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MIXED BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS OF THE SECOND ORDER IN THREE-DIMENSIONAL NONREGULAR DOMAINS

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We consider mixed boundary value problems for elliptic equations of the second order in $\mathbb{R}^3$, i.e. interior, exterior and problems when boundary conditions are given on an open Lipschitz surface. We show existence and uniqueness of solutions of the posed problems in appropriate functional spaces. By means of integral representation these differential problems are reduced to systems of integral and integro-differential equations with the same type for all problems. The equivalence of obtained systems and given differential problems has been proved. We have also shown that matrix operators over the boundary of the domain in all three cases are positive definite and this gives us a possibility to construct a steady numerical algorithm using Galerkin method.

Ю. М. Сыбиль. *Смешанные краевые задачи для эллиптических дифференциальных уравнений второго порядка в нерегулярных трехмерных областях* // Математичні Студії. – 2007. – Т.28, №1. – С.191–205.

Рассматриваются смешанные краевые задачи для эллиптических дифференциальных уравнений второго порядка в $\mathbb{R}^3$, т.е. внутренняя, внешняя задачи и задачи, в которых краевые условия заданы на разомкнутой липшицевой поверхности. Исследуются существование и единственность решений в подходящих функциональных пространствах. С помощью интегральных представлений все эти дифференциальные краевые задачи сводятся к системам интегральных и интегро-дифференциальных уравнений одного и того же типа. Доказывается эквивалентность полученных систем исходным краевым задачам. Доказывается также, что матричный оператор на границе области во всех трех случаях положительно определен, что дает возможность с помощью метода Галеркина построить устойчивый вычислительный алгоритм.

1. Introduction. In this paper we consider mixed boundary value problems for the second order elliptic equation in $\mathbb{R}^3$, i.e. the problems when on one part of the boundary are given conditions of Dirichlet type and on another one conditions of Neumann type. A boundary surface is supposed to be a closed or an open Lipschitz surface and it could be essentially unregular.

In the case of open surface we have Dirichlet condition on one side of the surface and Neumann condition on the other one. Such a posed mixed boundary value problem essentially differs from boundary value problems considered in [10] where we had problems with boundary conditions of only Dirichlet or only Neumann type given on both sides of open Lipschitz surface.

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The theory of mixed problem was started by the work of Zaremba ([12]) and includes now various types of differential equations and boundary conditions. The present approaches are based mostly on a calculus of pseudodifferential operators (for a closed surface see [4, 9] and for an open [3, 8]) or variational formulation of the problem ([1, 11]). All these methods require sufficient smoothness of the boundary. In the case of unregular domain we have additional problems connected with definitions of corresponding trace maps and appropriate functional spaces.

In this paper we present an approach based on the method of Green formula and using some orthogonal decomposition of the corresponding Hilbert spaces according to the type of boundary conditions. We consider interior, exterior problems as well as problems when the boundary conditions are given on an open Lipschitz surface. All these problems are reduced to the system of boundary integral and integro-differential equations which have the same structure for all types of problems and are given by positive definite matrix operators.

2. Some notation and functional spaces. Let $\Omega_+ \subset \mathbb{R}^3$ be a bounded Lipschitz domain. This means that its boundary Σ is locally the graph of a Lipschitz function ([2, 7]). Let us note that Σ can be piecewise smooth and have edges and corners. Then $\Omega_- = \mathbb{R}^3 \setminus \overline{\Omega}_+$ is an exterior unbounded domain where $\overline{\Omega}_+ = \Omega_+ \cup \Sigma$.

We suppose that $\Sigma = \overline{S}_1 \cup S_2$, where S_1 and S_2 are parts of Σ with common edge Γ.

In this paper we consider the different types of mixed boundary value problems for the elliptic operator of the second order

$$Lu = -\sum_{i=1}^3 \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + a_0 u,$$

where $a_{ij}, a_0 \in C^\infty(\mathbb{R}^3)$ are real functions bounded at infinity, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $a_{ij}(x) = a_{ji}(x)$.

Let us denote by $C_0^\infty(\Omega_+)$ the class of infinitely differentiable functions with compact support in Ω_+. $C^\infty(\overline{\Omega}_+)$ denotes the space of functions which are C^∞ up to the boundary Σ, i.e. every derivative has a limit on the boundary. Since Ω_+ is a Lipschitz domain, every function $u \in C^\infty(\overline{\Omega}_+)$ has an extension $pu \in C^\infty(\mathbb{R}^3)$ ([7]). Then analogously for Ω_- we denote by $C_0^\infty(\Omega_-)$ the class of infinitely differentiable functions with compact support in Ω_-. $C_0^\infty(\overline{\Omega}_-)$ denotes the space of functions from $C_0^\infty(\mathbb{R}^3)$ restricted to $\overline{\Omega}_-$.

Let $u(x)$ be a function defined in $\Omega' = \Omega_+ \cup \Omega_-$. Then we denote by $u_\pm(x) = r_{\Omega_\pm}u(x)$ the restriction of $u(x)$ on $\Omega_\pm$ respectively. We use the following functional spaces on $\Omega_\pm$ and Σ ([5, 7])

$$\|u_\pm\|^2_{H^1(\Omega_\pm)} = \int_{\Omega_\pm}\left\{|\nabla u_\pm|^2 + u_\pm^2\right\}dx, \quad (u_\pm, v_\pm)_{H^1(\Omega_\pm)} = \int_{\Omega_\pm}\left\{(\nabla u_\pm, \nabla v_\pm) + u_\pm v_\pm\right\}dx,$$

$$\|u_\pm\|^2_{H^1(\Omega_\pm,L)} = \|u_\pm\|^2_{H^1(\Omega_\pm)} + \|Lu_\pm\|^2_{L_2(\Omega_\pm)},$$

$$(u_\pm, v_\pm)_{H^1(\Omega_\pm,L)} = (u_\pm, v_\pm)_{H^1(\Omega_\pm)} + (Lu_\pm, Lv_\pm)_{L_2(\Omega_\pm)},$$

$$(\nabla u_\pm, \nabla v_\pm) = \sum_{i=1}^3 \frac{\partial u_\pm}{\partial x_i}\frac{\partial v_\pm}{\partial x_i}.$$

$H_0^1(\Omega_\pm)$ is the closure of $C_0^\infty(\Omega_\pm)$ with respect to the norm $\|\cdot\|_{H^1(\Omega_\pm)}$. $H_0^1(\Omega_\pm) = \{u_\pm \in H^1(\Omega_\pm)\colon \gamma_0^\pm u_\pm = 0\}$, $\gamma_0^\pm\colon H^1(\Omega_\pm) \to H^{1/2}(\Sigma)$ are continuous and surjective trace maps ([2]).Then $H_0^1(\Omega_\pm)$=ker $\gamma_0^\pm$.

As usual, we denote $H^{-1}(\Omega_\pm) = (H_0^1(\Omega_\pm))'$, $H^{-1/2}(\Sigma) = (H^{1/2}(\Sigma))'$. Let us note that $C^\infty(\overline{\Omega}_+)$ is dense in $H^1(\Omega_+)$ ([2, 7]) and in $H^1(\Omega_+, L)$([2], Lemma 3.3). On the other hand we see that $C_0^\infty(\overline{\Omega}_-)$ is dense in $H^1(\Omega_-)$ and $H^1(\Omega_-, L)$ ([6]).

For $u_\pm \in H^1(\Omega_\pm, L)$, $v_\pm \in H^1(\Omega_\pm)$ the first Green formula ([2])

$$\int_{\Omega_\pm} \{ \sum_{i,j=1}^3 a_{ij} \frac{\partial u_\pm}{\partial x_i} \frac{\partial v_\pm}{\partial x_j} + a_0 u_\pm v_\pm \} dx = \int_{\Omega_\pm} Lu_\pm \cdot v_\pm dx \pm \langle \gamma_1^\pm u_\pm, \gamma_0^\pm v_\pm \rangle \tag{1}$$

and for $u_\pm, v_\pm \in H^1(\Omega_\pm, L)$ the second Green formula

$$\int_{\Omega_\pm} (Lu_\pm \cdot v_\pm - Lv_\pm \cdot u_\pm) dx = \pm \langle \gamma_1^\pm v_\pm, \gamma_0^\pm u_\pm \rangle \mp \langle \gamma_1^\pm u_\pm, \gamma_0^\pm v_\pm \rangle. \tag{2}$$

hold. Here $\langle \cdot, \cdot \rangle$ is the relation of duality between $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$. $\gamma_1^+ u_+ \in H^{-1/2}(\Sigma)$ and coincides with $\frac{\partial u_+}{\partial n_x}$ for $u_+ \in C^\infty(\overline{\Omega}_+)$ where $\frac{\partial}{\partial n_x} = \sum_{i,j=1}^3 \cos(\vec{n}_x, \vec{x}_i) a_{ij} \frac{\partial}{\partial x_j}$ is a conormal derivative, $\cos(\vec{n}_x, x_i)$ are the coordinates of the almost everywhere defined outward pointing vector of the normal $\vec{n}_x$ to Σ. Let us note that $\gamma_1^\pm \colon H^1(\Omega_+, L) \to H^{-1/2}(\Sigma)$ is a continuous and surjective trace map ([2]).

If $\sum_{i,j=1}^3 a_{ij}(x) t_i t_j > 0$ and $a_0(x) > 0$ for arbitrary $x \in \Omega_+$, $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, we can connect with the operator L in $\Omega_\pm$ the following norm and inner product:

$$\|u_\pm\|_L^2 = \int_{\Omega_\pm} \{ \sum_{i,j=1}^3 a_{ij} \frac{\partial u_\pm}{\partial x_i} \frac{\partial u_\pm}{\partial x_j} + a_0 u_\pm^2 \} dx, (u_\pm, v_\pm)_L = \int_{\Omega_\pm} \{ \sum_{i,j=1}^3 a_{ij} \frac{\partial u_\pm}{\partial x_i} \frac{\partial v_\pm}{\partial x_j} + a_0 u_\pm v_\pm \} dx.$$

If the functions $a_{ij}(x)$, $a_0(x)$ satisfy the conditions $\sum_{i,j=1}^3 a_{ij} t_i t_j \geq c_1 \sum_{i=1}^3 t_i^2$, $t \in \mathbb{R}^3$, $c_1 > 0$, $a_0(x) \geq c_2 > 0$, for $x \in \Omega_\pm$, then the norms $\| \cdot \|_{H^1(\Omega_\pm)}$ and $\| \cdot \|_L$ are equivalent ([10]). Thus we can consider spaces $H^1(\Omega_\pm)$ with the norm $\| \cdot \|_L$ and inner product $(\cdot, \cdot)_L$ and rewrite the first Green formula (1) for $u_\pm \in H^1(\Omega_\pm, L)$, $v_\pm \in H^1(\Omega_\pm)$ as

$$(u_\pm, v_\pm)_L = (Lu_\pm, v_\pm)_{L_2(\Omega_\pm)} \pm \langle \gamma_1^\pm u_\pm, \gamma_0^\pm v_\pm \rangle. \tag{3}$$

The first Green formula (3) is valid also for $u_\pm \in H^1(\Omega_+)$, $v_\pm \in H_0^1(\Omega_+)$ [10], i.e. $(u_\pm, v_\pm)_L = \langle Lu_\pm, v_\pm \rangle$, where $\langle \cdot, \cdot \rangle$ is the relation of duality between $H_0^1(\Omega_+)$ and $H^{-1}(\Omega_+)$.

Let us introduce the space $H^1(\Omega')$ of functions $u(x) = (u_+(x), u_-(x))$, $x \in \Omega'$, with the norm and inner product

$$\|u\|_{H^1(\Omega')}^2 = \|u_+\|_L^2 + \|u_-\|_L^2, \qquad (u, v)_{H^1(\Omega')} = (u_+, v_+)_L + (u_-, v_-)_L,$$

and the space $H^1(\Omega', L)$ with the norm $\|u\|_{H^1(\Omega', L)}^2 = \|u\|_{H^1(\Omega')}^2 + \|u\|_{L_2(\Omega')}^2$.

In what follows we need some functional spaces and notation connected with S_1 and S_2 ([10]). Let us denote by $H^{1/2}(S_1)$ the restriction of functions from $H^{1/2}(\Sigma)$ on S_1, i.e. $f \in H^{1/2}(S_1)$ admits an extension pf onto Σ that belongs to $H^{1/2}(\Sigma)$. The norm in $H^{1/2}(S_1)$ is defined by

$$\|f\|_{H^{1/2}(S_1)} = \inf_{pf} \|pf\|_{H^{1/2}(\Sigma)},$$

where the infimum is taken over all extensions of f that belong to $H^{1/2}(\Sigma)$. Then we can define a continuous and surjective operator $r_{S_1} \colon H^{1/2}(\Sigma) \to H^{1/2}(S_1)$.

We denote $H_{\overline{S}_1}^{1/2}(\Sigma) = \{u \in H^{1/2}(\Sigma) \colon \text{supp } u \in \overline{S}_1\}$ and $H_{00}^{1/2}(S_1)$ is a restriction of functions from $H_{\overline{S}_1}^{1/2}(\Sigma)$ on S_1, i.e. if $f \in H_{00}^{1/2}(S_1)$ then there exists an extension $p_0 f$ by zero on Σ that belongs to $H_{\overline{S}_1}^{1/2}(\Sigma)$ and $r_{S_1} p_0 f = f$. The norm in $H_{00}^{1/2}(S_1)$ is defined by $\|f\|_{H_{00}^{1/2}(S_1)} = \|p_0 f\|_{H^{1/2}(\Sigma)}$. Let us note that $H_{00}^{1/2}(S_1)$ is essentially different from $H^{1/2}(S_1)$ and for smooth S_1 one can define an equivalent norm in $H_{00}^{1/2}(S_1)$ as [5]

$$\|u\|_{H_{00}^{1/2}(S_1)}^2 = \|u\|_{H^{1/2}(S_1)}^2 + \|\rho^{-1/2} u\|_{L_2(S_1)}^2,$$

where $\rho(x)$ is the distance from $x \in S_1$ to the smooth edge ∂S_1.

Let $f \in H^{-1/2}(\Sigma)$ and C be an arbitrary open set of Σ. We will say that $f = 0$ in C if a restriction of f to C is a null functional on $H_{00}^{1/2}(C)$, i.e. $\langle f, v \rangle = 0$ for all $v \in$

$H^{1/2}_{\overline{C}}(\Sigma)$. Let C_{max} be the largest open set on which $f = 0$. Then the complement $\Sigma \setminus C_{max}$ is denoted by supp$f$ and is called the support of the functional f. We denote $H^{-1/2}_{\overline{S}_1}(\Sigma) = \{f \in H^{-1/2}(\Sigma)\colon \text{supp} f \subset \overline{S}_1\}$. From the definition we have $\langle f, v\rangle = 0$ for all $f \in H^{-1/2}_{\overline{S}_1}(\Sigma)$, $v \in H^{1/2}_{\overline{S}_2}(\Sigma)$.

For every $f \in H^{-1/2}(\Sigma)$ we can consider a functional g defined on $H^{1/2}_{00}(S_1)$ and given for all $v \in H^{1/2}_{00}(S_1)$ by $\langle g, v\rangle = \langle f, p_0 v\rangle$, where $p_0 v \in H^{1/2}_{\overline{S}_1}(\Sigma)$. We see that $g \in (H^{1/2}_{00}(S_1))'$ and $\|g\|_{H^{-1/2}(S_1)} \le \|f\|_{H^{-1/2}(\Sigma)}$. The functional g is called the restriction of f on S_1. We denote $H^{-1/2}(S_1) = (H^{-1/2}_{00}(S_1))'$.

So far as the operator $r_{S_1}\colon H^{1/2}_{\overline{S}_1}(\Sigma) \to H^{1/2}_{00}(S_1)$ is an isometry then $r'_{S_1}\colon H^{-1/2}(S_1) \to (H^{1/2}_{\overline{S}_1}(\Sigma))'$ is also an isometry and we can identify $H^{-1/2}(S_1)$ and $(H^{1/2}_{\overline{S}_1}(\Sigma))'$. $H^{1/2}_{\overline{S}_1}(\Sigma)$ is a closed subspace of $H^{1/2}(\Sigma)$. Hence by the Hahn-Banach theorem every $g \in (H^{1/2}_{\overline{S}_1}(\Sigma))'$ can be extended by continuity (but not uniquely) to $f \in H^{-1/2}(\Sigma)$. Finally we get that the operator of restriction $\widetilde{r}_{S_1}\colon H^{-1/2}(\Sigma) \to H^{-1/2}(S_1)$ is continuous and surjective and $g = \widetilde{r}_{S_1} f$.

Let us denote $H^{-1/2}_{00}(S_1) = (H^{1/2}(S_1))'$. We see that $H^{-1/2}_{00}(S_1)$ is isometric to $H^{-1/2}_{\overline{S}_1}(\Sigma)$ ([10]) and if $f \in H^{-1/2}_{\overline{S}_1}(\Sigma)$ then $g = \widetilde{r}_{S_1} f \in H^{-1/2}_{00}(S_1)$

Let H be a Hilbert space with the inner product $(\cdot,\cdot)_H$, $M \subset H$. We denote $M^\perp = \{u \in H : (u, v)_H = 0,\ v \in M\}$, $\ker L_\pm = \{u_\pm \in H^1(\Omega_\pm)\colon\ Lu_\pm = 0\}$, $\ker L = \{u \in H^1(\Omega')\colon\ Lu_\pm = 0\}$, $\ker \gamma_1^\pm = \{u_\pm \in H^1(\Omega_\pm, L)\colon \gamma_1^\pm u_\pm = 0\}$. .

Lemma 1 ([10]). *$H^1(\Omega_\pm) = H^1_0(\Omega_\pm) \oplus kerL_\pm$. The operators $\gamma_0^\pm\colon$ ker $L_\pm \to H^{1/2}(\Sigma)$ and $L\colon H^1_0(\Omega_\pm) \to H^{-1}(\Omega_\pm)$ are isomorphisms.*

Lemma 2 ([10]). *The operators $\gamma_1^\pm\colon$ ker $L_\pm \to H^{-1/2}(\Sigma)$ and $L\colon$ ker $\gamma_1^\pm \to L_2(\Omega_\pm)$ are isomorphisms.*

We denote throughout this paper by $\gamma^\pm_{0,S_i}$, $\gamma^\pm_{1,S_i}$ the trace maps onto S_i from $\Omega_\pm$. We consider $\gamma^\pm_{0,S_i} u_\pm$, $u_\pm \in H^1(\Omega_\pm)$, as the restrictions of $\gamma_0^\pm u_\pm \in H^{1/2}(\Sigma)$ on S_i, i.e. $\gamma^\pm_{0,S_i} u_\pm = r_{S_i}\gamma_0^\pm u_\pm \in H^{1/2}(S_i)$. Analogously $\gamma^\pm_{1,S_i} u_\pm$, $u_\pm \in H^1(\Omega_\pm, L)$, are the restrictions of $\gamma_1^\pm u_\pm$ on S_i, i.e $\gamma^\pm_{1,S_i} u_\pm = \widetilde{r}_{S_i}\gamma_1^\pm u_\pm \in H^{-1/2}(S_i)$. By definition we have $\langle \gamma_1^\pm u_\pm, p_0 g\rangle = \langle \gamma^\pm_{1,S_i} u_\pm, g\rangle$, where $p_0 g \in H^{1/2}_{\overline{S}_i}(\Sigma)$, $g \in H^{1/2}_{00}(S_i)$.

Let us denote $[\gamma_0]u = \gamma_0^+ u - \gamma_0^- u$ if $u \in H^1(\Omega')$ and by $[\gamma_1]u = \gamma_1^+ u - \gamma_1^- u$ if $u \in H^1(\Omega', L)$. Then $[\gamma_0]u \in H^{1/2}(\Sigma)$, $[\gamma_1]u \in H^{-1/2}(\Sigma)$. Let $[\gamma_i]_{S_j} u = \gamma^+_{i,S_j} u_+ - \gamma^-_{i,S_j} u_-$, $i = 0, 1$, $j = 1, 2$, be the restrictions of $[\gamma_i]u$ on S_j, i.e. $[\gamma_0]_{S_i} u = r_{S_i}[\gamma_0]u \in H^{1/2}(S_i)$ and $[\gamma_1]_{S_i} u = \widetilde{r}_{S_i}[\gamma_1]u \in H^{-1/2}(S_i)$.

Let us denote ker $\gamma^\pm_{i,S_j} = \{u_\pm \in$ker $L_\pm\colon \gamma^\pm_{i,S_j} u_\pm = 0\}$, $i \in \{0,\ 1\}$, $j \in \{1,\ 2\}$.

Lemma 3. *ker $L_\pm$ =ker $\gamma^\pm_{0,S_1} \oplus$ker $\gamma^\pm_{1,S_2}$.*

Proof. Let $v \in$ker γ^+_{0,S_1}, $w \in$ker γ^+_{1,S_2}. Then we see that $\gamma_0^+ v \in H^{1/2}_{\overline{S}_2}(\Sigma)$, $\gamma_1^+ w \in H^{-1/2}_{\overline{S}_1}(\Sigma)$. Since ker γ^+_{0,S_1} is a closed subspace of ker L_+ we have ker L_+ =ker $\gamma^+_{0,S_1} \oplus$ (ker $\gamma^+_{0,S_1})^\perp$. By using (3) we obtain $(v, w)_L = \langle \gamma_1^+ w, \gamma_0^+ v\rangle = 0$. This means that ker $\gamma^+_{1,S_2} \subset$ (ker $\gamma^+_{0,S_1})^\perp$.

Let now $w \in$ (ker $\gamma^+_{0,S_1})^\perp$. Then we have $\langle \gamma_1^+ w, \gamma_0^+ v\rangle = (w, v)_L = 0$ for an arbitrary $\gamma_0^+ v \in H^{1/2}_{\overline{S}_2}(\Sigma)$. By definition we can get $\gamma^+_{1,S_2} w = \widetilde{r}_{S_2}\gamma_1^+ w = 0$ or $w \in$ker γ^+_{1,S_2}. Thus we

obtain that (ker $\gamma^+_{0,S_1})^\perp \subset$ ker γ^+_{1,S_2} or (ker $\gamma^+_{0,S_1})^\perp =$ker γ^+_{1,S_2} that yields us the orthogonal representation of ker L_+.
The same result we obtain for the domain Ω_-, i.e. ker L_- = ker $\gamma^-_{0,S_1}\oplus$ ker γ^-_{1,S_2}. □

3. Potentials and boundary operators. Let $Q(x,y)$ be the fundamental solution of L, i.e. $L_yQ(x,y) = \delta(x-y),\ x,y \in \mathbb{R}^3$. Applying the second Green formula (2) we can get for $u \in H^1(\Omega', L)$ ([2])

$$u(x) = (Lu, Q(x,\cdot))_{L_2(\Omega')} + \langle[\gamma_1]u, Q(x,\cdot)\rangle - \langle\gamma_1 Q(x,\cdot), [\gamma_0]u\rangle, \quad x \in \Omega'. \tag{4}$$

The representation formula (4) implies that every function $u \in H^1(\Omega', L)$, i.e. $Lu = f \in L_2(\Omega')$, is a sum of the following potentials

$$u(x) = Df(x) + V[\gamma_1]u(x) - W[\gamma_0]u(x), \tag{5}$$

where
$Df(x) = \int_{\Omega'} Q(x,y)f(y)dy,\ W[\gamma_0]u(x) = \int_\Sigma \frac{\partial Q(x,y)}{\partial n_y}[\gamma_0]u(y)ds_y,\ V[\gamma_1]u(x) = \langle[\gamma_1]u, Q(x,\cdot)\rangle$.
If $[\gamma_1]u \in L_p(\Sigma), p \geq 1$, then $V[\gamma_1]u(x) = \int_\Sigma Q(x,y)[\gamma_1]u(y)ds_y$.

Let us denote $N\tau = \frac{1}{2}(\gamma_1^+V\tau+\gamma_1^-V\tau),\ M\mu = \frac{1}{2}(\gamma_0^+W\mu+\gamma_0^-W\mu)$. We have the wellknown jump relations ([2]).

Lemma 4. *Let* $\tau \in H^{-1/2}(\Sigma)$ *and* $\mu \in H^{1/2}(\Sigma)$. *Then*

$$\gamma_0^+V\tau = \gamma_0^-V\tau,\ \gamma_1^\pm V\tau = \pm\frac{1}{2}\tau + N\tau;\ \gamma_1^+W\mu = \gamma_1^-W\mu,\ \gamma_0^\pm W\mu = \mp\frac{1}{2}\mu + M\mu.$$

If $u \in$ker L from (5) we can get the following representation

$$u(x) = V[\gamma_1]u(x) - W[\gamma_0]u(x). \tag{6}$$

Let us denote ker $[\gamma_0] = \{u \in H^1(\Omega')\colon \gamma_0^+u_+ = \gamma_0^-u_-\}$, ker $[\gamma_1] = \{u \in H^1(\Omega', L)\colon \gamma_1^+u_+ = \gamma_1^-u_-\}$, ker $[\gamma_0]_L$ = ker $[\gamma_0]\cap$ker L and ker $[\gamma_1]_L$ = ker $[\gamma_1]\cap$ker L.

Let $u, v \in$ker L. Then by using the following first Green formula for $u, v \in H^1(\Omega', L)$ $(u,v)_L = (Lu, v)_{L_2(\Omega')} + \langle\gamma_1^+u_+, \gamma_0^+v_+\rangle - \langle\gamma_1^-u_-, \gamma_0^-v_-\rangle$ we can obtain

$$(u,v)_L = \langle\gamma_1^+u_+, [\gamma_0]v\rangle + \langle[\gamma_1]u, \gamma_0^-v_-\rangle. \tag{7}$$

Lemma 5. *ker* L = *ker* $[\gamma_0]_L\oplus$ *ker* $[\gamma_1]_L$.

Proof. Since ker $[\gamma_0]_L$ is a closed subspace of $H^1(\Omega')$ we have representation ker L = ker $[\gamma_0]_L\oplus$ ker $[\gamma_0]_L^\perp$. Let $v \in$ ker $[\gamma_0]_L$, i.e. $v \in$ ker L and $\gamma_0^+v_+ = \gamma_0^-v_- = g$ on Σ. Then for $u \in$ ker L from (7) we have $(u,v)_L = \langle[\gamma_1]u, g\rangle$. If $u \in$ ker $[\gamma_0]_L^\perp$ it follows $\langle[\gamma_1]u, g\rangle = 0$ for an arbitrary $g \in H^{1/2}(\Sigma)$. Thus $u \in$ker $[\gamma_1]_L$ or ker $[\gamma_0]_L^\perp \subset$ker $[\gamma_1]_L$. Let now $u \in$ ker $[\gamma_1]_L$. Then $(u,v)_L = 0$ for all $v \in$ ker $[\gamma_0]_L$ or $u \in$ker $[\gamma_0]_L^\perp$. This means that ker $[\gamma_0]_L^\perp \subset$ ker $[\gamma_1]_L$ or ker $[\gamma_0]_L^\perp$ = ker $[\gamma_1]_L$. □

Lemma 6. *The operators* $V\colon H^{-1/2}(\Sigma) \to$ *ker* $[\gamma_0]_L$ *and* $W\ \colon H^{1/2}(\Sigma) \to$ *ker* $[\gamma_1]_L$ *are isomorphisms.*

Proof. From the representation (6) which is valid for all $u \in$ ker L we see that $W = [\gamma_0]^{-1}$ on ker $[\gamma_1]_L$ and $V = [\gamma_1]^{-1}$ on ker $[\gamma_0]_L$. Let us show that $[\gamma_0]\colon$ ker $[\gamma_1]_L \to H^{1/2}(\Sigma)$ and $[\gamma_1]\colon$ ker $[\gamma_0]_L \to H^{-1/2}(\Sigma)$ are isomorphisms. If we take $u \in$ ker L such that $u_+ \in$ ker L_+ and $u_- = 0$ then from Lemma 1 and Lemma 2 it follows the surjectivity of $[\gamma_0]\colon$ ker $L \to H^{1/2}(\Sigma)$ and $[\gamma_1]\colon$ ker $L \to H^{-1/2}(\Sigma)$. Lemma 5 gives us the injectivity of $[\gamma_0]$ on ker $[\gamma_1]_L$ and $[\gamma_1]$ on ker $[\gamma_0]_L$ respectively. The continuity of $[\gamma_0]$ and $[\gamma_1]$ follows from the continuity of $\gamma_0^\pm$ and $\gamma_1^\pm$. Thus W and V are isomorphisms. □

In what follows for convenience we use the next obvious result.

Proposition 1. *Let X, Y are the Hilbert spaces and the linear operator $A\colon X \to Y$ be an isomorphism. Then the norm $\|\cdot\|_X$ is equivalent to the norm $\|\cdot\|_*$, where $\|x\|_* = \|Ax\|_Y$ and the inner product is defined as $(y,z)_* = (Ay, Az)_Y$.*

Based on Proposition 1 and Lemma 6 we can introduce an equivalent norm in $H^{1/2}(\Sigma)$ as

$$\|\mu\|^2_{H^{1/2}(\Sigma)} = \|W\mu\|^2_L = \|w_+\|^2_L + \|w_-\|^2_L, \tag{8}$$

where $w_\pm$ are as usual the restrictions of $w = W\mu$ on $\Omega_\pm$. Analogously for $H^{-1/2}(\Sigma)$ we can use the following norm

$$\|\tau\|^2_{H^{-1/2}(\Sigma)} = \|V\tau\|^2_L = \|v_+\|^2_L + \|v_-\|^2_L.$$

Here $v_\pm$ are the restrictions of $v = V\tau$ on $\Omega_\pm$.

Let us denote $H = -\gamma_1^\pm W,\ K = \gamma_0^\pm V$. As a consequence of Lemma 1, Lemma 2 and Lemma 6 we can get the next corollary.

Corollary 1. *The operators $H\colon H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma)$ and $K\colon H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$ are isomorphisms.*

Moreover for K and H we have the following result.

Lemma 7. *$H\colon H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma)$ and $K\colon H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma)$ are positive definite, i.e. $\langle H\mu, \mu\rangle = \|\mu\|^2_{H^{1/2}(\Sigma)}$ and $\langle \tau, K\tau\rangle = \|\tau\|^2_{H^{-1/2}(\Sigma)}$.*

Proof. Let $\mu \in H^{1/2}(\Sigma),\ \tau \in H^{-1/2}(\Sigma)$ and $w = W\mu,\ v = V\tau$. Then from (7) we see that $\|w\|^2_L = \langle H\mu, \mu\rangle$, which together with (8) gives us the first assertion of the lemma. Analogously for K . □

By using Lemma 7 we can consider the space $H^{1/2}(\Sigma)$ with the following equivalent norm and the inner product

$$\|\mu\|^2_{H^{1/2}(\Sigma)} = \langle H\mu, \mu\rangle, \qquad (\mu, \nu)_{H^{1/2}(\Sigma)} = \langle H\mu, \nu\rangle = \langle H\nu, \mu\rangle.$$

Analogously for $H^{-1/2}(\Sigma)$

$$\|\tau\|^2_{H^{-1/2}(\Sigma)} = \langle \tau, K\tau\rangle, \qquad (\tau, \nu)_{H^{-1/2}(\Sigma)} = \langle \tau, K\nu\rangle = \langle \nu, K\tau\rangle.$$

Let us denote $H_i = -\gamma^\pm_{1,S_i} W = \widetilde{r}_{S_i} H, \quad H_i\colon H^{1/2}(\Sigma) \to H^{-1/2}(S_i)$ and introduce the following spaces

$$H^{1/2}_{00}(S_1 \cup S_2) = \{\mu \in H^{1/2}(\Sigma)\colon \mu_i = r_{S_i}\mu \in H^{1/2}_{00}(S_i),\ i = 1,2\},$$
$$H^{1/2}_{\alpha}(\Sigma) = \{\alpha \in H^{1/2}(\Sigma)\colon H_i\alpha = 0,\ i = 1,2\}.$$

As it was mentioned above the space $H^{1/2}_{00}(S_1)$ is essentially different from the space $H^{1/2}(S_1)$. Moreover $H^{1/2}_{00}(S_1)$ is dense in $H^{1/2}(S_1)$. Thus we can't obtain orthogonal decomposition of $H^{1/2}(S_1)$ analogously as in a case of $H^1(\Omega_+)$. But we can note some similar properties between the operator L in $H^1(\Omega_+)$ and the operator H in $H^{1/2}(\Sigma)$. We have the following decomposition.

Lemma 8. *$H^{1/2}(\Sigma) = H^{1/2}_{00}(S_1 \cup S_2) \oplus H^{1/2}_{\alpha}(\Sigma)$.*

Proof. Since $H^{1/2}_{\widetilde{S}_i}(\Sigma)$, $i = 1,2$, are the closed subspaces of $H^{1/2}(\Sigma)$ then $H^{1/2}_{00}(S_1 \cup S_2)$ is a closed subspace of $H^{1/2}(\Sigma)$ and we see that $H^{1/2}(\Sigma) = H^{1/2}_{00}(S_1 \cup S_2) \oplus H^{1/2}_{00}(S_1 \cup S_2)^\perp$. Let $\mu \in H^{1/2}_{00}(S_1 \cup S_2),\ \mu_i = r_{S_i}\mu$ and $\alpha \in H^{1/2}_{00}(S_1 \cup S_2)^\perp,\ \alpha_i = r_{S_i}\alpha$. It's obvious that

$\alpha_i \notin H_{00}^{1/2}(S_i)$. We have $(\mu,\alpha)_{H^{1/2}(\Sigma)} = \langle H\alpha,\mu\rangle = 0$. Let $\mu \in H_{\overline{S}_1}^{1/2}(\Sigma)$, i.e. $\mu_2 = 0, \mu_1 \in H_{00}^{1/2}(S_1)$. Then we obtain $\langle H\alpha,\mu\rangle = \langle H_1\alpha,\mu_1\rangle = 0$ for an arbitrary $\mu_1 \in H_{00}^{1/2}(S_1)$. This means that $H_1\alpha = 0$. In the same way we can get that $H_2\alpha = 0$. Thus $\alpha \in H_{\alpha}^{1/2}(\Sigma)$ and $H_{00}^{1/2}(S_1 \cup S_2)^{\perp} \subset H_{\alpha}^{1/2}(\Sigma)$.

Every $\mu \in H_{00}^{1/2}(S_1 \cup S_2)$ can be represented in the unique manner as $\mu = p_0\mu_1 + p_0\mu_2$, where $p_0\mu_i \in H_{\overline{S}_i}^{1/2}(\Sigma)$. Let $\alpha \in H_{\alpha}^{1/2}(\Sigma)$. Then we see that $(\alpha,\mu)_{H^{1/2}(\Sigma)} = \langle H\alpha,\mu\rangle = \langle H\alpha,p_0\mu_1\rangle + \langle H\alpha,p_0\mu_2\rangle = \langle H_1\alpha,\mu_1\rangle + \langle H_2\alpha,\mu_2\rangle = 0$. Hence $\alpha \in H_{00}^{1/2}(S_1 \cup S_2)^{\perp}$ or $H_{\alpha}^{1/2}(\Sigma) \subset H_{00}^{1/2}(S_1 \cup S_2)^{\perp}$ and finally $H_{00}^{1/2}(S_1 \cup S_2)^{\perp} = H_{\alpha}^{1/2}(\Sigma)$. □

If $\alpha \in H_{\alpha}^{1/2}(\Sigma)$ then $H\alpha \in H_{\Gamma}^{-1/2}(\Sigma)$ and we can consider $H_{\alpha}^{1/2}(\Sigma)$ as a some type of a kernel of H.

By using Lemma 8 we can infer that an arbitrary $\mu \in H^{1/2}(\Sigma)$ has a unique form $\mu = \tau + \alpha$, where $\tau \in H_{00}^{1/2}(S_1 \cup S_2)$, $\alpha \in H_{\alpha}^{1/2}(\Sigma)$. It's important that τ_1 and τ_2 are not connected with each other but α_2 depends on α_1. More precisely we can say that for every $\alpha_1 = \tau_1 - \mu_1$ where $\tau_1 \in H^{1/2}(S_1)$, $\mu_1 \in H_{00}^{1/2}(S_1)$ there exists a unique $\alpha_2 \in H^{1/2}(S_2)$ such that $\alpha = (\alpha_1,\alpha_2) \in H_{\alpha}^{1/2}(\Sigma)$.

Analogously we can consider the space $H^{-1/2}(\Sigma)$. Let us denote

$$H_2^{-1/2}(\Sigma) = \{\tau \in H^{-1/2}(\Sigma) \colon K\tau \in H_{\overline{S}_2}^{1/2}(\Sigma)\},$$
$$H_{\alpha}^{-1/2}(\Sigma) = \{\tau \in H^{-1/2}(\Sigma) \colon K\tau \in H_{00}^{1/2}(S_1 \cup S_2)\}.$$

It's easy to verify that $H_2^{-1/2}(\Sigma)$ and $H_{\alpha}^{-1/2}(\Sigma)$ are the closed subspaces of $H^{-1/2}(\Sigma)$.

Lemma 9. $H^{-1/2}(\Sigma) = H_{\overline{S}_1}^{-1/2}(\Sigma) \oplus H_2^{-1/2}(\Sigma)$, $H^{-1/2}(\Sigma) = H_{\Gamma}^{-1/2}(\Sigma) \oplus H_{\alpha}^{-1/2}(\Sigma)$.

Proof. Since $H_{\overline{S}_1}^{-1/2}(\Sigma)$ is a closed subspace of $H^{-1/2}(\Sigma)$ then we have $H^{-1/2}(\Sigma) = H_{\overline{S}_1}^{-1/2}(\Sigma) \oplus H_{\overline{S}_1}^{-1/2}(\Sigma)^{\perp}$. Let $\tau \in H_{\overline{S}_1}^{-1/2}(\Sigma)$, $\mu \in H_{\overline{S}_1}^{-1/2}(\Sigma)^{\perp}$. Then $(\tau,\mu)_{H^{-1/2}(\Sigma)} = \langle\tau,K\mu\rangle = 0$ or $K\mu \in H_{\overline{S}_2}^{1/2}(\Sigma)$, i.e. $\mu \in H_2^{-1/2}(\Sigma)$. Let $\mu \in H_2^{-1/2}(\Sigma)$. Then $(\tau,\mu)_{H^{-1/2}(\Sigma)} = \langle\tau,K\mu\rangle = 0$ or $\mu \in H_{\overline{S}_1}^{-1/2}(\Sigma)^{\perp}$. Thus we obtain $H_2^{-1/2}(\Sigma) = H_{\overline{S}_1}^{-1/2}(\Sigma)^{\perp}$.
For the second decomposition of the lemma we see that $H^{-1/2}(\Sigma) = H_{\Gamma}^{-1/2}(\Sigma) \oplus H_{\Gamma}^{-1/2}(\Sigma)^{\perp}$. Let $\tau \in H_{\Gamma}^{-1/2}(\Sigma)$, $\alpha \in H_{\Gamma}^{-1/2}(\Sigma)^{\perp}$. Then $(\tau,\alpha)_{H^{-1/2}(\Sigma)} = \langle\tau,K\alpha\rangle = 0$ or $K\alpha \in H_{00}^{1/2}(S_1 \cup S_2)$. If $\alpha \in H_{\alpha}^{-1/2}(\Sigma)$ then $(\tau,\alpha)_{H^{-1/2}(\Sigma)} = \langle\tau,K\alpha\rangle = 0$ or $\alpha \in H_{\Gamma}^{-1/2}(\Sigma)^{\perp}$. It yields us that $H_{\Gamma}^{-1/2}(\Sigma)^{\perp} = H_{\alpha}^{-1/2}(\Sigma)$. □

From (6) it follows for an arbitrary $u \in \ker L$:

$$u = V\tau - W\mu, \tag{9}$$

where $\mu \in H^{1/2}(\Sigma)$, $\tau \in H^{-1/2}(\Sigma)$. From (9) by using the jump relations we can get

$$\gamma_0^{\pm}u_{\pm} = K\tau \pm \tfrac{1}{2}\mu - M\mu, \qquad \gamma_1^{\pm}u_{\pm} = \pm\tfrac{1}{2}\tau + N\tau + H\mu. \tag{10}$$

If $u_+ \in \ker L_+$, $\mu = \gamma_0^+u_+$, $\tau = \gamma_1^+u_+$ from (10) we obtain $C\beta = \beta$, where $C = \frac{1}{2}I + B$, $\beta = (\mu,\tau)$, $B = \begin{pmatrix} -M & K \\ H & N \end{pmatrix}$. Here C is the well known *Calderon projector.*

Let us denote $X(\Sigma) = H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)$ with the the following norm

$$\|\beta\|_{X(\Sigma)}^2 = \|\beta_1\|_{H^{-1/2}(\Sigma)}^2 + \|\beta_2\|_{H^{1/2}(\Sigma)}^2,$$

where $\beta = (\beta_1, \beta_2) \in X(\Sigma)$. Then $X(\Sigma)' = H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$ and if $\alpha = (\alpha_1, \alpha_2) \in X(\Sigma)'$ we can define the relation of duality as $\langle \alpha, \beta \rangle = \langle \beta_1, \alpha_1 \rangle + \langle \alpha_2, \beta_2 \rangle$. It's easy to verify that $B\colon X(\Sigma)' \to X(\Sigma)'$ is an isomorphism and $B^{-1} = \frac{1}{4}B$.

In the theory of mixed boundary value problems the following matrix operator

$$A = \begin{pmatrix} K & -M \\ N & H \end{pmatrix} \tag{11}$$

plays an important role. Since B is an isomorphism the operator $A\colon X(\Sigma) \to X(\Sigma)'$ is also an isomorphism.

Theorem 1. *The operator $A\colon X(\Sigma) \to X(\Sigma)'$ is positive definite, i.e.*

$$\langle A\alpha, \alpha \rangle = \|\mu\|^2_{H^{1/2}(\Sigma)} + \|\tau\|^2_{H^{-1/2}(\Sigma)} = \|\alpha\|^2_{X(\Sigma)},\ \alpha = (\tau, \mu).$$

Proof. By using Lemma 7 we can obtain $\langle A\alpha, \alpha \rangle = \langle \tau, K\tau \rangle - \langle \tau, M\mu \rangle + \langle N\tau, \mu \rangle + \langle H\mu, \mu \rangle = \|\mu\|^2_{H^{1/2}(\Sigma)} + \|\tau\|^2_{H^{-1/2}(\Sigma)} - \langle \tau, M\mu \rangle + \langle N\tau, \mu \rangle$. We have to show that for an arbitrary $\mu \in H^{1/2}(\Sigma)$ and $\tau \in H^{-1/2}(\Sigma)$ it holds $\langle \tau, M\mu \rangle = \langle N\tau, \mu \rangle$. Let $v = V\tau,\ w = W\mu$. By applying the second Green formula (2) we can get $\pm\langle \gamma_1^{\pm} w_{\pm}, \gamma_0^{\pm} v_{\pm} \rangle \mp \langle \gamma_1^{\pm} v_{\pm}, \gamma_0^{\pm} w_{\pm} \rangle = 0$. So far as $\gamma_1^{+} w_{+} = \gamma_1^{-} w_{-}$ and $\gamma_0^{+} v_{+} = \gamma_0^{-} v_{-}$ we see that $\langle \gamma_1^{+} v_{+}, \gamma_0^{+} w_{+} \rangle - \langle \gamma_1^{-} v_{-}, \gamma_0^{-} w_{-} \rangle = 0$. According to Lemma 4 we obtain $\langle \frac{1}{2}\tau + N\tau, -\frac{1}{2}\mu + M\mu \rangle - \langle -\frac{1}{2}\tau + N\tau, \frac{1}{2}\mu + M\mu \rangle = 0$ or finally $\langle \tau, M\mu \rangle = \langle N\tau, \mu \rangle$, what was to be proved. □

4. Mixed boundary value problems for a closed surface. As usual we pose the following boundary value problem (b.v.p.) in Ω_+: we look for a function $u_+ \in H^1(\Omega_+, L)$ such that

$$Lu_+ = h,\ \gamma^+_{0,S_1} u_+ = g_1,\ \gamma^+_{1,S_2} u_+ = f_2,$$

where $h \in L_2(\Omega_+),\ g_1 \in H^{1/2}(S_1),\ f_2 \in H^{-1/2}(S_2)$.

If we put $u_+ = w_+ + v_+$ then we can split this problem into two independent ones:

$$Lv_+ = 0,\ \gamma^+_{0,S_1} v_+ = g_1,\ \gamma^+_{1,S_2} v_+ = f_2, \quad v_+ \in H^1(\Omega_+, L), \tag{12}$$

$$Lw_+ = h,\ \gamma^+_{0,S_1} w_+ = 0,\ \gamma^+_{1,S_2} w_+ = 0, \quad w_+ \in H^1(\Omega_+, L). \tag{13}$$

Let us denote $\gamma^{\pm}_{0,1} = (\gamma^{\pm}_{0,S_1}, \gamma^{\pm}_{1,S_2})$, $\ker \gamma^{\pm}_{0,1} = \{u_{\pm} \in H^1(\Omega_{\pm}, L)\colon \gamma^{\pm}_{0,S_1} u_{\pm} = 0,\ \gamma^{\pm}_{1,S_2} u_{\pm} = 0\}$.

Theorem 2. *The b.v.p. (12) has a unique solution $u_+ \in H^1(\Omega_+, L)$ for arbitrary $g_1 \in H^{1/2}(S_1),\ f_2 \in H^{-1/2}(S_2)$. The operator $\gamma^+_{0,1}\colon\ \ker L_+ \to H^{1/2}(S_1) \times H^{-1/2}(S_2)$ is an isomorphism.*

Proof. Let $g_1 \in H^{1/2}(S_1),\ f_2 \in H^{-1/2}(S_2)$. We denote by $g \in H^{1/2}(\Sigma)$ and $f \in H^{-1/2}(\Sigma)$ the extensions of g_1 and f_2 on Σ respectively. Then from Lemma 1 we have a unique $v_+ \in \ker L_+$ such that $\gamma_0^+ v_+ = g$. Lemma 3 gives us a unique representation $v_+ = v_1 + v_2$ where $v_1 \in \ker \gamma^+_{0,S_1}$, $v_2 \in \ker \gamma^+_{1,S_2}$. Thus we obtain a unique $v_2 \in \ker \gamma^+_{1,S_2}$ with the boundary condition $\gamma^+_{0,S_1} v_2 = \gamma^+_{0,S_1}(v_+ - v_1) = g_1$. Analogously for f by applying Lemma 2 we can obtain a unique $w_+ \in \ker L_+$ such that $\gamma_1^+ w_+ = f$. So far as we have a unique representation $w_+ = w_1 + w_2$ where $w_1 \in \ker \gamma^+_{0,S_1}$, $w_2 \in \ker \gamma^+_{1,S_2}$ this gives us a unique function $w_1 \in \ker \gamma^+_{0,S_1}$ with the boundary condition $\gamma^+_{1,S_2} w_1 = f_2$.

Hence for arbitrary $g_1 \in H^{1/2}(S_1),\ f_2 \in H^{-1/2}(S_2)$ we can get a unique function $u_+ = v_2 + w_1$ where $v_2 \in \ker \gamma^+_{1,S_2}$, $w_1 \in \ker \gamma^+_{0,S_1}$ with the boundary conditions $\gamma^+_{0,S_1} u_+ = g_1,\ \gamma^+_{1,S_2} u_+ = f_2$ and this proves the first assertion of the lemma. If we take to attention the continuity of the operator $\gamma^+_{0,1}$ we obtain that $\gamma^+_{0,1}\colon\ \ker L_+ \to H^{1/2}(S_1) \times H^{-1/2}(S_2)$ is an isomorphism. Let us note that the injectivity of $\gamma^+_{0,1}$ also follows directly from Lemma 3. □

Theorem 3. *The b.v.p. (13) has a unique solution $u_+ \in H^1(\Omega_+, L)$ for an arbitrary $h \in L_2(\Omega_+)$. The operator $L\colon$ ker $\gamma^+_{0,1} \to L_2(\Omega_+)$ is an isomorphism.*

Proof. Let $h \in L_2(\Omega_+) \subset H^{-1}(\Omega_+)$. Then from Lemma 1 it follows that there exists a unique $v \in H^1_0(\Omega_+)$ such that $Lv = h$. By Theorem 2 for $f = \gamma^+_1 v \in H^{-1/2}(\Sigma)$ there exists a unique $w \in \ker \gamma^+_{0,S_1}$ with $\gamma^+_{1,S_2} w = \widetilde{r}_{S_2} f \in H^{-1/2}(S_2)$. Thus for an arbitrary $h \in L_2(\Omega_+)$ we get a unique function $u_+ = v - w$ such that $Lu_+ = Lv = h$ and $\gamma^+_{0,S_1} u_+ = 0$, $\gamma^+_{1,S_2} u_+ = \widetilde{r}_{S_2} f - \widetilde{r}_{S_2} f = 0$. Hence the operator $L\colon$ ker $\gamma^+_{0,1} \to L_2(\Omega_+)$ is bijective. It's continuity completes the proof. □

Let us remind that an arbitrary $u_+ \in \ker L_+$ has the following representation

$$u_+ = V\tau - W\mu, \qquad \tau = \gamma^+_1 u_+,\ \mu = \gamma^+_0 u_+. \tag{14}$$

As usual we denote $\tau_i = \widetilde{r}_{S_i}\tau$, $\mu_i = r_{S_i}\mu$. Let f and g be arbitrary extensions of f_2 and g_1 on Σ respectively, i.e. $f_2 = \widetilde{r}_{S_2} f$, $g_1 = r_{S_1} g$. From the boundary conditions in (12) we see that $\mu_1 = g_1$, $\tau_2 = f_2$. If we take to attention the properties of $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$ we can get a unique representation $\tau = \tau_0 + f$, $\mu = \mu_0 + g$ (for smooth Σ see [9]). Here $\tau_0 \in H^{-1/2}_{\overline{S}_1}(\Sigma)$, $\mu_0 \in H^{1/2}_{\overline{S}_2}(\Sigma)$ and it's obvious that τ_0 and μ_0 depend on the extensions f of f_1 and g of g_2 respectively. Hence from (14) we obtain the following expression for u_+

$$u_+ = V\tau_0 - W\mu_0 + Vf - Wg. \tag{15}$$

Let us denote $K_i = \gamma^\pm_{0,S_i} V = r_{S_i} K$, $M_i = r_{S_i} M$, $N_i = \widetilde{r}_{S_i} N$, $i = 1, 2$. Since $\widetilde{r}_{S_2}\tau_0 = 0$, $r_{S_1}\mu_0 = 0$ by using the boundary conditions in (12) and Lemma 4 we can get

$$\gamma^+_{0,S_1} u_+ = \gamma^+_{0,S_1} V\tau_0 - M_1\mu_0 + \gamma^+_{0,S_1} Vf + \tfrac{1}{2} g_1 - M_1 g = g_1,$$
$$\gamma^+_{1,S_2} u_+ = N_2\tau_0 - \gamma^+_{1,S_2} W\mu_0 + \tfrac{1}{2} f_2 + N_2 f - \gamma^+_{1,S_2} Wg = f_2.$$

Thus we obtain the following system

$$\begin{cases} K_1\tau_0 - M_1\mu_0 = \widetilde{g}_1, \\ N_2\tau_0 + H_2\mu_0 = \widetilde{f}_2, \end{cases} \tag{16}$$

where

$$\widetilde{g}_1 = \tfrac{1}{2} g_1 - K_1 f + M_1 g, \qquad \widetilde{f}_2 = \tfrac{1}{2} f_2 - N_2 f + H_2 g. \tag{17}$$

From the properties of the trace maps γ^+_{0,S_1} and γ^+_{1,S_2} it follows that $\widetilde{g}_1 \in H^{1/2}(S_1)$ and $\widetilde{f}_2 \in H^{-1/2}(S_2)$. Let us denote $A_0 = \begin{pmatrix} K_1 & -M_1 \\ N_2 & H_2 \end{pmatrix}$. Then we can rewrite the system (16) as $A_0\alpha = \beta$, where $\alpha = (\tau_0, \mu_0)$, $\beta = (\widetilde{g}_1, \widetilde{f}_2)$.

For the sake of brevity we denote by $\tau_0 \in H^{-1/2}_{00}(S_1)$ the restriction of $\tau_0 \in H^{-1/2}_{\overline{S}_1}(\Sigma)$ to S_1. Analogously $\mu_0 \in H^{1/2}_{00}(S_2)$ is the restriction of $\mu_0 \in H^{1/2}_{\overline{S}_2}(\Sigma)$ to S_2. Since $\tau_0 \in H^{-1/2}_{\overline{S}_1}(\Sigma)$ and $\mu_0 \in H^{1/2}_{\overline{S}_2}(\Sigma)$ we can consider the operators K_1 and N_2 in (16) as the boundary operators upon the open surface S_1 and the operators M_1 and H_2 as the boundary operators upon the open surface S_2, i.e. $K_1\colon H^{-1/2}_{00}(S_1) \to H^{1/2}(S_1)$, $H_2\colon H^{1/2}_{00}(S_2) \to H^{-1/2}(S_2)$ and so on.

Let us denote $X_0(\Sigma) = H^{-1/2}_{00}(S_1) \times H^{1/2}_{00}(S_2)$. Then $X_0(\Sigma)' = H^{1/2}(S_1) \times H^{-1/2}(S_2)$ and for $\alpha = (\alpha_1, \alpha_2) \in X_0(\Sigma)$, $\beta = (\beta_1, \beta_2) \in X_0(\Sigma)'$ we use the norm

$$\|\alpha\|^2_{X_0(\Sigma)} = \|\alpha_1\|^2_{H^{-1/2}_{00}(S_1)} + \|\alpha_2\|^2_{H^{1/2}_{00}(S_2)}$$

and the relation of duality $\langle\beta, \alpha\rangle = \langle\alpha_1, \beta_1\rangle + \langle\beta_2, \alpha_2\rangle$. Finally we see that

$$A_0\colon X_0(\Sigma) \to X_0(\Sigma)'.$$

Theorem 4. *The operator* $A_0\colon X_0(\Sigma)\to X_0(\Sigma)'$ *is positive definite, i.e.* $\langle A_0\alpha,\alpha\rangle=\|\alpha\|^2_{X_0(\Sigma)}$ *for all* $\alpha\in X_0(\Sigma)$.

Proof. By using the properties of $H^{1/2}_{\overline{S}_2}(\Sigma)$ and $H^{-1/2}_{\overline{S}_1}(\Sigma)$ it's easy to verify that $\langle A\beta,\beta\rangle=\langle A_0\alpha,\alpha\rangle$, where $\beta=(\tau,\mu)\in H^{-1/2}_{\overline{S}_1}(\Sigma)\times H^{1/2}_{\overline{S}_2}(\Sigma)\subset X(\Sigma)$, $\alpha=(\widetilde{r}_{S_1}\tau,r_{S_2}\mu)\in X_0(\Sigma)$ and A is a matrix operator given by (11). Since for $\tau\in H^{-1/2}_{\overline{S}_1}(\Sigma)$ and $\mu\in H^{1/2}_{\overline{S}_2}(\Sigma)$ we have $\|\tau\|_{H^{-1/2}(\Sigma)}=\|\widetilde{r}_{S_1}\tau\|_{H^{-1/2}_{00}(S_1)}$ and $\|\mu\|_{H^{1/2}(\Sigma)}=\|r_{S_2}\mu\|_{H^{1/2}_{00}(S_2)}$ the lemma is a consequence of Theorem 1. □

Corollary 2. *The operator* $A_0\colon X_0(\Sigma)\to X_0(\Sigma)'$ *is an isomorphism.*

We have shown that the system (16) has a unique solution for an arbitrary $(\widetilde{g}_1,\widetilde{f}_2)\in X_0(\Sigma)'$. The next important question is a connection between the b.v.p. (12) and the system (16).

Theorem 5. *Let* f *and* g *are arbitrary extensions of* f_2 *and* g_1 *on* Σ *respectively. Then the b.v.p. (12) is equivalent to the system (16), i.e. the solution* $u_+\in H^1(\Omega_+,L)$ *of the b.v.p. (12) has the form (15), where* (τ_0,μ_0) *is a solution of the system (16) with* $(\widetilde{g}_1,\ \widetilde{f}_2)$ *given by (17), and vice versa if* (τ_0,μ_0) *is a solution of the system (16) where* $(\widetilde{g}_1,\ \widetilde{f}_2)$ *is given by (17), then the function* u_+ *defined by (15) belongs to* $H^1(\Omega_+,L)$ *and is a solution of the b.v.p. (12).*

Proof. As it was mentioned above the representation (15) is a consequence of (14) and the boundary conditions in (12). Here $\tau_0+f=\gamma_1^+u_+$, $\mu_0+g=\gamma_0^+u_+$. By using the jump relations, the boundary conditions in (12) and the representation (15) we can reduce the b.v.p. (12) to the system (16).

Let now function u_+ be given by (15) where (τ_0,μ_0) is a solution of (16). Since $\tau_0+f\in H^{-1/2}(\Sigma)$ and $\mu_0+g\in H^{1/2}(\Sigma)$ from Lemma 6 it follows that $u_+\in\ker L_+\subset H^1(\Omega_+,L)$. If we apply jump relations and take to attention that $A_0(\tau_0,\mu_0)=(\widetilde{g}_1,\widetilde{f}_2)$ where $(\widetilde{g}_1,\widetilde{f}_2)$ are given by (17) then we obtain the boundary conditions $\gamma^+_{0,S_1}u_+=g_1$ and $\gamma^+_{1,S_2}u_+=f_2$. Thus u_+ is a solution of the b.v.p. (12). □

Let's consider the b.v.p. (13). From (5) we have

$$u_+=Dh+V\tau-W\mu,\qquad \tau=\gamma_1^+u_+,\ \mu=\gamma_0^+u_+. \tag{18}$$

Since $u_+\in\ker\gamma^+_{0,1}$ it follows that $\tau\in H^{-1/2}_{\overline{S}_1}(\Sigma)$ and $\mu\in H^{1/2}_{\overline{S}_2}(\Sigma)$. By using the jump relations we can get

$$\gamma^+_{0,S_1}u_+=\gamma^+_{0,S_1}Dh+\gamma^+_{0,S_1}V\tau-M_1\mu=0,\quad \gamma^+_{1,S_2}u_+=\gamma^+_{1,S_2}Dh+N_2\tau-\gamma^+_{1,S_2}W\mu=0.$$

Let us denote $\tau_0=\widetilde{r}_{S_1}\tau$, $\mu_0=r_{S_2}\mu$ and

$$\widetilde{g}=-\gamma^+_{0,S_1}Dh,\ \widetilde{f}=-\gamma^+_{1,S_2}Dh. \tag{19}$$

Thus we obtain the following system

$$\begin{cases} K_1\tau_0-M_1\mu_0=\widetilde{g},\\ N_2\tau_0+H_2\mu_0=\widetilde{f}, \end{cases} \tag{20}$$

or $A_0\alpha=\beta$, where $\alpha=(\tau_0,\mu_0)\in X_0(\Sigma)$, $\beta=(\widetilde{g},\widetilde{f})\in X_0(\Sigma)'$.
Corollary 2 gives us that the system (20) has a unique solution for all $(\widetilde{g},\widetilde{f})\in X_0(\Sigma)'$.

Theorem 6. *The b.v.p. (13) is equivalent to the system (20), i.e. the solution $u_+ \in H^1(\Omega_+, L)$ of the b.v.p. (13) has the form (18), where $\tau_0 = \widetilde{r}_{S_1}\tau$, $\mu_0 = r_{S_2}\mu$ is a solution of the system (20) with $(\widetilde{g}, \widetilde{f})$ given by (19), and vice versa if (τ_0, μ_0) is a solution of the system (20) where $(\widetilde{g}, \widetilde{f})$ is given by (19), then the function u_+ defined by (18) belongs to $H^1(\Omega_+, L)$ and is a solution of the b.v.p. (13).*

Proof. The representation (18) follows from (5) with $u_- = 0$. Since $u_+ \in \ker\,\gamma^+_{0,1}$ by virtue of the jump relations we can reduce (13) to the system (20) where $(\widetilde{g}, \widetilde{f})$ are given by (19).

Let function u_+ be defined via (18) where $\tau \in H^{-1/2}_{\overline{S}_1}(\Sigma)$, $\mu \in H^{1/2}_{\overline{S}_2}(\Sigma)$, $h \in L_2(\Omega_+)$ and $\tau_0 = \widetilde{r}_{S_1}\tau$, $\mu_0 = r_{S_2}\mu$ is a solution of the system (20). Then $u_+ \in H^1(\Omega_+, L)$, $Lu_+ = h$ and by using the jump relations we can verify that $u_+ \in \ker\,\gamma^+_{0,1}$. Thus u_+ is a solution of the b.v.p. (13). □

In order to obtain the numerical solution of the posed mixed boundary value problems we may consider Galerkin approximation which is the most natural approximate scheme for solving of the system (16) if we take to attention the positive definiteness of the operator A_0.

Let us recall that we consider the spaces $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$ with the norms and the inner products defined by the operators H and K respectively, i.e. $\|\tau\|^2_{H^{-1/2}(\Sigma)} = \langle \tau, K\tau\rangle$, $\|\mu\|^2_{H^{1/2}(\Sigma)} = \langle H\mu, \mu\rangle$, $\quad (\tau_1, \tau_2)_{H^{-1/2}(\Sigma)} = \langle \tau_1, K\tau_2\rangle$, $\quad (\mu_1, \mu_2)_{H^{1/2}(\Sigma)} = \langle H\mu_1, \mu_2\rangle$.

Let $\{\varphi_i\}_{i=1}^{\infty}$ be a basis in $H^{1/2}_{00}(S_2)$ and $\{\psi_i\}_{i=1}^{\infty}$ be a basis in $H^{-1/2}_{00}(S_1)$. We denote $\sigma_i = (\psi_i, \varphi_i)$ and by definition $\sigma_i \in X_0(\Sigma)$.

Let $\alpha = (\tau_0, \mu_0)$. An approximate solution $\alpha_n = (\tau_n, \mu_n)$ of the system (16) we search as

$$\alpha_n = \textstyle\sum_{i=1}^{n} a_i\sigma_i \tag{21}$$

and unknown meaning of a_i we get from the following system

$$\textstyle\sum_{i=1}^{n} a_i\langle A_0\sigma_i, \sigma_j\rangle = \langle \beta, \sigma_j\rangle,\ j \in \{1, \dots, n\}. \tag{22}$$

Since $\langle A_0\sigma_i, \sigma_j\rangle = (\sigma_i, \sigma_j)_{X_0(\Sigma)} = (\varphi_i, \varphi_j)_{H^{1/2}_{00}(S_2)} + (\psi_i, \psi_j)_{H^{-1/2}_{00}(S_1)}$ and $\langle \beta, \sigma_j\rangle = \langle A_0\alpha, \sigma_j\rangle = (\alpha, \sigma_j)_{X_0(\Sigma)}$ we can rewrite the system (22) as

$$\textstyle\sum_{i=1}^{n} a_i(\sigma_i, \sigma_j)_{X_0(\Sigma)} = \langle \beta, \sigma_j\rangle = (\alpha, \sigma_j)_{X_0(\Sigma)},\ j \in \{1, \dots, n\}. \tag{23}$$

From (23) we can infer that in the case when we consider the operators K and H as canonical isometries in corresponding spaces Galerkin method gives us an element of the best approximation, i.e. α_n given by (21) where a_i we obtain from (23) is an orthogonal projection of α on the subspace $X_0^n(\Sigma)$ with $\{\sigma_i\}_{i=1}^{n}$ as a basis and we have

$$\|\alpha_n - \alpha\|_{X_0(\Sigma)} = \inf_{u \in X_0^n(\Sigma)} \|u - \alpha\|_{X_0(\Sigma)}.$$

Remark 1. *All assertions considered in this section are also valid for the exterior mixed boundary value problem: find a function $u_- \in H^1(\Omega_-, L)$ such that*

$$Lu_- = h,\ \gamma^-_{0,S_1}u_- = g_1,\ \gamma^-_{1,S_2}u_- = f_2,$$

where $h \in L_2(\Omega_-)$, $g_1 \in H^{1/2}(S_1)$, $f_2 \in H^{-1/2}(S_2)$.

5. Mixed boundary value problems for an open Lipschitz surface. According to our notation the open Lipschitz surface S_1 is a part of some closed Lipschitz surface Σ. S_1 is considered as a doublesided surface and orientation of S_1 is fixed in such a way that $\vec{n}_x$ is pointed into Ω_-.

We denote $\Omega = \mathbb{R}^3 \setminus \overline{S}_1$, $\overline{S}_1 = S_1 \cup \Gamma$. Let $C_0^\infty(\Omega)$ be a class of infinitely differentiable functions with compact support in Ω. We assume that $u \in H^1(\Omega)$ if $u \in L_2(\Omega)$ and $|\nabla u| \in$

$L_2(\Omega)$. Here derivatives we mean in the distributional sense, i.e. if $u \in L_2(\Omega)$ then for all $\varphi \in C_0^\infty(\Omega)$

$$\langle \tfrac{\partial u}{\partial x_i}, \varphi\rangle = -\int_\Omega u \tfrac{\partial \varphi}{\partial x_i} dx = -\Big(u, \tfrac{\partial \varphi}{\partial x_i}\Big)_{L_2(\Omega)}.$$

Then $H^1(\Omega)$ is a Hilbert space of functions $u(x), x \in \Omega$, with the norm

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega \{|\nabla u|^2 + u^2\}\, dx \tag{24}$$

and the inner product

$$(u, v)_{H^1(\Omega)} = \int_\Omega \{(\nabla u, \nabla v) + uv\}\, dx. \tag{25}$$

We denote by $H_0^1(\Omega)$ a closure of $C_0^\infty(\Omega)$ in the norm (24), $H^{-1}(\Omega) = (H_0^1(\Omega))'$.

If $u_\pm(x) = r_{\Omega_\pm} u(x),\ u(x) \in H^1(\Omega)$, then the norm (24) and the inner product (25) are equivalent to the following ones

$$\|u\|^2_{H^1(\Omega)} = \|u_+\|^2_{H^1(\Omega_+)} + \|u_-\|^2_{H^1(\Omega_-)} \tag{26}$$

$$(u, v)_{H^1(\Omega)} = (u_+, v_+)_{H^1(\Omega_+)} + (u_-, v_-)_{H^1(\Omega_-)} \tag{27}$$

It's obvious that the norm (26) and the inner product (27) are not depend on the choice of S_2.

Analogously as in section 2 we see that the norm $\|\cdot\|_{H^1(\Omega)}$ defined by (24) and the following norm $\|u\|^2_L = \|u_+\|^2_L + \|u_-\|^2_L$ are equivalent. Thus we can consider $H^1(\Omega)$ for a given operator L as a Hilbert space with the norm $\|\cdot\|_L$ and the inner product

$$(u, v)_L = (u_+, v_+)_L + (u_-, v_-)_L.$$

Let us introduce the space

$$H^1(\Omega, L) = \{u \in H^1(\Omega)\colon Lu \in L_2(\Omega)\}, \quad \|u\|^2_{H^1(\Omega,L)} = \|u\|^2_{H^1(\Omega)} + \|Lu\|^2_{L_2(\Omega)}.$$

As a consequence of Lemma 4.1 ([10]) we have the following corollary.

Corollary 3. *$H^1(\Omega)$ is a subspace of $H^1(\Omega')$ and coincides with ker $[\gamma_0]_{S_2} = \{u \in H^1(\Omega')\colon [\gamma_0]_{S_2} u = 0\}$. On its turn $H^1(\Omega, L)$ is a subspace of $H^1(\Omega', L)$ and coincides with ker $[\gamma_{0,1}]_{S_2} = \{u \in H^1(\Omega', L)\colon [\gamma_0]_{S_2} u = 0,\ [\gamma_1]_{S_2} u = 0\}$.*

For $u \in H^1(\Omega, L)$, $v \in H^1(\Omega)$ we have the first Green formula ([10])

$$(u, v)_L = (Lu, v)_{L_2(\Omega)} + \langle \gamma^+_{1,S_1} u, [\gamma_0]_{S_1} v\rangle + \langle [\gamma_1]_{S_1} u, \gamma^-_{0,S_1} v\rangle \tag{28}$$

and for $u, v \in H^1(\Omega, L)$ the second Green formula $(Lu, v)_{L_2(\Omega)} - (u, Lv)_{L_2(\Omega)} = \langle \gamma^+_{1,S_1} v, [\gamma_0]_{S_1} u\rangle + \langle -[\gamma_1]_{S_1} v, \gamma^-_{0,S_1} u\rangle - \langle \gamma^+_{1,S_1} u, [\gamma_0]_{S_1} v\rangle - \langle [\gamma_1]_{S_1} u, \gamma^-_{0,S_1} v\rangle$.

In this section we consider the following mixed b.v.p.: find a function $u \in H^1(\Omega, L)$ such that

$$Lu = w, \quad \gamma^-_{0,S_1} u = g, \quad \gamma^+_{1,S_1} u = f,$$

where $w \in L_2(\Omega)$, $g \in H^{1/2}(S_1)$, $f \in H^{-1/2}(S_1)$.

As usual we put $u = v_1 + v_2$ where $v_1, v_2 \in H^1(\Omega, L)$ and split this problem into two independent ones:

$$Lv_1 = 0, \quad \gamma^-_{0,S_1} v_1 = g, \quad \gamma^+_{1,S_1} v_1 = f, \tag{29}$$

$$Lv_2 = w, \quad \gamma^-_{0,S_1} v_2 = 0, \quad \gamma^+_{1,S_1} v_2 = 0, \tag{30}$$

We denote ker $L = \{u \in H^1(\Omega)\colon Lu = 0\}$, ker $\gamma^-_{0,L} = \{u \in \text{ker } L\colon \gamma^-_{0,S_1} u = 0\}$, ker $\gamma^+_{1,L} = \{u \in \text{ker } L\colon \gamma^+_{1,S_1} u = 0\}$.

Since the inner product in $H^1(\Omega)$ is given by $(\cdot,\cdot)_L$ we have the following decomposition.

Lemma 10. *ker L = ker $\gamma^-_{0,L} \oplus$ ker $\gamma^+_{1,L}$.*

Proof. So far as ker $\gamma^-_{0,L}$ is a closed subspace of ker L we have that ker L = ker $\gamma^-_{0,L} \oplus$(ker

$\gamma_{0,L}^-)^\perp$. Let $v \in \ker\, \gamma_{0,L}^-$. Then if $u \in \ker\, \gamma_{1,L}^+$ from (28) we can get $(u,v)_L = 0$, i.e. $\ker\, \gamma_{1,L}^+ \subset (\ker\, \gamma_{0,L}^-)^\perp$.

Let now $u \in (\ker\, \gamma_{0,L}^-)^\perp$. Then for $v \in \ker\, \gamma_{0,L}^-$ we have $\gamma_{0,S_1}^+ v = p \in H_{00}^{1/2}(S_1)$ and $(u,v)_L = \langle \gamma_{1,S_1}^+ u, [\gamma_0]_{S_1} v\rangle = \langle \gamma_{1,S_1}^+ u, p\rangle = 0$. Since for an arbitrary $p \in H_{00}^{1/2}(S_1)$ there exists a unique $v \in \ker\, \gamma_{0,L}^-$ that $\gamma_{0,S_1}^+ v = p$ ([10], Corollary 4.4) from $\langle \gamma_{1,S_1}^+ u, p\rangle = 0$ it follows $\gamma_{1,S_1}^+ u = 0$ or $(\ker\, \gamma_{0,L}^-)^\perp \subset \ker\, \gamma_{1,L}^+$. Hence $(\ker\, \gamma_{0,L}^-)^\perp = \ker\, \gamma_{1,L}^+$. □

Let us denote $\gamma = (\gamma_{0,S_1}^-, \gamma_{1,S_1}^+)$, $\ker\, \gamma = \{u \in H^1(\Omega, L)\colon \gamma_{0,S_1}^- u = 0,\ \gamma_{1,S_1}^+ u = 0\}$, $Z(S_1) = H^{1/2}(S_1) \times H^{-1/2}(S_1)$.

Lemma 11. *The trace maps $\gamma_{0,S_1}^\pm\colon\ \ker L \to H^{1/2}(S_1)$ and $\gamma_{1,S_1}^\pm\colon\ \ker L \to H^{-1/2}(S_1)$ are continuous and surjective.*

Proof. The lemma is a consequence of the fact that the trace maps $(\gamma_{0,S_1}^-, [\gamma_0]_{S_1})\colon$ $\ker L \to H^{1/2}(S_1) \times H_{00}^{1/2}(S_1)$ and $(\gamma_{1,S_1}^+, [\gamma_1]_{S_1})\colon\ \ker L \to H^{-1/2}(S_1) \times H_{00}^{-1/2}(S_1)$ are the isomorphisms ([10]). □

Theorem 7. *The trace map $\gamma\colon\ \ker L \to Z(S_1)$ is an isomorphism.*

Proof. At first we show that γ is surjective. Let $(g,f) \in Z(S_1)$, i.e. $g \in H^{1/2}(S_1)$, $f \in H^{-1/2}(S_1)$. Since γ_0^- is surjective there exists $u \in \ker L$ that $\gamma_{0,S_1}^- u = g$. From Lemma 10 we have $u = u_0 + u_1$ where $u_0 \in \ker\, \gamma_{0,L}^-$, $u_1 \in \ker\, \gamma_{1,L}^+$. Thus there exists $u_1 \in \ker\, \gamma_{1,L}^+$ that $\gamma_{0,S_1}^- u_1 = g$ for an arbitrary $g \in H^{1/2}(S_1)$. This means that the trace map $\gamma_{0,S_1}^-\colon$ $\ker\, \gamma_{1,L}^+ \to H^{1/2}(S_1)$ is surjective.

Analogously we can show that $\gamma_{1,S_1}^+\colon\ \ker\, \gamma_{0,L}^- \to H^{-1/2}(S_1)$ is also surjective, i.e. for an arbitrary $f \in H^{-1/2}(S_1)$ there exists $u_0 \in \ker\, \gamma_{0,L}^-$ that $\gamma_{1,S_1}^+ u_0 = f$. And finally for all $(g,f) \in Z(S_1)$ we have $u = u_0 + u_1$ where $u_0 \in \ker\, \gamma_{0,L}^-$, $u_1 \in \ker\, \gamma_{1,L}^+$ such that $\gamma u = (g,f)$.

The injectivity of γ on $\ker L$ follows from (28). Indeed if $u \in \ker\, \gamma_{0,L}^- \cap \ker\, \gamma_{1,L}^+$ we have $\|u\|_L^2 = 0$ or $u = 0$ in Ω. Thus if we take to attention the continuity of γ on $\ker L$ (Lemma 11) we get what was to be proved. □

Corollary 4. *The b.v.p. (29) has a unique solution for an arbitrary $(g,f) \in Z(S_1)$.*

Theorem 8. *The operator $L\colon\ \ker\gamma \to L_2(\Omega)$ is an isomorphism.*

Proof. The injectivity of L on $\ker\gamma$ follows from (28) because $(Lu,u)_{L_2(\Omega)} = \|u\|_L^2$. Since the operator $L\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is positive definite [10], i.e. $\langle Lu, u\rangle = \|u\|_L^2$, $u \in H_0^1(\Omega)$, from inclusion $L_2(\Omega) \subset H^{-1}(\Omega)$ it follows that for an arbitrary $w \in L_2(\Omega)$ there exists a unique $u_0 \in H^1(\Omega, L) \cap H_0^1(\Omega)$ and $Lu_0 = w$. From Theorem 7 we can get a unique function $u_1 \in \ker L$ such that $\gamma_{0,S_1}^- u_1 = 0$, $\gamma_{1,S_1}^+ u_1 = -\gamma_{1,S_1}^+ u_0 \in H^{-1/2}(S_1)$. Thus for $w \in L_2(\Omega)$ we obtain a unique function $u = u_0 + u_1$ such that $Lu = Lu_0 = w$ and $\gamma_{0,S_1}^- u = \gamma_{0,S_1}^- u_0 + \gamma_{0,S_1}^- u_1 = 0$, $\gamma_{1,S_1}^+ u = \gamma_{1,S_1}^+ u_0 - \gamma_{1,S_1}^+ u_0 = 0$. This means that the operator $L\colon\ \ker\gamma \to L_2(\Omega)$ is surjective. The continuity of L on $\ker\gamma$ is obvious. □

Corollary 5. *The b.v.p. (30) has a unique solution for an arbitrary $w \in L_2(\Omega)$.*

By means of the second Green formula in Ω we can obtain the following integral representation ([10]) $u(x) = (Lu, Q(x,\cdot))_{L_2(\Omega)} + \langle [\gamma_1]_{S_1} u, Q(x,\cdot)\rangle -\ \langle \gamma_{1,S_1}^\pm Q(x,\cdot), [\gamma_0]_{S_1} u\rangle$, for $u \in H^1(\Omega, L)$, $x \in \Omega$, or

$$u(x) = Dw(x) + V[\gamma_1]_{S_1} u(x) - W[\gamma_0]_{S_1} u(x), \quad Dw(x) = \int_\Omega Q(x,y) w(y) dy, \tag{31}$$

$$W[\gamma_0]_{S_1} u(x) = \int_{S_1} \frac{\partial Q(x,y)}{\partial n_y} [\gamma_0]_{S_1} u(y) ds_y,\ \ V[\gamma_1] u(x) = \langle [\gamma_1] u, Q(x,\cdot)\rangle.$$

If $[\gamma_1]u \in L_p(S_1), p \geq 1$, then $V[\gamma_1]u(x) = \int_{S_1} Q(x,y)[\gamma_1]u(y)ds_y$.

For $u \in \ker L$ we have

$$u = V\tau - W\mu, \qquad \tau = [\gamma_1]_{S_1}u, \ \mu = [\gamma_0]_{S_1}u. \tag{32}$$

Lemma 4.1 in [10] gives us that $\tau \in H_{00}^{-1/2}(S_1)$ and $\mu \in H_{00}^{1/2}(S_1)$. By using the boundary conditions from (29) and the jump relations we can get $\gamma_{0,S_1}^{-}u = \gamma_{0,S_1}^{-}V\tau - \frac{1}{2}\mu - M_1\mu = g$, $\gamma_{1,S_1}^{+}u = \frac{1}{2}\tau + N_1\tau - \gamma_{1,S_1}^{+}W\mu = f$, or

$$\begin{cases} K_1\tau - \dfrac{1}{2}\mu - M_1\mu = g, \\ \dfrac{1}{2}\tau + N_1\tau + H_1\mu = f. \end{cases} \tag{33}$$

Let us remind that we denoted $K_1 = \gamma_{0,S_1}^{-}V = r_{S_1}K$, $M_1 = r_{S_1}M$, $N_1 = \widetilde{r}_{S_1}N$, $H_1 = -\gamma_{1,S_1}^{\pm}W = \widetilde{r}_{S_1}H$. Let us denote $A^* = \begin{pmatrix} K_1 & -\frac{1}{2}I - M_1 \\ \frac{1}{2}I + N_1 & H_1 \end{pmatrix}$, $Y(S_1) = H_{00}^{-1/2}(S_1) \times H_{00}^{1/2}(S_1)$, $\alpha = (\tau,\mu)$, $\beta = (g,f)$, $\|\alpha\|^2_{Y(S_1)} = \|\tau\|^2_{H_{00}^{-1/2}(S_1)} + \|\mu\|^2_{H_{00}^{1/2}(S_1)}$. Then $Y(S_1)' = H^{1/2}(S_1) \times H^{-1/2}(S_1)$ and we can rewrite the system (33) as $A^*\alpha = \beta$, where $A^* \colon Y(S_1) \to Y(S_1)'$.

Lemma 12. *The operators $H_1 \colon H_{00}^{1/2}(S_1) \to H^{-1/2}(S_1)$ and $K_1 \colon H_{00}^{-1/2}(S_1) \to H^{1/2}(S_1)$ are positive definite, i.e. $\langle H_1\mu,\mu\rangle = \|\mu\|^2_{H_{00}^{1/2}(S_1)}$ and $\langle \tau, K_1\tau\rangle = \|\tau\|^2_{H_{00}^{-1/2}(S_1)}$*

Proof. Let $\mu \in H_{00}^{1/2}(S_1)$, $\mu_0 = p_0\mu \in H^{1/2}(\Sigma)$, $r_{S_1}\mu_0 = \mu$. From Lemma 7 we can get $\langle H\mu_0,\mu_0\rangle = \langle H_1\mu,\mu\rangle = \|\mu_0\|^2_{H^{1/2}(\Sigma)} = \|\mu\|^2_{H_{00}^{1/2}(S_1)}$. Analogously for K_1 □

As we have defined above $\langle \alpha,\beta\rangle = \langle \tau, g\rangle + \langle f,\mu\rangle$. By using Lemma 12 we can define the inner products in $H_{00}^{1/2}(S_1)$ and $H_{00}^{-1/2}(S_1)$ as $(\mu_1,\mu_2)_{H_{00}^{1/2}(S_1)} = \langle H_1\mu_1,\mu_2\rangle$ and $(\tau_1,\tau_2)_{H_{00}^{-1/2}(S_1)} = \langle \tau_1, K_1\tau_2\rangle$ respectively.

Theorem 9. *The operator $A^* \colon Y(S_1) \to Y(S_1)'$ is positive definite, i.e. $\langle A^*\alpha,\alpha\rangle = \|\alpha\|^2_{Y(S_1)}$ for all $\alpha = (\tau,\mu) \in Y(S_1)$.*

Proof. By applying Lemma 12 we can obtain $\langle A^*\alpha,\alpha\rangle = \langle \tau, K_1\tau\rangle - \langle \tau, M_1\mu\rangle + \langle N_1\tau,\mu\rangle + \langle H_1\mu,\mu\rangle = \|\tau\|^2_{H_{00}^{-1/2}(S_1)} + \|\mu\|^2_{H_{00}^{1/2}(S_1)} - \langle \tau, M_1\mu\rangle + \langle N_1\tau,\mu\rangle$. Let $u = V\tau$, $v = W\mu$. Then from second Green formula we see that $-\langle \tau, \frac{1}{2}\mu + M_1\mu\rangle + \langle \frac{1}{2}\tau + N_1\tau,\mu\rangle = 0$ or $\langle \tau, M_1\mu\rangle = \langle N_1\tau,\mu\rangle$ for all $\tau \in H_{00}^{-1/2}(S_1)$ and $\mu \in H_{00}^{1/2}(S_1)$. □

Corollary 6. *The operator $A^* \colon Y(S_1) \to Y(S_1)'$ is an isomorphism.*

This corollary yields us that system (33) has a unique solution $(\tau,\mu) \in Y(S_1)$ for an arbitrary $(g,f) \in Y(S_1)'$.

Theorem 10. *The b.v.p. (29) is equivalent to the system (33), i.e. the solution u of the b.v.p. (29) has the form (32), where $(\tau,\mu) \in Y(S_1)$ is a solution of the system (33) and vice versa if (τ,μ) is a solution of the system (33), then the function u given by (32) is a solution of the b.v.p. (29).*

Proof. As it was mentioned above an arbitrary $u \in \ker L$ has the form (32). The system (33) is a consequence of (32), the boundary conditions in (29) and the jump relations. Let us note that from Lemma 4.1 [10] we see that $(\tau,\mu) \in Y(S_1)$. If $(\tau,\mu) \in Y(S_1)$ then the function u given by (32) belongs to $\ker L$. Since (τ,μ) is a solution of (33) it's easy to verify that u satisfies the boundary conditions in (29). □

Let us consider the b.v.p. (30). From (31) it follows that the solution u of this problem has the form $u = Dw + V\tau - W\mu$. By using the boundary conditions in (30) we obtain $\gamma^-_{0,S_1}u = K_1\tau - \frac{1}{2}\mu - M_1\mu + \gamma^-_{0,S_1}Dw = 0$, $\gamma^+_{1,S_1}u = \frac{1}{2}\tau + N_1\tau + H_1\mu + \gamma^+_{1,S_1}Dw = 0$, or

$$\begin{cases} K_1\tau - \dfrac{1}{2}\mu - M_1\mu = \widetilde{g} \quad (\equiv -\gamma^-_{0,S_1}Dw), \\ \dfrac{1}{2}\tau + N_1\tau + H_1\mu = \widetilde{f} \quad (\equiv -\gamma^+_{1,S_1}Dw). \end{cases} \tag{34}$$

We can rewrite system (34) as $A^*\alpha = \widetilde{\beta}$, where $\alpha = (\tau, \mu)$, $\widetilde{\beta} = (\widetilde{g}, \widetilde{f})$. From Corollary 6 it follows that the system (34) has a unique solution for an arbitrary $(\widetilde{g}, \widetilde{f}) \in Y(S_1)'$.

Theorem 11. *The b.v.p. (30) is equivalent to the system (34), i.e. the solution u of the b.v.p. (30) has the form $u = Dw + V\tau - W\mu$,where $(\tau, \mu) \in Y(S_1)$ is a solution of the system (34) and vice versa if (τ, μ) is a solution of the system (34), then the function $u = Dw + V\tau - W\mu$ is a solution of the b.v.p. (30).*

Proof is obvious if we take to attention the boundary conditions in (30) and the fact that the solution (τ, μ) of the system (34) for given $(\widetilde{g}, \widetilde{f}) \in Y(S_1)'$ belongs to $Y(S_1)$. Then $u = Dw + V\tau - W\mu$ belongs to $H^1(\Omega, L)$.

REFERENCES

1. Aubin J.-P. Approximation of elliptic boundary-value problems. Wiley-Interscience, New York, 1972.
2. Costabel M. *Boundary integral operators on Lipschitz domains: elementary results.* SIAM J. Math. Anal. **19** (1988), 613–626.
3. Costabel M., Stephan E.P. *An improved boundary element Galerkin method for three-dimensional crack problems.* Integral Equation and Operator Theory. **10** (1987), 467–504.
4. Eskin G.I. *Boundary problems for elliptic pseudo-differential operators.* Transl. of Math. Mon., American Mathematical Society, Providence, Rhode Island. V. 52. Providence, Rhode Island, 1981.
5. Lions J.L., Magenes E. Nonhomogeneuos boundary-value problems and applications. V.1. Springer-Verlag, Berlin, 1972.
6. von Petersdorff T. *Boundary integral equations for the mixed, Dirichlet, Neumann and transmission problems.* Math. Meth. in Appl. Sci. **11** (1989), 185–213.
7. Schneider R. *Reduction of order for pseudodifferential operators on Lipschitz domain.* Commun. in Partial Differ. Equations. **16** (1991), 1263–1286.
8. Stephan E.P. *Boundary integral equations for the screen problems in R^3.* Int. Equat. Oper. Theory. **10** (1987), 236–257.
9. Stephan E.P. *Boundary integral equations for the mixed boundary value problems in R^3.* Math. Nachr. **134** (1987), 21–53.
10. Sybil Yu. *Three-dimensional elliptic boundary value problems for an open Lipschitz surface.* Matematychni Studii. **8**, №1 (1997), 79–96.
11. Wendland W.L., Stephan E.P., G.C.Hsiao G.C. *On the integral equation method for the plane mixed boundary value problem of the Laplacian.* Math. Meth. Appl. Sci. **1** (1979), 265–321.
12. Zaremba M.S. *Sur un problème mixte relatif a l'équation de Laplace.* Bull. int. df l'Academie de Cracovie, Class. math. et nat. Ser. A. (1910), 313–344.

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