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ESTIMATES OF THE LOGARITHMIC DERIVATIVE OF MEROMORPHIC FUNCTIONS WITH A LOGARITHMIC SINGULARITY AT ∞ AND THEIR APPLICATIONS

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We obtain estimates of the logarithmic derivative of meromorphic functions with a logarithmic singularity. Their applications to the theory of complex differential equations are given.

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Получены оценки логарифмической производной мероморфных функций с логарифмической особой точкой и указаны их применения в аналитической теории дифференциальных уравнений.

The following theorem is proved in ([1]): *let all coefficients of the differential equation*

$$w^{(n)} + a_{n-1}(z)\, w^{(n-1)} + \ldots + a_{\mu+1}(z)\, w^{(\mu+1)} + a_\mu(z)\, w^{(\mu)} + \ldots + a_0(z)\, w = 0, \qquad (1)$$

be entire functions, and $a_{n-1}(z), \ldots, a_{\mu+1}(z), z \in \mathbb{C}$, be polynomials, $a_\mu(z)$, $z \in \mathbb{C}$, be a transcedental function. Then equation (1) has at most μ linearly independent entire solutions of finite order.

For a polynomial $a(z), z \in \mathbb{C}$, Nevanlinna's characteristic is defined by $m(r,a) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |a(re^{i\varphi})|\, d\varphi = O(\ln r), (r \to +\infty)$. An entire transcendental function $f(z), z\in\mathbb{C}$, has an essential singularity in ∞, and we have $m(r,f)/\ln r \to +\infty, r \to +\infty$ ([2, p. 40, 50]). The mentioned theorem can be formulated in a more general form in terms of Nevanlinna's characteristics: *let the coefficients $a_i(z), z \in \mathbb{C}, i \in \{0,1,\ldots,n-1\}$, of equation (1) be meromorphic functions. If*

$$m(r,a_{n-1}) = O(\ln r), m(r,a_{n-2}) = O(\ln r), \ldots, m(r,a_{\mu+1}) = O(\ln r),$$

$$m(r,a_\mu) \neq O(\ln r), (r \to +\infty),$$

then equation (1) has at most μ linearly independent meromorphic solutions of finite order. Here we prove that an analogous statement is true in the case when coefficients and solutions of equation (1) have a logarithmic singularity (Theorem 1). The proof is based on estimates of the logarithmic derivative of a function with logarithmic singularity at ∞ (Theorems 2, 3).

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Let us recall the definition of a meromorphic function with logarithmic singularity at ∞. By A_l we denote the set of analytic functions in $G = \{z : r_0 \le |z| < \infty\}$ for which ∞ is the unique singular point, namely logarithmic singular point. The set A_l is a commutative ring without divisors of zero (complete ring). The field of quotients of the ring A_l is denoted by M_l (each complete ring can be embedded in some field [3, p. 52, 58]) $A_l \subset M_l$. If $f \in A_l$ then we shall say, that $f(z), z \in G$, is an analytic function with an isolated logarithmic singular point at ∞. If $f \in M_l$, then the function $f(z), z \in G$, is called a *meromorphic function with logarithmic singularity at* ∞.

In [4, p. 12] an equivalent definition of a meromorphic function is considered. This definition is based on a concept of analytic continuation.

Let $f \in M_l$. For any $\alpha, \beta, \ -\infty < \alpha < \beta < +\infty$ (it is possible that $\beta - \alpha > 2\pi$) we shall denote by

$$f(z), \quad z \in g_{\alpha,\beta} = \left\{ z = re^{i\varphi} : \alpha \le \varphi \le \beta, r_0 \le r < +\infty \right\}, \tag{2}$$

a single-valued branch of the function $f \in M_l$ (see [4, p. 12]).

We shall consider Nevanlinna's characteristics of the function $f(z), z \in g_{\alpha,\beta}$ ([2, p. 40]). We write $\ln^+ x = \max(\ln x, 0)$, $x \ge 0$; $k = \pi/(\beta - \alpha) > 0$. Let $b_l = |b_l| \exp(i\varphi_l)$ be poles of the function $f(z), z \in g_{\alpha,\beta}$. We put

$$\begin{gathered} A_{\alpha,\beta}(r,f) = \frac{k}{\pi} \int\limits_{r_0}^{r} \left(\frac{1}{t^{k+1}} - \frac{t^{k-1}}{r^{2k}} \right) \left(\ln^+ |f(te^{i\alpha})| + \ln^+ \left| f\left(te^{i\beta}\right) \right| \right) dt, \\ B_{\alpha,\beta}(r,f) = \frac{2k}{\pi r^k} \int\limits_{\alpha}^{\beta} \ln^+ |f(re^{i\varphi})| \sin k(\varphi - \alpha) d\varphi, \\ C_{\alpha,\beta}(r,f) = 2k \int\limits_{r_0}^{r} c_{\alpha,\beta}(t,f) \left(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}} \right) dt, \end{gathered} \tag{3}$$

where $c_{\alpha,\beta}(t,f) = c_{\alpha,\beta}(t,\infty) = \sum\limits_{r_0 < |b_l| \le t, \ \alpha \le \varphi_l \le \beta} \sin k(\varphi_l - \alpha)$ is the counting function of the poles; each pole is counted according to its multiplicity,

$$S_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f), \quad r_0 \le r < \infty. \tag{4}$$

For any single-valued branch $f(z), z \in g_{\alpha,\beta}$ of a function $f \in M_l$ we define

$$\rho_{\alpha,\beta} = \overline{\lim_{r \to +\infty}} \ln^+ S_{\alpha,\beta}(r,f) / \ln r. \tag{5}$$

The value $\rho = \rho[f] = \sup\{\rho_{\alpha,\beta} \colon -\infty < \alpha < \beta < +\infty\}$ is called the order of growth of the function $f(z), z \in G$.

For a single-valued branch $f(z), z \in g_{\alpha,\beta}$ of a meromorphic function $f(z), z \in G$ with a logarithmic singularity in ∞ we define

$$m_{\alpha,\beta}(r,f) \overset{\text{def}}{=} \frac{1}{\beta - \alpha} \int\limits_{\alpha}^{\beta} \ln^+ |f(z)|_{z = re^{i\varphi} \in g_{\alpha,\beta}} d\varphi. \tag{6}$$

Let the coefficients $a_j(z), z \in G = \{z : 0 < r_0 \le |z| < +\infty\}$, $j \in \{0, 1, \dots, n-1\}$, of equation (1) be holomorphic functions. The holomorphic solution $w(z), z \in g_0$ of equation (1) such that $w^{(j)}(z_0) = w_j, j \in \{0, 1, \dots, n-1\}$, exists in some neighborhood $g_0 = \{z : |z - z_0| < \varepsilon\}$ by the theorem on existence and uniqueness of a solution of the Cauchy problem [5, p. 34]. This solution can be analytically continued along any curve $L : z = \lambda(t), t_1 \le t \le t_2, \lambda(t_1) = z_0, \lambda(t_2) = z_1$, $[L] = \{z : z = \lambda(t), t_1 \le t \le t_2\} \subset G$, and the result of this continuation is a regular element $w_1(z), z \in g_1 = \{z : |z - z_1| < \varepsilon\}$ with the center at a point z_1, which also satisfies equation (1) ([5, p. 36]). The set of all such elements is denoted by $w(z), z \in G$. Since G is not a simply connected domain, $w(z), z \in G$, is an analytic function with logarithmic singularity at ∞, in general. In the considered case equation (1) has n linearly independent solutions of the form $w(z) = z^{\lambda} \sum_{j=0}^{k} f_j(z) \ln^j z, \lambda \in \mathbb{R}$; $f_j(z), z \in G$, are holomorphic functions for which ∞ is an essential singular point ([5], [6, p. 184]). For example, the general integral of the equation $zw' = 1$ is the function $w = \ln \frac{z}{z_0}$.

Let now a_j be coefficients of equation (1), and $a_j \in M_l, j \in \{0, 1, \dots, n-1\}$. We choose arbitrary $\alpha, \beta, -\infty < \alpha < \beta < +\infty$. We denote by $a_j(z), z \in g_{\alpha,\beta}$, single-valued branches of the coefficients $a_j \in M_l, j \in \{0, 1, \dots, n-1\}$, of equation (1) on the part $g_{\alpha,\beta}$ of the Riemann surface (see definition), respectively. The following theorems hold.

Theorem 1. *Let a_j be coefficients of equation (1), and $a_j \in M_l, j \in \{0, 1, \dots, n-1\}$. If for some $\alpha, \beta, -\infty < \alpha < \beta < +\infty$ single-valued branches $a_j(z), z \in g_{\alpha,\beta}, j \in \{0, 1, \dots, n-1\}$, are such that as $r \to \infty$*

$$\begin{gathered} m_{\alpha,\beta}(r, a_k) = O(\ln r) \quad (\mu + 1 \le k \le n - 1), \\ m_{\alpha,\beta}(r, a_\mu) \ne O(\ln r), \end{gathered} \tag{7}$$

then equation (1) has at most μ linearly independent solutions f, $f \in M_l$, of a finite order.

Theorem 2. *If $f(z), z \in G = \{z : r_0 \le |z| < +\infty\}$ is a meromorphic function with a logarithmic singularity in ∞, which has finite order ρ, then for any branch $f(z), z \in g_{\alpha,\beta}$ and $(\forall\, \varepsilon > 0)$ $(\exists\, d = d(\alpha, \beta, \varepsilon))$:*

$$\left| \frac{f^{(n)}(z)}{f(z)} \right| < |z|^{2n(\rho+1+\varepsilon)}, \quad z \in g_{\alpha,\beta} \backslash E, \quad |z| \ge d, \quad n \in \mathbb{N}, \tag{8}$$

where E is a set of disks on the part of the Riemann surface $g_{\alpha,\beta}$ with centers at zeros and poles of a branch $f(z), z \in g_{\alpha,\beta}$, and the sum of their radii is finite.

Proof of Theorem 2. The following theorem is proved in [4, p. 12]: *if $f(z), z \in G$ is a meromorphic function with logarithmic singularity at ∞, which has finite order ρ, then for any single-valued branch $f(z), z \in g_{\alpha,\beta}$ and $\forall\, \varepsilon > 0\ \exists d = d(\alpha, \beta, \varepsilon)$:*

$$\left| \frac{d^q \ln f(z)}{dz^q} \right| < |z|^{(q+1)(\rho+1)+\varepsilon}, \quad z \in g_{\alpha,\beta} \backslash E, \ |z| \ge d, \ q \in \mathbb{N}, \tag{9}$$

E is a set of disks on the part of the Riemann surface $g_{\alpha,\beta}$ with centers at zeros and poles of a branch $f(z), z \in g_{\alpha,\beta}$, and the sum of their radii is finite.

We use the following known lemma ([6, p. 150]). *Let $f(z)$ be a meromorphic function in a domain $D \subset \mathbb{C}$. Then for $n \in \mathbb{N}$*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{f'(z)}{f(z)} \right)^n + \sum_{\sum q i_q = n} B_{i_1 \dots i_{n-1}} \prod_{q=1}^{n-1} \left(\frac{d^q \ln f(z)}{dz^q} \right)^{i_q} + \frac{d^n \ln f(z)}{dz^n}, \tag{10}$$

holds, where the sum is taken for all $i_1, \dots, i_{n-1}$, with $0 \le i_1, \dots, i_{n-1} < n$, $\sum q i_q = 1 i_1 + 2 i_2 + \dots + (n-1) i_{n-1} = n$; $B_{i_1 \dots i_{n-1}}$ are nonnegative numbers.

From (9) and (10) we have ($z \in g_{\alpha,\beta} \backslash E$)

$$\left| \prod_{q=1}^{n-1} \left(\frac{d^q \ln f(z)}{dz^q} \right)^{i_q} \right| < \prod_{q=1}^{n-1} |z|^{(q+1)(\rho+1+\varepsilon) i_q} = |z|^{\sum (q+1)(\rho+1+\varepsilon) i_q}. \tag{11}$$

Since $i_1 < n$, we have $i_1 + i_2 + \ldots + i_{n-1} < i_1 + 2i_2 + \ldots + (n-1) i_{n-1} = n$, i $\sum (q+1) \times (\rho + 1 + \varepsilon) i_q = (\rho + 1 + \varepsilon) \left(\sum q i_q + \sum i_q \right) \le (\rho + 1 + \varepsilon)(2n - 1)$. Thus from (11) we obtain

$$\left| \prod_{q=1}^{n-1} \left(\frac{d^q \ln f(z)}{dz^q} \right)^{i_q} \right| < |z|^{(\rho+1+\varepsilon)(2n-1)}, \quad z \in g_{\alpha,\beta} \backslash E. \tag{12}$$

For $q = 1$ and $q = n$ from (9) we deduce

$$\left| \frac{f'(z)}{f(z)} \right|^n < |z|^{2n(\rho+1+\varepsilon)}, \quad \left| \frac{d^n \ln f(z)}{dz^n} \right| < |z|^{(n+1)(\rho+1+\varepsilon)}, \quad z \in g_{\alpha,\beta} \backslash E. \tag{13}$$

Inequality (8) is a consequence of (10), (12), (13). □

We denote $\Delta = \{ r : \ r = |z|, z \in E \}$. Since E is a set of disks on the part of the Riemann surface $g_{\alpha,\beta}$ of the function $f(z), z \in G$, with finite sum of the radii, then Δ is a set of intervals on $[r_0, +\infty)$, with finite sum of their lengths (mes $\Delta < +\infty$). Then from (8), (13) it follows that

$$\begin{aligned} \left| \frac{d^n \ln f(z)}{dz^n} \right|_{|z|=r} &< r^{(n+1)(\rho+1+\varepsilon)}, \quad r \in [r_0, +\infty) \backslash \Delta, \ \varepsilon > 0, \\ \left| \frac{f^{(n)}(re^{i\varphi})}{f(re^{i\varphi})} \right| &< r^{2n(\rho+1+\varepsilon)}, \qquad r \in [r_0, +\infty) \backslash \Delta, \ \varepsilon > 0, n \in \mathbb{N}. \end{aligned} \tag{14}$$

By (14), (6) the following inequalities hold for meromorphic functions of finite order ρ with a logarithmic singularity at ∞:

$$\begin{aligned} m_{\alpha,\beta}\left(r, \frac{d^n \ln f(z)}{dz^n}\right) &< (n+1)(\rho + 1 + \varepsilon) \ln r, \quad r \notin \Delta, \ \text{mes}\ \Delta < \infty; \\ m_{\alpha,\beta}\left(r, \frac{f^{(n)}}{f}\right) &< 2n(\rho + 1 + \varepsilon) \ln r, \qquad r \notin \Delta, \ \text{mes}\ \Delta < \infty, \ n \in \mathbb{N}. \end{aligned} \tag{15}$$

Estimates (15) hold on $[r_0, +\infty)$ outside a set of intervals Δ with finite sum of lengths. The following estimates hold everywhere on some interval $[d, +\infty)$.

Theorem 3. *If $f(z), z \in G = \{ z : r_0 \le |z| < +\infty \}$ is a meromorphic function with a logarithmic singularity at ∞, which has finite order ρ, then for any single-valued branch $f(z), z \in g_{\alpha,\beta} = \{ z = re^{i\varphi} : \alpha \le \varphi \le \beta, r_0 \le r < +\infty \}$ and $(\forall\ \varepsilon > 0)$ $(\exists\ d = d(\alpha, \beta, \varepsilon))$:*

$$m_{\alpha,\beta}\left(r, \frac{d^n \ln f(z)}{dz^n}\right) < (2n\rho + \varepsilon) \ln r, \quad r \ge d, n \in \mathbb{N}. \tag{16}$$

Moreover,

$$m_{\alpha,\beta}\left(r, \frac{f^{(n)}}{f}\right) = O(\ln r), \quad r \to \infty, n \in \mathbb{N}. \tag{17}$$

We denote by $\{c_q\}$ the set of zeros and poles of a single-valued branch $f(z), z \in g_{\alpha,\beta}, \; c_q = |c_q| e^{i\varphi_q}$;

$$c_{\alpha,\beta}(s,f) = \sum_{r_0<|c_q|\le s, \; \alpha\le\varphi_q\le\beta} \sin k(\varphi_q - \alpha) \tag{18}$$

is the counting function of zeros and poles of the branch $f(z), z \in g_{\alpha,\beta}$.

We need the following lemma.

Lemma 1. *Let $r_0 < s < R$. Then*

$$c_{\alpha,\beta}(s,f) < \frac{s^k R^{2k} S_{\alpha,\beta}(R,f)}{2(R^{2k} - s^{2k})}, \quad k = \frac{\pi}{\beta - \alpha}, \tag{19}$$

where $S_{\alpha,\beta}(R,f)$ is Nevanlinna's characteristic.

Proof of the lemma 1. From the definition of the characteristic $C_{\alpha,\beta}(R,f)$ it follows that

$$C_{\alpha,\beta}(R,f) \ge 2k \int_s^R c_{\alpha,\beta}(t,f) \left(\tfrac{1}{t^{k+1}} + \tfrac{t^{k-1}}{R^{2k}}\right) dt \ge 2c_{\alpha,\beta}(s,f) \left(\tfrac{1}{s^k} - \tfrac{1}{R^k} + \tfrac{R^k - s^k}{R^{2k}}\right) =$$
$$= 2c_{\alpha,\beta}(s,f) \tfrac{R^{2k} - s^{2k}}{s^k R^{2k}}.$$

Since $C_{\alpha,\beta}(R,f) \le S_{\alpha,\beta}(R,f)$, the latter inequalities yield (19). □

Let us note that the special case of a lemma at к=1 is Lemma 2 of [4, p. 14].

Proof of Theorem 3. Let $\alpha \in]-\infty, +\infty[$. We consider the case $\beta = \alpha + \pi$. We write $A = \alpha - \frac{\pi}{2}$, $B = \alpha + \frac{3\pi}{2}$. Consider single-valued branches $f(z), \; z \in g_{\alpha,\alpha+\pi}$ and $f(z), \; z \in g_{A,B}$, of the function $f(z), z \in G$. The set of zeros and poles of the branch $f(z), z \in g_{A,B}$, is denoted by $\{c_q\}$ again, $c_q = |c_q| e^{i\varphi_q}$. Then $k = \frac{\pi}{B-A} = \frac{1}{2}$, $\sin k\left(\varphi - \alpha + \frac{\pi}{2}\right) = \sin \frac{1}{2}\left(\varphi - \alpha + \frac{\pi}{2}\right)$.

$$c_{A,B}(s,0,\infty) \stackrel{\text{def}}{=} \sum_{r_0<|c_q|\le s, \; A\le\varphi_q\le B} \sin \frac{1}{2}\left(\varphi_q - \alpha + \frac{\pi}{2}\right)$$

is the counting function of zeros and poles of the branch $f(z), z \in g_{A,B}$;

$$n_{\alpha,\alpha+\pi}(s,0,\infty) \stackrel{\text{def}}{=} \sum_{r_0<|c_q|\le s, \; \alpha\le\varphi_q\le\alpha+\pi} 1 \tag{20}$$

is the number of zeros and poles of the branch $f(z), z \in g_{\alpha,\alpha+\pi}$ in the domain $g_{\alpha,\alpha+\pi}$. If $\alpha \le \varphi \le \alpha + \pi$, then $\frac{\pi}{4} \le \frac{1}{2}\left(\varphi - \alpha + \frac{\pi}{2}\right) \le \frac{3\pi}{4}$. Therefore

$$\sin \tfrac{1}{2}\left(\varphi - \alpha + \tfrac{\pi}{2}\right) \ge \tfrac{1}{\sqrt{2}}, \; \alpha \le \varphi \le \alpha + \pi. \tag{21}$$

From (20), (21) it follows that

$$\tfrac{1}{\sqrt{2}} n_{\alpha,\alpha+\pi}(s,0,\infty) = \tfrac{1}{\sqrt{2}} \sum_{r_0<|c_q|\le s, \; \alpha\le\varphi_q\le\alpha+\pi} 1 \le \sum_{r_0<|c_q|\le s, \; \alpha\le\varphi_q\le\alpha+\pi} \sin \tfrac{1}{2}\left(\varphi_q - \alpha + \tfrac{\pi}{2}\right) \le$$
$$\le \sum_{r_0<|c_q|\le s, \; A\le\theta_q\le B} \sin \tfrac{1}{2}\left(\varphi_q - \alpha + \tfrac{\pi}{2}\right) = c_{A,B}(s,0,\infty),$$

or

$$n_{\alpha,\alpha+\pi}(s,0,\infty) \le \sqrt{2}\, c_{A,B}(s,0,\infty). \tag{22}$$

Let us apply estimate (19) to the function $c_{A,B}(s,0,\infty)$. In this case $k = \frac{\pi}{B-A} = \frac{1}{2}$ and from (19) it follows that

$$c_{A,B}(s,0,\infty) \le \frac{\sqrt{s}\ R\ S_{A,B}(R,f)}{2(R-s)}. \tag{23}$$

The function $z = \zeta e^{i\alpha}$ is a bijection of the closed domain $\overline{D} = \{\zeta = re^{i\theta} : 0 \le \theta \le \pi, r_0 \le r < +\infty\}$ on $g_{\alpha,\alpha+\pi}$. Under this bijection the single-valued branch $f(z)$, $z \in g_{\alpha,\alpha+\pi}$ corresponds to the meromorphic function

$$w(\zeta) \stackrel{\text{def}}{=} f\left(\zeta e^{i\alpha}\right), \zeta \in \overline{D}; \tag{24}$$

any zero (pole) $c_q \in \{c_q\}$ of the branch $f(z), z \in g_{\alpha,\alpha+\pi}$ corresponds to the zero (pole) $\zeta_q = c_q e^{-i\alpha}, \zeta_q = |\zeta_q| e^{i\theta_q} = |c_q| e^{i(\varphi_q-\alpha)}$ of the function $w(\zeta), \zeta \in \overline{D}$. From (24) it follows that

$$w'(\zeta) = f'(\zeta e^{i\alpha})\, e^{i\alpha},\ \tfrac{d}{d\zeta}\ln w(\zeta) = \tfrac{w'(\zeta)}{w(\zeta)} = \tfrac{f'\left(\zeta e^{i\alpha}\right)}{f(\zeta e^{i\alpha})} e^{i\alpha} = \tfrac{d}{dz}\ln f(z)\,\Big|_{z=\zeta e^{i\alpha}} e^{i\alpha}, \dots,$$
$$\tfrac{d^l}{d\zeta^l}\ln w(\zeta) = \tfrac{d^l}{dz^l}\ln f(z)\,\Big|_{z=\zeta e^{i\alpha}} e^{il\alpha},\ l \in \mathbb{N}.$$

So

$$\left|\frac{d^l}{dz^l}\ln f(z)\right| = \left|\frac{d^l}{d\zeta^l}\ln w(\zeta)\right|,\quad re^{i\varphi} = z = \zeta e^{i\alpha} = re^{i(\theta+\alpha)}, \tag{25}$$

$\theta = \varphi - \alpha, \theta_q = \varphi_q - \alpha$. Therefore ($\varkappa \in (0,1)$)

$$\int_0^\pi \Big|\frac{d^l}{d\zeta^l}\ln w(\zeta)\Big|^{\varkappa}_{\zeta=re^{i\theta}} d\theta = \int_0^\pi \Big|\frac{d^l}{dz^l}\ln f(z)\Big|^{\varkappa}_{z=re^{i(\theta+\alpha)}} d\theta = \int_\alpha^{\alpha+\pi} \Big|\frac{d^l}{dz^l}\ln f(z)\Big|^{\varkappa}_{z=re^{i\varphi}} d\varphi. \tag{26}$$

From (24) it follows ([2, p. 41]) that

$$S_{0,\pi}(r,w) = S_{\alpha,\alpha+\pi}(r,f). \tag{27}$$

The following estimate is proved in [7] (in the sequel K stands for different constants: $K = \text{const} > 0$): if $w(\zeta), \zeta \in \overline{D} = \left\{\zeta = re^{i\theta} :\ 0 \le \theta \le \pi,\ r_0 \le r < +\infty\right\}$ is a meromorphic function, $\{\zeta_q\}$ is the set of its zeros and poles, $\zeta_q = |\zeta_q| e^{i\theta_k}$, then

$$K^{-1}\left|\frac{d^l}{d\zeta^l}\ln w(\zeta)\right| < \frac{S_{0,\pi}(2r,w)+1}{r^{l-1}\sin^{l+1}\theta} + \sum_{r_0<|\zeta_q|<2r} \frac{\sin\theta_q}{\sin^l\theta\,|\zeta-\zeta_q|^l} + 1, \tag{28}$$

$K = \text{const} > 0, l \in \mathbb{N}, S_{0,\pi}(s,w)$ is Nevanlinna's characteristic of the function $w(\zeta), \zeta \in \overline{D}$; the sum is taken over all $\zeta_q, r_0 < |\zeta_q| < 2r$.

The following inequality is well-known: $\left(\sum_q d_q\right)^{\varkappa} \le \sum_q (d_q)^{\varkappa},\ \ 0 < \varkappa < 1,\ d_q > 0$. Let us integrate (28) on the interval $[0,\pi]$ with $\frac{1-\varepsilon}{2l} < \varkappa < \frac{1}{2l}$, where $\varepsilon > 0,\ \varepsilon$ is a given number.

We obtain ($0 \le \sin\theta_q \le 1, 0 \le \theta_q \le \pi$)

$$K^{-1}\int\limits_0^{\pi}\left|\frac{d^l \ln w(\zeta)}{d\zeta^l}\right|^{\varkappa}_{\zeta=re^{i\theta}} d\theta \le \frac{(S_{0,\pi}(2r,w)+1)^{\varkappa}}{r^{(l-1)\varkappa}}\int\limits_0^{\pi}\frac{d\theta}{\sin^{(l+1)\varkappa}\theta}+ \\ +\sum_{r_0<|\zeta_q|\le 2r,\ 0\le\theta_q\le\pi}\int\limits_0^{\pi}\frac{d\theta}{\sin^{l\varkappa}\theta|\zeta-\zeta_q|^{l\varkappa}}+\pi,\quad l\in\mathbb{N}. \tag{29}$$

Using the Cauchy-Bunyakovski inequality and inequalities $\frac{2x}{\pi} < \sin x,\ 0 < x < \frac{\pi}{2};\ \ \varkappa < \frac{1}{2l},\ 1-(l+1)\varkappa > 0$, we obtain

$$\int\limits_0^{\pi}\frac{d\theta}{\sin^{l\varkappa}\theta|\zeta-\zeta_q|^{l\varkappa}} \le \left(\int\limits_0^{\pi}\frac{d\theta}{\sin^{2l\varkappa}\theta}\right)^{\frac12}\left(\int\limits_0^{\pi}\frac{d\theta}{|\zeta-\zeta_q|^{2l\varkappa}}\right)^{\frac12} < \\ < \left(2\int\limits_0^{\frac{\pi}{2}}\frac{d\theta}{\sin^{2l\varkappa}\theta}\right)^{\frac12}\left(\int\limits_0^{2\pi}\frac{d\theta}{\left|re^{i\theta}-|\zeta_q|\right|^{2l\varkappa}}\right)^{\frac12} < \left(2\left(\frac{\pi}{2}\right)^{2l\varkappa}\int\limits_0^{\frac{\pi}{2}}\frac{d\theta}{\theta^{2l\varkappa}}\right)^{\frac12}\left(\int\limits_0^{2\pi}\frac{d\theta}{(r\sin\theta)^{2l\varkappa}}\right)^{\frac12} = \\ < \frac{2}{r^{l\varkappa}}\left(\frac{\pi}{1-2l\varkappa}\right)^{\frac12}\left(\int\limits_0^{\frac{\pi}{2}}\frac{d\theta}{\theta^{2l\varkappa}}\right)^{\frac12}\left(\frac{\pi}{2}\right)^{l\varkappa} = \frac{\pi\sqrt{2}}{(1-2l\varkappa)r^{l\varkappa}}. \tag{30}$$

From (29), (30) it follows that $\left(\sum_{r_0<|\zeta_q|\le 2r,\ 0\le\theta_q\le\pi} 1 = n_{0,\pi}(2r,0,\infty)\right)$

$$K^{-1}\int\limits_0^{\pi}\left|\frac{d^l\ln w(\zeta)}{d\zeta^l}\right|^{\varkappa}_{\zeta=re^{i\theta}} d\theta < \frac{(S_{0,\pi}(2r,w)+1)^{\varkappa}}{r^{(l-1)\varkappa}}+\frac{n_{0,\pi}(2r,0,\infty)}{r^{l\varkappa}}+1, \tag{31}$$

$K = K(\varkappa) = \text{const} > 0,\ l \in \mathbb{N}$. From (26), (27), (31) we have

$$K^{-1}\int\limits_{\alpha}^{\alpha+\pi}\left|\frac{d^l}{dz^l}\ln f(z)\right|^{\varkappa}_{z=re^{i\varphi}} d\varphi \le \frac{(S_{\alpha,\alpha+\pi}(2r,f)+1)^{\varkappa}}{r^{(l-1)\varkappa}}+\frac{n_{\alpha,\alpha+\pi}(2r,0,\infty)}{r^{l\varkappa}}+1, \tag{32}$$

where $n_{\alpha,\alpha+\pi}(s,0,\infty)$ is determined by (20).

It is well-known ([2, p.116] that if $f(x) \ge 0, a \le x \le b$, is a measurable function then $\frac{1}{b-a}\int\limits_a^b \ln^+ f(x)\,dx \le \ln^+\left\{\frac{1}{b-a}\int\limits_a^b f(x)\,dx\right\}+\ln 2$. Therefore from (32) we have

$$\frac{1}{\pi}\int\limits_{\alpha}^{\alpha+\pi}\ln^+\left|\frac{d^l}{dz^l}\ln f(z)\right|_{z=re^{i\varphi}} d\varphi = \frac{1}{\varkappa\pi}\int\limits_{\alpha}^{\alpha+\pi}\ln^+\left|\frac{d^l}{dz^l}\ln f(z)\right|^{\varkappa}_{z=re^{i\varphi}} d\varphi \le \\ \le \frac{1}{\varkappa}\ln^+\left\{\frac{1}{\pi}\int\limits_{\alpha}^{\alpha+\pi}\left|\frac{d^l}{dz^l}\ln f(z)\right|^{\varkappa}_{z=re^{i\varphi}} d\varphi\right\}+\frac{\ln 2}{\varkappa}. \tag{33}$$

If the function $f(z), z \in G$, has finite order ρ, then for any $\alpha, \beta,\ -\infty < \alpha < \beta < +\infty$ the following inequality holds:

$$S_{\alpha,\beta}(r,f) < Kr^{\rho+\frac{\varepsilon}{2}},\quad \varepsilon > 0;\ K = \text{const} > 0. \tag{34}$$

From (22), (23) it follows that ($R = 3r, s = 2r$)

$$n_{\alpha,\alpha+\pi}(2r,0,\infty) \le \sqrt{2}c_{A,B}(2r,0,\infty) \le \frac{\sqrt{2r}\ 3rS_{A,B}(3r,f)}{2r} = \frac{3\sqrt{r}S_{A,B}(3r,f)}{\sqrt{2}}.$$

Hence, using (32), (34), we have

$$K^{-1}\int\limits_{\alpha}^{\alpha+\pi}\Big|\frac{d^l}{dz^l}\ln f(z)\Big|^{\varkappa}_{z=re^{i\varphi}}d\varphi<r^{\varkappa\left(\rho+\frac{\varepsilon}{2}\right)-(l-1)\varkappa}+r^{\rho+\frac{\varepsilon}{2}+\frac{1}{2}-l\varkappa}.$$

So $(\frac{1}{2}-\frac{\varepsilon}{2}<\varkappa\, l<\frac{1}{2})$

$$\int\limits_{\alpha}^{\alpha+\pi}\Big|\frac{d^l}{dz^l}\ln f(z)\Big|^{\varkappa}_{z=re^{i\varphi}}d\varphi<Kr^{\rho+\varepsilon},\quad \varepsilon>0, K=\text{const}>0. \tag{35}$$

From (33), (35) it follows that $\frac{1}{\pi}\int\limits_{\alpha}^{\alpha+\pi}\ln^+\Big|\frac{d^l}{dz^l}\ln f(z)\Big|^{\varkappa}_{z=re^{i\varphi}}d\varphi<\frac{\rho+\varepsilon}{\varkappa}\ln r+K$, $\varepsilon>0$, $K=$ const >0, $l\in\mathbb{N}$, $\frac{1-\varepsilon}{2l}<\varkappa<\frac{1}{2l}$. Thus,

$$m_{\alpha,\alpha+\pi}\left(r,\frac{d^l}{dz^l}\ln f(z)\right)=\frac{1}{\pi}\int\limits_{\alpha}^{\alpha+\pi}\ln^+\Big|\frac{d^l}{dz^l}\ln f(z)\Big|_{z=re^{i\varphi}}d\varphi<(2l\rho+\varepsilon)\ln r+K \tag{36}$$

for the function $f(z), z\in G$ which has finite order ρ, $\varepsilon>0$, $K=\text{const}>0$, $l\in\mathbb{N}$, $r\geq r_0$. We have proved estimate (16) for the branch $f(z), z\in g_{\alpha,\alpha+\pi}$. The proof of estimate (16) in the case of any branch $f(z), z\in g_{\alpha,\beta}$ is analogous.

Since ([2, p. 25]) $\ln^+\left|\prod\limits_{\nu=1}^{n}x_\nu\right|\leq\sum\limits_{\nu=1}^{n}\ln^+|x_\nu|$, $\ln^+\left|\sum\limits_{\nu=1}^{n}x_\nu\right|\leq\sum\limits_{\nu=1}^{n}\ln^+|x_\nu|+\ln n$, from the definition of the characteristic $m_{\alpha,\beta}(r,f)$ it follows that for meromorphic functions $f(z)$, $g(z), z\in G$ with a logarithmic singularity in ∞ the inequalities

$$\begin{gathered}m_{\alpha,\beta}(r,fg)\leq m_{\alpha,\beta}(r,f)+m_{\alpha,\beta}(r,g),\\ m_{\alpha,\beta}(r,f+g)\leq m_{\alpha,\beta}(r,f)+m_{\alpha,\beta}(r,g)+\ln 2,\end{gathered} \tag{37}$$

are valid. Estimate (17) is a consequence of (10), (37). □

Proof of Theorem 1. Let us write $m(r,f)$ instead of $m_{\alpha,\beta}(r,f)$ for simplicity. Let $\mu=0$, that is

$$m(r,a_{n-1})=O(\ln r),\dots,m(r,a_1)=O(\ln r),m(r,a_0)\neq O(\ln r). \tag{38}$$

Then let us prove that $\forall n\in\mathbb{N}$ equation (1) has no solutions $f\in M_l$ of finite order. In fact, assume that $w=f\in M_l$ is a solution of finite order of differential equation (1). Then from (1) it follows that $a_0=-\frac{f^{(n)}}{f}-a_{n-1}\frac{f^{(n-1)}}{f}-\dots-a_1\frac{f'}{f}$. Hence, using (38) and (17) we have the inequality $m(r,a_0)\leq m\left(r,\frac{f^{(n)}}{f}\right)+m(r,a_{n-1})+m\left(r,\frac{f^{(n-1)}}{f}\right)+\dots+m(r,a_1)+m\left(r,\frac{f'}{f}\right)=O(\ln r)$, which contradicts to (38).

Let us assume that for $\mu=m-1$ and $\forall n\in\mathbb{N}$ the statement of the theorem is proved, that is, if

$$\begin{gathered}m(r,a_j)=O(\ln r),(r\to\infty),m\leq j\leq n-1,\\ m(r,a_{m-1})\neq O(\ln r),\end{gathered} \tag{39}$$

then equation (1) has at most $m-1$ linearly independent solutions $f_j\in M_l$, $j\in\{1,\dots,m-1\}$, of finite order.

Let us prove that for $\mu = m$ equation (1) cannot have more than m linearly independent solutions $f_j \in M_l, j \in \{1, \ldots, m\}$, of finite order. Assume the contrary. Let us use procedure of downturn of the order. Suppose that

$$m\,(r, a_j) = O\,(\ln r)\,, (r \to \infty), m + 1 \le j \le n - 1, \\ m\,(r, a_m) \ne O\,(\ln r)\,, \tag{40}$$

and equation (1) has $m + 1$ linearly independent solutions $f_j \in M_l, j \in \{1, 2, \ldots, m + 1\}$ of orders $\rho[f_j] < +\infty$, respectively. Let us make the substitution $w = u \cdot f_1$. Then by Leibnitz' formula (further we write $f_1 = f$) $w^{(k)} = \sum_{j=0}^{k} C_k^j u^{(j)} f^{(k-j)}$, $k \in \{0, 1, \ldots, n\}$, and equation (1) can be rewritten in the form $\sum_{k=0}^{n} a_k \sum_{j=0}^{k} C_k^j u^{(j)} f^{(k-j)} = 0$, $a_n = 1$, or

$$\sum_{k=1}^{n} a_k \sum_{j=1}^{k} C_k^j u^{(j)} f^{(k-j)} + u \sum_{k=0}^{n} a_k f^{(k)} = 0. \tag{41}$$

Since $f = f_1$ is a solution of (1), we have $\sum\limits_{k=0}^{n} a_k f^{(k)} = 0$. Then (41) becomes

$$\sum_{k=1}^{n} a_k \sum_{j=1}^{k} C_k^j u^{(j)} f^{(k-j)} = 0. \tag{42}$$

Dividing both parts of (42) by f and gathering summands with $u^{(s)}$, $s \in \{1, \ldots, n\}$, we obtain

$$\sum_{s=1}^{n} u^{(s)} \sum_{k=s}^{n} a_k C_k^s \frac{f^{(k-s)}}{f} = 0,$$

or

$$\sum_{s=2}^{n} (u')^{(s-1)} \sum_{k=s}^{n} a_k C_k^s \frac{f^{(k-s)}}{f} + u' \sum_{k=1}^{n} a_k C_k^1 \frac{f^{(k-1)}}{f} = 0. \tag{43}$$

We denote $u' = \upsilon, t = s - 1$. Then equation (43) takes the form

$$\sum_{t=0}^{n-1} \upsilon^{(t)} \sum_{k=t+1}^{n} a_k C_k^{t+1} \frac{f^{(k-t-1)}}{f} = 0. \tag{44}$$

Let

$$b_t = \sum_{k=t+1}^{n} a_k C_k^{t+1} \frac{f^{(k-t-1)}}{f}, \ t \in \{0, 1, \ldots, n-1\}, \tag{45}$$

where $a_n = 1$. In particular,

$$b_{m-1} = \sum_{k=m}^{n} a_k C_k^m \frac{f^{(k-m)}}{f} = a_m + \sum_{k=m+1}^{n} a_k C_k^m \frac{f^{(k-m)}}{f}. \tag{46}$$

From (17),(37),(45) it follows that

$$m\,(r, b_t) \le \sum_{k=t+1}^{n} m\,(r, a_k) + \sum_{k=t+1}^{n} m\left(r, \frac{f^{(k-t-1)}}{f}\right) + O\,(1) = \sum_{k=t+1}^{n} m\,(r, a_k) + O\,(\ln r)\,.$$

Therefore, taking into account equations (40), we deduce

$$m\left(r,b_j\right)=O\left(\ln r\right),(r\to\infty),m\le j\le n-1. \tag{47}$$

From (46) it follows that $a_m=b_{m-1}-\sum_{k=m+1}^{n}a_kC_k^m\frac{f^{(k-m)}}{f}$, therefore, taking into account equations (17), (37), (40), ($a_n=1$) we have $m_{\alpha,\beta}\left(r,a_m\right)\le m\left(r,b_{m-1}\right)+\sum_{k=m+1}^{n}m\left(r,a_k\right)+\sum_{k=m+1}^{n}m\left(r,\frac{f^{(k-m)}}{f}\right)+O\left(1\right)=m\left(r,b_{m-1}\right)+O\left(\ln r\right)$.
Since (see (40)), $m\left(r,a_m\right)\ne O\left(\ln r\right)$, the latter inequality gives

$$m\left(r,b_{m-1}\right)\ne O\left(\ln r\right),(r\to\infty). \tag{48}$$

b_{m-1} is the last from the coefficients $b_0,\dots,b_{m-1},b_m,\dots,b_{n-1}$, for which (48) holds, and for factors with large indices (47) is valid. This follows from (47) and (48). Therefore equation (44) cannot have more than $m-1$ linearly independent solutions $v_j\in M_l$ of finite order by the assumption of the induction.

The linearly independent solutions $f_2\left(z\right),\dots,f_{m+1}\left(z\right)$ of equations (1) correspond to the linearly independent solutions $u_1=f_2/f_1,\dots,u_m=f_{m+1}/f_1$ of equations (42) by the formula $w=u\cdot f_1$, and these solutions after replacement $u'=v$ transform in linearly independent solutions $v_j=u_j'=\left(\frac{f_{j+1}}{f_1}\right)'$, $j\in\{1,\dots,m\}$, of equations (44) with (as it will be shown below) the orders of growth

$$\rho[v_j]=\rho[(f_{j+1}/f_1)']<+\infty,\ j\in\{1,\dots,m\}; \tag{49}$$

We have that equation (44) has m linearly independent solutions of the finite order that contradicts the assumption.

Let us prove (49). The functions $f_j\in M_l$, $j\in\{1,\dots,m\}$ have finite order of growth by our assumption. As in [2, p. 45] $S_{\alpha,\beta}\left(r,\frac{f_{j+1}}{f_j}\right)\le S_{\alpha,\beta}\left(r,f_{j+1}\right)+S_{\alpha,\beta}\left(r,f_j\right)+O\left(1\right)$, then the ratio $\frac{f_{j+1}}{f_j}$ has also finite order of growth. Then the derivative $\left(\frac{f_{j+1}}{f_j}\right)'$ has finite order as well. Indeed, if the function $f\in M_l$ has finite order of growth, then ([8, p. 481]) for any α,β,A,B, $-\infty<A<\alpha<\beta<B<+\infty$ there exist $\alpha_1,\beta_1,-\infty<A<\alpha_1<\alpha<\beta<\beta_1<B<+\infty$ such that for characteristics $S_{\alpha,\beta}\left(r,f\right),S_{A,B}\left(r,f\right)$ and $S_{\alpha_1,\beta_1}\left(r,f\right),S_{\alpha_1,\beta_1}\left(r,f'\right)$ of single-valued branches $f\left(z\right),z\in g_{\alpha,\beta}$, $f\left(z\right),z\in g_{A,B}$, and $f\left(z\right),f'\left(z\right),z\in g_{\alpha_1,\beta_1}$, the inequalities

$$S_{\alpha,\beta}\left(r,f\right)\le S_{A,B}\left(r,f\right),\quad r_0\le r<+\infty,\ \ -\infty<A<\alpha_1<\alpha<\beta<\beta_1<B<+\infty, \tag{50}$$

$$S_{\alpha_1,\beta_1}\left(r,f'\right)\le 2S_{\alpha_1,\beta_1}\left(r,f\right),\ r\notin\Delta,\ \text{mes}\ \Delta<\infty \tag{51}$$

hold. From (50), (51) and it definition of the order the function f and its derivative f', it follows that f' has finite order too. □

Note that we cannot apply the lemma on the logarithmic derivative ([2, p. 137]) to the proof of inequality (51), because for the meromorphic function in an angular domain this lemma is not valid ([9]), in general.

Remark. *Let $f\left(z\right),z\in G$, is a single-valued meromorphic function. Let us write its argument in the exponential form; the function $f\left(re^{i\theta}\right),r_0\le r<+\infty,-\infty<\theta<+\infty$ has the period 2π in θ. It allows us to consider the single-valued meromorphic function $f\left(re^{i\theta}\right),re^{i\theta}\in G$, as a version of meromorphic function with a logarithmic singular point at*

∞, *which has the period* 2π *in* θ. *Hence the statements which are proved for any function* $f \in M_l$ *are valid for single-valued meromorphic functions.*

In particular, we apply Theorem 1 in the case when in equation (1) the coefficients $a_j(z)$, $z \in G,\ j \in \{0, \ldots, n-1\}$, are single-valued meromorphic functions such that for $\alpha = 0$, $\beta = 2\pi$ conditions (7) are satisfied. Then the statement of the theorem of Margaret Frei: (equation (1) has at most μ linearly independent single-valued solutions of the finite order) follows from Theorem 1.

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