)s - s(1-u) = -s$. Therefore

$$e \in \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Hence R is indecomposable.

It is clear that $\begin{pmatrix} 0 & P & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is an ideal of R and $\begin{pmatrix} 0 & P & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = 0$. Hence $\begin{pmatrix} 0 & P & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subseteq J(R)$. It is obvious that the factor-ring $R/\begin{pmatrix} 0 & P & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is isomorphic to the ring $P \times \begin{pmatrix} P & P \\ P & P \end{pmatrix}$. But $J\left(P \times \begin{pmatrix} P & P \\ P & P \end{pmatrix}\right) = J(P) \times J\left(\begin{pmatrix} P & P \\ P & P \end{pmatrix}\right) = 0 \times 0 = 0$. Hence $J\left(R/\begin{pmatrix} 0 & P & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 0$. It follows from this that $J(R) \subseteq \begin{pmatrix} 0 & P & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus $J(R) = \begin{pmatrix} 0 & P & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Now apply Theorem 4 ([5]). □

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