УДК 517.5+517.9

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EXISTENCE OF A SOLUTION FOR A HIGHER ORDER PARABOLIC EQUATION IN UNBOUNDED DOMAIN, BY THE METHOD OF INTRODUCING A PARAMETER

L. Zaręba. *Existence of a solution for a higher order parabolic equation in unbounded domain, by the method of introducing a parameter* , Matematychni Studii, **28** (2007) 183–190.

In this paper we consider the initial boundary value problem for the equation $u_{tt} + A_1 u + A_2 u_t + g(u_t) = f(x,t)$ in an unbounded domain, where A_1 is a linear elliptic operator of the fourth order and A_2 is a linear elliptic operator of the second order. Using the method of introducing a parameter we obtain the conditions of the existence of the weak solution for this problem.

Л. Заремба. *Существование решений параболических уравнений высших порядков в неограниченной области и метод введения параметра* // Математичні Студії. – 2007. – Т.28, №2. – С.183–190.

Рассматривается краевая задача с начальными условиями для уравнения $u_{tt} + A_1 u + A_2 u_t + g(u_t) = f(x,t)$ в неограниченной области, где A_1 — линейный эллиптический оператор четвертого порядка, а A_2 — линейный эллиптический оператор второго порядка. С помощью метода введения параметра получены условия существования слабого решения этой задачи.

In [1]–[3] and [5]–[6], the authors have considered the Cauchy problem and initial boundary value problem for a parabolic equation of a high order and the properties of its solutions. The method of introducing a parameter has been used first of all to the uniqueness of the solution. For example in [7], the authors by introducing a parameter, have shown the uniqueness of a weak solution in the class of functions which do not grow faster than the function $e^{a|x|^\alpha}$ for $|x| \to \infty$. In [8] there was considered the long–time behaviour of a solution of the boundary value problem for the equation div $\sigma(Du) + \triangle u_t - \delta^2 \triangle^2 u = u_{tt}$. In [9] there was considered the long–time behaviour of a solution of a class of nonlinear partial differential equations which model the dynamics of phase transitions in van der Waals fluid of the form div $\sigma(\nabla u) + v\triangle u - \delta^2 \triangle^2 u = \rho u_{tt}$. In [10], the author proved using this method that the solution of problem (1)–(3) is unique. The main goal of this paper is to obtain by (introducing a parameter) some conditions for the existence of a weak solution for a parabolic equation of the fourth order with the second derivative with respect to time.

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain and $\partial\Omega \in C^1$, $\Omega \cap B_R = \Omega^R$ be a regular domain for all $R > 0$, where $B_R = \{x \in R^n, |x| < R\}$ and $Q_T = \Omega \times (0,T)$, $Q_T^R = \Omega^R \times (0,T)$, $\Omega_\tau = Q_\tau \cap \{t = \tau\}$, $Q_{\tau_0,\tau_1} = \Omega \times (\tau_0, \tau_1)$.

2000 *Mathematics Subject Classification:* 35K55, 35K60.

We shall consider the equation of the form

$$
\begin{aligned}
&u_{tt}(x,t)+\sum_{i,j,k,l=1}^{n}(a_{ij}^{kl}(x,t)u_{x_ix_j}(x,t))_{x_kx_l}-\sum_{i,j=1}^{n}(a_{ij}(x,t)u_{x_i}(x,t))_{x_j}-\\
&-\sum_{i,j=1}^{n}(b_{ij}(x,t)u_{tx_i}(x,t))_{x_j}+a(x,t)u(x)+g(x,u_t)=f(x,t)
\end{aligned}
\tag{1}
$$

in the domain Q_T. For this equation, we put the following boundary and initial conditions

$$
u|_{S_T}=0,\ \ \frac{\partial u}{\partial\nu}\Big|_{S_T}=0, \tag{2}
$$

$$
u|_{t=0}=u_0(x),\ \ u_t|_{t=0}=u_1(x),\ x\in\Omega, \tag{3}
$$

where $S_T=\partial\Omega\times(0,T)$ and ν is a normal vector for S_T.

Let us start with some notation

$$
H^{0,k}_{loc}(\overline{\Omega})=\{u\in H^k(\Omega^R): u|_{\partial\Omega\cap B_R}=0,\frac{\partial^{k-1}u}{\partial\nu^{k-1}}|_{\partial\Omega\cap B_R}=0,(\forall\ R>0)\},\ k\in\{1,2\},
$$

$$
L^p_{loc}(\overline{\Omega})=\{u\in L^p(\Omega^R),\forall\ R>0\},
$$

$H^{0,k}(\Omega^R)=\big\{u:\int\limits_{\Omega^R}\sum\limits_{|\alpha|\le k}|D^\alpha u|^2dx<\infty,\ u|_{\partial\Omega^R}=0,\frac{\partial^{k-1}u}{\partial\nu^{k-1}}|_{\partial\Omega^R}=0\big\}$, and $k\in\{1,2\}$;

$D^\alpha=\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$, $|\alpha|=\alpha_1+\dots+\alpha_n$, and ν is the normal vector on $\partial\Omega$.

For equation (1), we adapt the following system of assumptions:

(**A**) $a_{ij}^{kl},(a_{ij}^{kl})_{x_kx_l},(a_{ij}^{kl})_{tt}\in L^\infty(Q_T)$; $a_{ij}^{kl}(x,t)=a_{kl}^{ij}(x,t)$, $\{i,j,k,l\}\subset\{1,\dots,n\}$ for almost all $(x,t)\in Q_T$;

$\sum\limits_{i,j,k,l=1}^{n}a_{ij}^{kl}(x,t)\xi_{ij}\xi_{kl}\ge a_2\sum\limits_{i,j=1}^{n}\xi_{ij}^2,\quad \{i,j,k,l\}\subset\{1,\dots,n\}$,

$\sum\limits_{i,j,k,l=1}^{n}(a_{ij}^{kl}(x,t))_t\xi_{ij}\xi_{kl}\le a_2^1\sum\limits_{i,j=1}^{n}\xi_{ij}^2$ for almost all $(x,t)\in Q_T$ and for all $\xi\in R^{\frac{n(n+1)}{2}}$,

where $a_2>0$ is a constant;

$a_{ij},(a_{ij})_{x_j},(a_{ij})_t\in L^\infty(Q_T),\quad\{i,j\}\subset\{1,\dots,n\};\quad a,a_t\in L^\infty(Q_T)$,

(**B**) $b_{ij},(b_{ij})_{x_j},(b_{ij})_t\in L^\infty(Q_T),\quad\{i,j\}\subset\{1,\dots,n\};\quad\sum\limits_{ij=1}^{n}b_{ij}(x,t)\xi_i\xi_j\ge b_0\sum\limits_{i=1}^{n}\xi_i^2$

for almost all $(x,t)\in Q_T$ and for all $\xi\in\mathbb{R}^n$, where $b_0>0$ is a constant;

(**G**) The functions $x\mapsto g(x,\xi)$, $x\mapsto g_\xi(x,\xi)$ are measurable for every $\xi\in\mathbb{R}$ and the functions $\xi\mapsto g(x,\xi)$, $\xi\mapsto g_\xi(x,\xi)$ are continuous for almost all $x\in\Omega$ and satisfy the following inequalities: $(g(x,\xi)-g(x,\mu))(\xi-\mu)\ge g_0|\xi-\mu|^q$ for almost all $x\in\Omega$ and for all $\xi,\mu\in\mathbb{R}$, $g_0=\mathrm{const}>0$;

$|g(x,\xi)|\le g_1|\xi|^{q-1}$ for almost all $x\in\Omega$ and for all $\xi\in\mathbb{R}$, and some $q\in(2,+\infty)$.

Under these assumptions in paper [10], we proved that there exists a unique weak solution of the following problem

$$
A(u)=f^R(x,t) \tag{4}
$$

in the domain $Q_T^R=\Omega^R\times(0,T)$, $R>1$, where

$$
f^R(x,t)=\begin{cases} f(x,t), & \text{for}\quad (x,t)\in Q_T^R,\\ 0, & \text{for}\quad (x,t)\in Q_T\setminus Q_T^R,\quad R\in\{2,3,4\dots\},\end{cases}
$$

with the following boundary and initial conditions

$$u|_{t=0} = u_0^R(x), \quad u_t|_{t=0} = u_1^R(x),\ x \in \Omega^R, \tag{5}$$

$$u|_{\partial\Omega^R \times (0,T)} = \frac{\partial u}{\partial \nu}|_{\partial\Omega^R \times (o,T)} = 0, \tag{6}$$

where $u_0^R(x) = u_0(x) \cdot \zeta^R(x)$, $u_1^R(x) = u_1(x)\zeta^R(x)$ for $0 \le \zeta^R(x) \le 1$, $x \in \mathbb{R}^n$, $\zeta \in C^2(\mathbb{R}^n)$ and $\zeta^R(x) = \begin{cases} 1, & \text{for} \quad |x| \le R-1, \\ 0, & \text{for} \quad |x| \ge R. \end{cases}$ We obtained the existence of a weak solution of the problem (1)–(3.)

We call a function u^R *a weak solution of problem* (4)–(6) if $u^R \in L^\infty\left((0,T); H^{0,2}(\Omega^R)\right)$, $u_{tt}^R \in L^2\left((0,T); L^2(\Omega^R)\right)$, $u_t^R \in L^2\left((0,T); H^{0,1}(\Omega^R)\right) \cap L^q(Q_T^R)$ and u^R satisfies the following integral identity:

$$\int\limits_{Q_\tau^R} \Big[u_{tt}^R w + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t) u_{x_i x_j}^R w_{x_k x_l} + \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i}^R w_{x_j} + a(x,t) u^R w +$$
$$+ \sum_{i,j=1}^n b_{ij}(x,t) u_{x_i t}^R w_{x_j} + g(x,u^R) w \Big] dxdt = \int\limits_{Q_\tau^R} f^R(x,t) w dxdt$$

$(\forall\ \tau \in (0,T])$ $\left(\forall\ w \in L^2\big((0,T); H^{0,2}(\Omega^R)\big) \cap L^q(Q_T^R)\right)$ and the initial conditions (5).

We call *a function u a weak solution of problem (1)–(3)* if $u \in L^2\big((0,T); H_{loc}^{0,2}(\overline{\Omega})\big)$, $u_t \in L^2\big((0,T); H_{loc}^{0,1}(\overline{\Omega})\big) \cap L^p\big((0,T); L_{loc}^p(\overline{\Omega})\big) \cap C\big([0,T]; L_{loc}^2(\overline{\Omega})\big)$, and u satisfies the following integral identity:

$$\int\limits_\Omega u_t(x,\tau) w(x,\tau) dx + \int\limits_{Q_\tau} \big[- u_t w_t + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j} w_{x_k x_l} + + \sum_{i,j=1}^n a_{ij}(x) u_{x_i} w_{x_j} +$$

$$+ a(x) u w + \sum_{i,j=1}^n b_{ij}(x,t) u_{x_i t} w_{x_j} + g(x,u_t) w \big] dxdt = \int\limits_{Q_\tau} f(x,t) v dxdt + \int\limits_\Omega u_1(x) w(x,0) dx, \tag{7}$$

$(\forall\ \tau \in (0,T])\big(\forall\ w \in L^2\big((0,T); H_{loc}^{0,2}(\overline{\Omega})\big) \cap L^p\big((0,T); L_{loc}^p(\overline{\Omega})\big), w_t \in L^2\big((0,T); L_{loc}^2(\overline{\Omega})\big)\big)$, where supp w is bounded.

In order to prove the existence of a weak solution of the problem (1)-(3) in unbounded domain we will use the method of introducing a parameter.

Theorem 1. *If conditions* (**A**) , (**B**) *and* (**G**) *hold, and*

$$\int\limits_0^T \int\limits_{B_R} |f|^2 dxdt + \int\limits_{B_R} \big[|u_1| + |u_0| + \sum_{i,j=1}^n |u_{0 x_i x_j}|^2 \big] dx < b e^{\alpha R^2} \quad (\forall\ R > 1),$$

where $b, \alpha > 0$ are constants, then problem (1)–(3) has a weak solution on Q_{τ_0}.

Proof. Let $R = 2^p$, $p \in N$. We extend every solution u^R of problem (4)–(6) by zero into Q_T. Hence, we obtain the sequence $\{u^p\}_{p=2}^\infty$. We prove that this sequence is a Cauchy sequence in some spaces.

Let $R_0 > 1$, $\chi > 0$. We define the functions Φ ,Ψ where the function $\Phi \in C^2(R)$ is the following $\Phi_{R_0}(\zeta) = \begin{cases} 1, & \text{for} \quad \zeta \le 0, \\ 0, & \text{for} \quad \zeta \ge 1, \end{cases}$ and $\Psi_{R_0}(x) = \left[\Phi_{R_0}\left(\frac{|x| - R_0}{\chi}\right)\right]^\gamma$, $\gamma > 4$.

Now we give some properties of the function Ψ: $\Psi_{R_0} \in C^2(\mathbb{R}^n)$, $\Psi_{R_0} = 1$ for $|x| \le R_0$, $\Psi_{R_0} = 0$ for $|x| \ge R_0 + \chi$, $0 \le \Psi_{R_0} \le 1$, for $R_0 < |x| < R_0 + \chi$. There exists a constant C such that $|\Psi_{R_0}(x)_x| \le \frac{C}{\chi}[\Phi_{R_0}]^{\gamma-1}$, $|\Psi_{R_0}(x)_{xx}| \le \frac{C}{\chi^2}[\Phi_{R_0}]^{\gamma-2}$.

Let $p > R_0 + 1$, $h > R_0 + 1$, $h \in N$, and $\sigma = p$, $\sigma = h$. From equality (7) we obtain for u^σ, $\sigma \in \{2, 3, \ldots\}$, $\tau \in (0,T]$, the following equality

$$\int\limits_{\Omega_\tau} u_t^\sigma(x,\tau)w(x,\tau)dx + \int\limits_{Q_\tau} \Big(- u_t^\sigma w_t + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t)u^\sigma_{x_ix_j} w_{x_kx_l} + \sum_{i,j=1}^n a_{ij}(x,t)u^\sigma_{x_i}w_{x_j} +$$

$$+a(x,t)u^\sigma w + \sum_{i,j=1}^n b_{ij}(x,t)u^\sigma_{x_it}w_{x_j} + g(x,u_t^\sigma)w\Big)dxdt = \int\limits_{Q_\tau} f(x,t)wdxdt + \int\limits_\Omega u_1^f(x)w(x,0)dx, \tag{8}$$

where $w \in L^2\big((0,T); H^{0,2}_{loc}(\overline{\Omega})\big) \cap C\big([0,T]; L^2_{loc}(\overline{\Omega})\big)$, $w_t \in L^2\big((0,T); H^{0,1}_{loc}(\overline{\Omega})\big)$, supp $w \subset Q_T^{R_0}$. From (8), if we choose w such that supp $w \subset \Omega_{R_0} \times (0,T]$ and denote $u^{p,h} = u^p - u^h$ then we obtain

$$\int\limits_{\Omega_\tau} u_t^{p,h}(x,\tau)w(x,\tau)dx + \int\limits_{Q_\tau} \Big(- u_t^{p,h} w_t + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t)u^{p,h}_{x_ix_j} w_{x_kx_l} + + \sum_{i,j=1}^n a_{ij}(x,t)u^{p,h}_{x_i}w_{x_j} +$$

$$+a(x,t)u^{p,h} w + \sum_{i,j=1}^n b_{ij}(x,t)u^{p,h}_{x_it}w_{x_j} + [g(x,u_t^p) - g(x,u_t^h)]w\Big)dxdt = 0. \tag{9}$$

In (10) we take $w = u_t^{p,h}(x,t)\Psi_{R_0}(x)e^{-\mu t}$, $\mu > 0$. Hence, we will have

$$\begin{aligned} &\int\limits_{\Omega_\tau} u_t^{p,h}(x,\tau)u_t^{p,h}(x,\tau)\Psi_{R_0}e^{-\mu t}dx + \int\limits_{Q_\tau} \Big(- u_t^{p,h}(u_t^{p,h}\Psi_{R_0}e^{-\mu t})_t + \\ &+ \sum_{i,j,k,l=1}^n a_{ij}^{p,h}(x,t)u^{p,h}_{x_ix_j}(u_t^{p,h}\Psi_{R_0}e^{-\mu t})_{x_kx_l} + \sum_{i,j=1}^n a_{ij}(x,t)u^{p,h}_{x_i}(u_t^{p,h}\Psi_{R_0}e^{-\mu t})_{x_j} + \\ &+a(x,t)u^{p,h}u_t^{p,h}\Psi_{R_0}e^{-\mu t} + \sum_{i,j=1}^n b_{ij}(x,t)u^{p,h}_{x_it}(u_t^{p,h}\Psi_{R_0}e^{-\mu t})_{x_j} + \\ &+[g(x,u_t^p) - g(x,u_t^h)][u_t^p - u_t^h]\Psi_{R_0}e^{-\mu t}\Big)dxdt = 0. \end{aligned} \tag{10}$$

If we consider the respective components of the last expression and we use that $u_0^{p,h}(x) = 0$, $x \in \Omega^{R_0}$, $f^{p,h}(x,t) = 0$, in $Q_T^{R_0}$ we will have

$$\begin{aligned} I_1 + I_2 :=& \int\limits_{\Omega_\tau} u_t^{p,h}(x,\tau)u_t^{p,h}(x,\tau)\Psi_{R_0}e^{-\mu t}dx - \int\limits_{Q_\tau} u_t^{p,h}(u_t^{p,h}\Psi_{R_0}e^{-\mu t})_t dxdt = \\ =& \tfrac{1}{2}\int\limits_{\Omega_\tau} |u_t^{p,h}|^2\Psi_{R_0}e^{-\mu\tau}dx + \tfrac{\mu}{2}\int\limits_{Q_\tau} |u_t^{p,h}|^2\Psi_{R_0}e^{-\mu t}dxdt. \end{aligned}$$

$$I_3 := \int\limits_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t)u^{p,h}_{x_ix_j}(u_t^{p,h}\Psi_{R_0}e^{-\mu t})_{x_kx_l}dxdt = I_3^1 + I_3^2 + I_3^3,$$

where, from (**A**),

$$I_3^1 := \int\limits_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t)u^{p,h}_{x_ix_j}u^{p,h}_{tx_kx_l}\Psi_{R_0}e^{-\mu t}dxdt \geq$$

$$\geq \tfrac{a_2}{2}\int\limits_{\Omega_\tau} \sum_{i,j=1}^n |u^{p,h}_{x_ix_j}|^2\Psi_{R_0}e^{-\mu\tau}dx + \tfrac{1}{2}(\mu a_2 - a_2^1)\int\limits_{Q_\tau} \sum_{i,j=1}^n |u^{p,h}_{x_ix_j}|^2\Psi_{R_0}e^{-\mu t}dxdt,$$

$$I_3^2 := \int\limits_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t)u^{p,h}_{x_ix_j}u^{p,h}_{tx_k}(\Psi_{R_0})_{x_l}e^{-\mu t}dxdt \leq$$

$$\leq \tfrac{A_2\delta_0}{2}\int\limits_{Q_\tau} \sum_{i=1}^n |u^{p,h}_{tx_i}|^2\Psi_{R_0}e^{-\mu\tau}dxdt + \tfrac{C}{2\delta_0\chi^2}\int\limits_{Q_\tau} \sum_{i,j=1}^n |u^{p,h}_{x_ix_j}|^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu t}dxdt,$$

where $\delta_0 > 0$ and $A_2 = \operatorname{ess\,sup}\limits_{Q_T} \sum\limits_{i,j,k,l=1}^n [a_{ij}^{kl}(x,t)]^2$.

$$I_3^3 := \int\limits_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t)u^{p,h}_{x_ix_j}u_t^{p,h}(\Psi_{R_0})_{x_kx_l}e^{-\mu t}dxdt \leq$$

$$\leq \tfrac{C\delta}{2\chi^2}\int\limits_{Q_\tau} |u_t^{p,h}|^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu\tau}dxdt + \tfrac{A_2}{2\delta\chi^2}\int\limits_{Q_\tau} \sum_{i,j=1}^n |u^{p,h}_{x_ix_j}|^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu t}dxdt.$$

It is easy to prove that

$$\int\limits_{Q_\tau}(u^{p,h}(x,t))^2\Psi_{R_0}(x)e^{-\mu t}dxdt \le 2T^2\int\limits_{Q_\tau}(u_t^{p,h}(x,t))^2\Psi_{R_0}(x)e^{-\mu t}dxdt,$$

$$\int\limits_{Q_\tau}(u_{x_i}^{p,h}(x,t))^2\Psi_{R_0}(x)e^{-\mu t}dxdt \le 2T^2\int\limits_{Q_\tau}(u_{x_it}^{p,h}(x,t))^2\Psi_{R_0}(x)e^{-\mu t}dxdt.$$

Next, we can observe that $\int\limits_{Q} u_{x_ix_i}^{p,h}u^{p,h}\Psi_{R_0}e^{-\mu t}dxdt = \int\limits_{\Omega}(u_{x_i}^{p,h}u^{p,h}\Psi_{R_0}e^{-\mu t})_{x_i}dx - -\int\limits_{Q}(u_{x_i}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt - \int\limits_{Q}u_{x_i}^{p,h}u^{kl}\Psi_{R_0x_i}e^{-\mu t}dxdt$, hence we obtain for function Ψ that

$$\int\limits_{Q}(u_{x_i}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt \le \delta\int\limits_{\Omega}(u_{x_ix_j}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt+$$
$$+\frac{1}{\delta}\int\limits_{Q}(u^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt + \frac{M}{\chi^2}\int\limits_{Q}(u^{p,h})^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu t}dxdt,\ \delta>0.$$

Now, we consider

$$I_4 := \int\limits_{Q_\tau}\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}^{p,h}(u_t^{p,h}\Psi_{R_0}e^{-\mu t})_{x_j}dxdt = I_4^1 + I_4^2.$$

We obtain the following estimation:

$$I_4^1 := \int\limits_{Q_\tau}\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}^{p,h}u_{tx_j}^{p,h}\Psi_{R_0}e^{-\mu t}dxdt \le$$

$$\le \frac{\delta_0}{2}\int\limits_{Q_\tau}\sum_{i=1}^{n}(u_{tx_i}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt + \frac{A_1}{2\delta_0}\int\limits_{Q_\tau}\sum_{i=1}^{n}(u_{x_i}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt \le \frac{\delta_0}{2}\int\limits_{Q_\tau}\sum_{i=1}^{n}(u_{tx_i}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt +$$
$$\frac{A_1}{2\delta_0}\int\limits_{Q_\tau}\sum_{i,j=1}^{n}(u_{x_ix_j}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt + +\frac{A_1 n}{2\delta_0}\int\limits_{Q_\tau}(u^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt + \frac{Mn}{\chi^2}\int\limits_{Q_\tau}(u^{p,h})^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu t}dxdt \le$$
$$\frac{\delta_0}{2}\int\limits_{Q_\tau}\sum_{i=1}^{n}(u_{tx_i}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt + \frac{A_1}{2\delta_0}\int\limits_{Q_\tau}\sum_{i,j=1}^{n}(u_{x_ix_j}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt + +\frac{A_1 n}{2\delta_0}T^2\int\limits_{Q_\tau}(u_t^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt+$$
$$+\frac{MnT^2}{\chi^2}\int\limits_{Q_\tau}(u_t^{p,h})^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu t}dxdt,$$

$$I_4^2 := \int\limits_{Q_\tau}\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}^{p,h}u_t^{p,h}(\Psi_{R_0})_{x_j}e^{-\mu t}dxdt \le \frac{A_1}{2}\int\limits_{Q_\tau}\sum_{i=1}^{n}|u_{x_i}^{p,h}|^2\Psi_{R_0}e^{-\mu\tau}dxdt+$$

$$+\frac{C}{2\chi^2}\int\limits_{Q_\tau}|u_t^{p,h}|^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu t}dxdt \le \frac{A_1}{2}\int\limits_{Q_\tau}\sum_{i,j=1}^{n}|u_{x_ix_j}^{p,h}|^2\Psi_{R_0}e^{-\mu\tau}dxdt + \frac{T^2A_1n}{2}\int\limits_{Q_\tau}|u_t^{p,h}|^2\Psi_{R_0}e^{-\mu t}dxdt$$
$$+\frac{MT^2n}{2}\int\limits_{Q_\tau}|u_t^{p,h}|^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu\tau}dxdt + \frac{C}{2\chi^2}\int\limits_{Q_\tau}|u_t^{p,h}|^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu t}dxdt.$$

Hence

$$I_4 \le \frac{\delta_0}{2}\int\limits_{Q_\tau}\sum_{i=1}^{n}(u_{tx_i}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt + \frac{A_1(1+\delta_0)}{2\delta_0}\int\limits_{Q_\tau}\sum_{i,j=1}^{n}(u_{x_ix_j}^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt+$$
$$+\frac{A_1T^2n(1+\delta_0)}{2\delta_0}\int\limits_{Q_\tau}(u_t^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt + \frac{3MT^2n+C}{2\chi^2}\int\limits_{Q_\tau}|u_t^{p,h}|^2[\Phi_{R_0}]^{\gamma-2}e^{-\mu\tau}dxdt,$$

where $A_1 = \operatorname*{ess\,sup}_{Q_T}\sum_{i,j=1}^{n}a_{ij}^2(x,t)$. Next, from assumption (**A**) and the initial condition we have

$$I_5 := \int\limits_{Q_\tau}a(x,t)u^{p,h}u_t^{p,h}\Psi_{R_0}e^{-\mu t}dxdt \le$$
$$\le \frac{A_0}{2}\int\limits_{Q_\tau}\left[(u^{p,h})^2 + (u_t^{ph})^2\right]\Psi_{R_0}e^{-\mu t}dxdt \le \frac{(T^2+1)A_0}{2}\int\limits_{Q_\tau}(u_t^{p,h})^2\Psi_{R_0}e^{-\mu t}dxdt,$$

where $A_0 = \operatorname*{ess\,sup}_{Q_T}|a(x,t)|$. Next, from (**B**) we have

$$I_6 := \int\limits_{Q_\tau}\sum_{i,j=1}^{n}b_{ij}(x,t)u_{x_it}^{p,h}(u_t^{p,h}\Psi_{R_0}e^{-\mu t})_{x_j}dxdt = I_6^1 + I_6^2,$$

where

$$I_6^1 := \int\limits_{Q_\tau} \sum_{i,j=1}^{n} b_{ij}(x,t) u^{p,h}_{tx_j} u^{p,h}_{tx_i} \Psi_{R_0} e^{-\mu t} dxdt \geq b_0 \int\limits_{Q^R_\tau} \sum_{i,j=1}^{n} |u^{p,h}_{tx_i}|^2 \Psi_{R_0} e^{-\mu t} dxdt,$$

$$I_6^2 := \int\limits_{Q_\tau} \sum_{i,j=1}^{n} b_{ij}(x,t) u^{p,h}_{tx_i} u^{p,h}_{t} (\Psi_{R_0})_{x_j} e^{-\mu t} dxdt \leq$$

$$\leq \tfrac{C}{2\delta_0 \chi^2} \int\limits_{Q_\tau} |u^{p,h}_t|^2 [\Phi_{R_0}]^{\gamma-2} e^{-\mu\tau} dxdt + \tfrac{B_1\delta_0}{2} \int\limits_{Q_\tau} \sum_{i=1}^{n} |u^{p,h}_{tx_i}|^2 \Psi_{R_0} e^{-\mu t} dxdt.$$

where $B_1 = \operatorname{ess\,sup}\limits_{Q_T} \sum\limits_{i,j=1}^{n} b^2_{ij}(x,t)$. Moreover, by virtue of condition $(\mathbf{G})$ we get

$$I_7 := \int\limits_{Q_\tau} [g(x,u^p_t) - g(x,u^h_t)][u^p_t - u^h_t] \Psi_{R_0} e^{-\mu t} dxdt \geq g_0 \int\limits_{Q_\tau} |u^{p,h}_t|^q \Psi_{R_0} e^{-\mu} dxdt.$$

From the estimates of the integrals $I_1, \ldots, I_7$ and (10), we obtain

$$\tfrac{1}{2}\int\limits_{\Omega_\tau} \big[|u^{p,h}_t|^2 + a_2 \sum_{i,j=1}^{n} |u^{p,h}_{x_ix_j}|^2\big] \Psi_{R_0} e^{-\mu\tau} dx + \tfrac{1}{2}\big[\mu - (T^2+1)A_0 - \tfrac{A_1 T^2 n(1+\delta_0)}{\delta_0}\big] \int\limits_{Q_\tau} |u^{p,h}_t|^2 \Psi_{R_0} e^{-\mu t} dxdt$$

$$+ g_0 \int\limits_{Q_\tau} |u^{p,h}_t|^q \Psi_{R_0} e^{-\mu} dxdt + \tfrac{1}{2}\big[\mu a_2 - a^1_2 - \tfrac{A_1(1+\delta_0)}{\delta_0}\big] \int\limits_{Q_\tau} \sum_{i,j=1}^{n} |u^{p,h}_{x_ix_j}|^2 \Psi_{R_0} e^{-\mu t} dxdt +$$

$$+ \tfrac{1}{2}\big[2b_0 - (B_1 + A_2 + 1)\delta_0\big] \int\limits_{Q^R_\tau} \sum_{i,j=1}^{n} |u^{p,h}_{tx_i}|^2 \Psi_{R_0} e^{-\mu t} dxdt \leq \tfrac{1}{2}\big[\tfrac{C + A_2}{\delta_0 \chi^2}\big] \int\limits_{Q_\tau} \sum_{i,j=1}^{n} |u^{p,h}_{x_ix_j}|^2 [\Phi_{R_0}]^{\gamma-2} e^{-\mu t} dxdt$$

$$+ \tfrac{1}{2}\big[C\delta_0 + \tfrac{C}{\delta_0} + C + 3MT^2 n\big] \int\limits_{Q_\tau} |u^{p,h}_t|^2 [\Phi_{R_0}]^{\gamma-2} e^{-\mu t} dxdt. \tag{11}$$

Now, we choose $\delta_0 = \frac{2b_0}{(B_1 + A_2 + A_1)}$, $\mu a_2 = (\mu_0 + \mu_1) - \gamma$, $\mu_1 a_2 = \gamma$, where $\gamma = a^1_2 + \frac{A_1(1+\delta_0)}{2\delta}$. From (12) we obtain the following inequality

$$\int\limits_{Q_\tau} \sum_{i,j=1}^{n} \big[|u^{p,h}_t|^2 + |u^{p,h}_{x_ix_j}|^2\big] \Psi_{R_0} e^{-\mu_0 t} dxdt \leq \tfrac{C}{\chi^2 \mu_0} \int\limits_0^\tau \sum_{i,j=1}^{n} \big[|u^{p,h}_t|^2 + |u^{p,h}_{x_ix_j}|^2\big] [\Phi_{R_0}]^{\gamma-2} e^{-\mu_0 t} dxdt. \tag{12}$$

If we denote $v^{p,h} = \sum\limits_{i,j=1}^{n} \big[|u^{p,h}_t|^2 + |u^{p,h}_{x_ix_j}|^2\big]$, then from (13) and assumption on the function Φ_{R_0} we obtain the following inequality

$$\int\limits_0^\tau \int\limits_{B_{R_0}} v^{p,h} e^{-\mu_0 t} dxdt \leq \tfrac{C}{\chi^2 \mu_0} \int\limits_0^\tau \int\limits_{B_{R_0}+\chi} v^{p,h} e^{-\mu_0 t} dxdt. \tag{13}$$

Now, we divide χ into s parts $\dfrac{\chi}{s}$ and assume that $\dfrac{Cs^2}{\chi^2 \mu_0} \leq e^{-1}$. From paper [7] and (14) we obtain

$$\int\limits_0^\tau \int\limits_{B_{R_0}} v^{p,h} dxdt \leq e^{-s + \mu_0 \tau} \int\limits_0^\tau \int\limits_{B_{R_0}+\chi} v^{p,h} dxdt. \tag{14}$$

Let $R_0 = 2^m$, $\chi = 2^m$, $R_0 + \chi = 2^{m+1}$, $\mu_0 = b \cdot 2^{2m}$, where b is some constant $s = \lambda 2^{2m}$ where $\lambda \in N$. Hence and from our assumption we get $\dfrac{Cs^2}{\chi^2 \mu_0} = \dfrac{C\lambda^2 2^{4m}}{2^{2m} 2^{2m} \cdot b} \leq e^{-1}$, and $\lambda^2 \leq \dfrac{e^{-1} \cdot b}{C}$. Moreover, from (14) we obtain

$$\int\limits_0^\tau \int\limits_{B_{2m}} v^{p,h} dxdt \leq e^{-s + \mu_0 \tau} \int\limits_0^\tau \int\limits_{B_{2m+1}} v^{p,h} dxdt.$$

Therefore if $p = m + 1$, $h = m + 2$, we have

$$\int_0^\tau \int_{B_{2m}} v^{m+1,m+2} dxdt \le e^{-s+\mu_0\tau} \int_0^\tau \int_{B_{2m+1}} \left[|v^{m+1}|^2 + |v^{m+2}|^2\right] dxdt. \tag{15}$$

If in (7) we put $w = u_t^m$ and integrating over the domain Q_τ^m we obtain
$\frac{1}{2}\int_{B_{2m}} \sum_{i,j=1}^n \left[|u_t^m|^2 + |u_{x_ix_j}^m|^2\right]\|_{t=\tau} dx \le \frac{1}{2}\int_0^\tau \int_{B_{2m}} |f|^2 dxdt + \frac{1}{2}\int_{B_{2m}} \sum_{i,j=1}^n \left[u_1^m + u_0^m|u_{0x_ix_j}^m|^2\right] dx.$
Hence

$$\int_0^\tau \int_{B_{2m}} w_t^m \le C\Big(\int_0^\tau \int_{B_{2m}} |f|^2 dxdt + \frac{1}{2}\int_{B_{2m}} \sum_{i,j=1}^n \left[u_1^m + u_0^m|u_{0x_ix_j}^m|^2\right] dx\Big).$$

The last inequality implies that

$$\int_0^\tau \int_{B_{2m}} w^{m+1,m+2} dxdt \le Ce^{-s+\mu_0\tau}\Big(\int_0^\tau \int_{B_{2m+2}} |f|^2 dxdt + \frac{1}{2}\int_{B_{2m+2}} \left[u_1^{m+2} + u_0^{m+2} + \sum_{i,j=1}^n |u_{0x_ix_j}^{m+2}|^2\right] dx\Big).$$

From assumption of Theorem we obtain $\int_0^\tau \int_{B_{2m}} w^{m+1,m+2} dxdt \le Ce^{-s+\mu_0\tau+\alpha 2^{2m+4}} = Ce^{-\lambda 2^{2m}} +$ $2^{2m}b\tau + \alpha 2^{2m+4} - \lambda 2^{2m} + 2^{2m}\tau + \alpha 2^{2m+4} = -2^{2m}(\lambda - b\tau - 16\alpha) = -2^{2m}(16 - b\tau - 16\{\alpha\}) =$ $-\alpha_1 2^{2m}$, where $\lambda = 16([\alpha]+1)$, $\alpha_0 = 16(1 - \{\alpha\})$. We choose $\tau_0 < \frac{\alpha_0}{b}$ and $\alpha_1 > 0$.

Let $\epsilon > 0$ be arbitrary small. Then we can choose R_0 such that

$$\Big(\int_0^\tau \int_{B_{R_0}} w^{m+1,m+\kappa} dxdt\Big)^{\frac{1}{2}} \le \sum_{s=1}^{\kappa-1} \Big(\int_0^\tau \int_{B_{R_0}} w^{m+s,m+s+1} dxdt\Big)^{\frac{1}{2}} \le C\sum_{s=1}^{\kappa-1} e^{-\alpha_1 2^m} < \epsilon. \tag{16}$$

It means that the sequence $\{u^m\}_{m=2}^\infty$ satisfies the following conditions:

$$u^m \to u \quad \text{in} \quad L^2\big((0,T); H_{loc}^{0,2}(\overline{\Omega})\big),$$

$$u_t^m \to u_t \quad \text{in} \quad L^2\big((0,T); H_{loc}^{0,1}(\overline{\Omega})\big) \cap L^q\big((0,T); L_{loc}^q(\overline{\Omega})\big) \cap C\big([0,T]; L_{loc}^2(\overline{\Omega})\big),$$

If we pass with m to infinity then from (8) we obtain

$$\int_{\Omega_\tau} u_t(x,\tau)w(x,\tau)dx + \int_{Q_\tau} \Big[- u_t w_t + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_ix_j} w_{x_kx_l} + \sum_{i,j=1}^n a_{ij}(x) u_{x_i} w_{x_j} +$$
$$+ \sum_{i,j=1}^n b_{ij}(x) u_{tx_i} w_{x_i} + a_0(x)uw + g(x,u_t))w\Big] dxdt = \int_{Q_\tau} f(x,t) w dxdt + \int_{\Omega_0} u_1(x) w(x,0) dx, \tag{17}$$

$\forall\ w \in L^2\big((0,T); H_{,loc}^{0,2}(\overline{\Omega})\big) \cap L^p\big((0,T); L_{loc}^p(\overline{\Omega})\big)$, $w_t \in L^2\big((0,T); L_{loc}^2(\overline{\Omega})\big)$, where supp w is bounded. In addition $u^m(x,0) = u_0^m(x)$, $u_0^m \to u_0$ in $L_{loc}^2(\overline{\Omega})$ hence $u(x,0) = u_0(x)$. Moreover, $u_t^m(x,0) = u_1^m(x)$, $u_1^m \to u_1$ in $L_{loc}^p(\overline{\Omega})$, hence $u_t(x,0) = u_1(x)$. This means that the function u is a solution of problem (1)–(3). The proof is complete. □

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Received 23.06.2007