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ON THE LOWER Φ-TYPE OF THE MAXIMAL TERM OF DIRICHLET SERIES

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For Dirichlet series conditions on coefficients and exponents are found in order that the lower Φ-type of its maximal term is positive and, in particular, infinite.

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Для ряда Дирихле найдены достаточные условия на коэффициенты и показатели для того, чтобы нижний Φ-тип его максимального члена был положительным и, в частности, бесконечным.

1. Introduction. Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be the increasing to $+\infty$ sequence of nonnegative numbers ($\lambda_0 = 0$) and let $S(\Lambda, A)$ be a class of Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

with the abscissa of absolutely convergence $\sigma_a = A \in (-\infty, +\infty]$. For $\sigma < A$ let $\mu(\sigma, F) = \max\{|a_n| \exp(\sigma\lambda_n) : n \geq 0\}$ be the maximal term of series (1).

By $\Omega(A)$ we denote a class of positive unbounded on $(-\infty, A)$ functions Φ such that the derivative Φ' is continuously differentiable, positive and increasing to $+\infty$ on $(-\infty, A)$. Let φ be the inverse function to Φ' and $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be the function associated with Φ in the sense of Newton. Then [1-2] Ψ is continuously differentiable and increasing to A on $(-\infty, A)$, and φ is continuously differentiable and increasing to A on $(0, +\infty)$.

We put

$$\tau_\Phi = \varliminf_{\sigma\uparrow A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)}, \quad T_\Phi = \varlimsup_{\sigma\uparrow A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)},$$

and

$$k_\Phi = \varliminf_{n\to\infty} \frac{\lambda_n}{\Phi'\left(\Psi^{-1}\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)\right)}, \quad K_\Phi = \varlimsup_{n\to\infty} \frac{\lambda_n}{\Phi'\left(\Psi^{-1}\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)\right)}.$$

The quantities T_Φ and τ_Φ are called Φ-type and lower Φ-type of $\ln \mu(\sigma, F)$.

It is known [1-2] that $\ln \mu(\sigma, F) \leq T\Phi(\sigma)$ for all $\sigma \in [\sigma_0, A)$ if and only if $\ln |a_n| \leq -\lambda_n \Psi(\varphi(\lambda_n/T))$ for all $n \geq n_0$. Hence, $T_\Phi = K_\Phi$.

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Here we investigate conditions on a_n and λ_n, under which $\tau_\Phi > 0$ and, in particular, $\tau_\Phi = +\infty$. If $\lambda_n = o\left(\Phi'\left(\Psi^{-1}\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)\right)\right)$, $n\to\infty$, then $\tau_\Phi = T_\Phi = 0$. Therefore, in order that $\tau_\Phi > 0$, it is necessary that $\Phi'\left(\Psi^{-1}\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|a_{n_k}|}\right)\right) = O(\lambda_{n_k})$, $k\to\infty$, for some subsequence (λ_{n_k}) of (λ_n). By analogy, in order that $\tau_\Phi = +\infty$, it is necessary that $\Phi'\left(\Psi^{-1}\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|a_{n_k}|}\right)\right) = o(\lambda_{n_k})$, $k\to\infty$, for some subsequence (λ_{n_k}) of λ_n. Thus, it is necessary to find a condition on the density of this sequence (λ_{n_k}). Below we show that such condition is $\lambda_{n_{k+1}} = O(\lambda_{n_k})$, $k\to\infty$.

We say that $F\in S(\Lambda, A, k_\Phi > 0)$ $(F\in S(\Lambda, A, k_\Phi = +\infty))$ if $F\in S(\Lambda, A)$ and $k_\Phi > 0$ $(k_\Phi = +\infty)$, and by $\Omega_1(A)$ we denote a class of functions $\Phi\in\Omega(A)$ such that Φ/Φ' is nonincreasing function.

Theorem. *Let $A\in(-\infty,+\infty]$. The condition $\lambda_{n+1} = O(\lambda_n)$, $n\to\infty$, is necessary and sufficient in order that $\tau_\Phi > 0$ for every $\Phi\in\Omega_1(A)$ and $F\in S(\Lambda, A, k_\Phi > 0)$, and is necessary and sufficient in order that $\tau_\Phi = +\infty$ for every $\Phi\in\Omega_1(A)$ and $F\in S(\Lambda, A, k_\Phi = = +\infty)$.*

Since $\max\{|a_n|\exp(\sigma\lambda_n) : n\ge 0\} \ge \max\{|a_{n_k}|\exp(\sigma\lambda_{n_k}) : k\ge 1\}$, from the theorem it follows that if there exists a subsequence (λ_{n_k}) of (λ_n) such that $\lambda_{n_{k+1}} = O(\lambda_{n_k})$, $k\to\infty$, and $\Phi'\left(\Psi^{-1}\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|a_{n_k}|}\right)\right) = O(\lambda_{n_k})$, $k\to\infty$, (or $\Phi'\left(\Psi^{-1}\left(\frac{1}{\lambda_{n_k}}\ln\frac{1}{|a_{n_k}|}\right)\right) = o(\lambda_{n_k})$, $k\to\infty$) then $\tau_\Phi > 0$ (respectively $\tau_\Phi = +\infty$) provided $\Phi\in\Omega(A)$.

2. Proof of the sufficiency. For $\Phi\in\Omega(A)$ we put

$$G_1(\lambda_n,\lambda_{n+1},\Phi) = \frac{\lambda_{n+1}\lambda_n}{\lambda_{n+1}-\lambda_n}\int\limits_{\lambda_n}^{\lambda_{n+1}}\frac{\Phi(\varphi(t))}{t^2}dt,\quad G_2(\lambda_n,\lambda_{n+1},\Phi) = \Phi\Big(\frac{1}{\lambda_{n+1}-\lambda_n}\int\limits_{\lambda_n}^{\lambda_{n+1}}\varphi(t)dt\Big).$$

From Theorem 3.1 in [2] it follows that if $\ln|a_n| \ge -\lambda_n\Psi(\varphi(\lambda_n))$ $(n\ge n_0)$ then $\ln\mu(\sigma, F) \ge \ge \Phi(\sigma)\frac{G_1(\lambda_n,\lambda_{n+1},\Phi)}{G_2(\lambda_n,\lambda_{n+1},\Phi)}$ for all $\sigma\in[\varphi(\lambda_n),\varphi(\lambda_{n+1})]$ and $n\ge n_0$. Since $\Phi(\varphi(t))/t$ is nonincreasing function, we have

$$\ln\mu(\sigma, F) \ge \Phi(\sigma)\frac{1}{\Phi(\varphi(\lambda_{n+1}))}\frac{\lambda_{n+1}\lambda_n}{\lambda_{n+1}-\lambda_n}\frac{\Phi(\varphi(\lambda_{n+1}))}{\lambda_{n+1}}\ln\frac{\lambda_{n+1}}{\lambda_n} = \Phi(\sigma)\frac{\lambda_n}{\lambda_{n+1}-\lambda_n}\ln\frac{\lambda_{n+1}}{\lambda_n}. \quad (2)$$

Now, let $\lambda_{n+1} = O(\lambda_n)$, $n\to\infty$, and $k_\Phi > 0$. Then there exist numbers $L\in(1,+\infty)$ and $K\in(0,+\infty)$ such that $\lambda_{n+1}\le L\lambda_n$ and $\ln|a_n|\ge -\lambda_n\Psi(\varphi(\lambda_n/K))$ for $n\ge n_0$. Since $\frac{\ln x}{x-1}\downarrow 0$ on $(1,+\infty)$, we have $\frac{\lambda_n}{\lambda_{n+1}-\lambda_n}\ln\frac{\lambda_{n+1}}{\lambda_n}\ge\frac{\ln L}{L-1}$ and (2) implies

$$\ln\mu(\sigma, F) \ge \frac{K\ln L}{L-1}\Phi(\sigma),\quad \sigma\in[\sigma_0, A), \quad (3)$$

that is $\tau_\Phi > 0$. If $k_\Phi = +\infty$ then $\ln|a_n|\ge -\lambda_n\Psi(\varphi(\lambda_n/K))$ for arbitrary $K\in(0,+\infty)$ and all $n\ge n_0(K)$. Therefore, (3) holds for an arbitrary $K\in(0,+\infty)$ and, thus, $\tau_\Phi = +\infty$.

3. Necessity.

Proof. At first let $A = +\infty$ and there exist a sequence (λ_{n_k}) such that $\lambda_{n_k+1}/\lambda_{n_k}\to\infty$, $k\to\infty$. Let $\Phi(\sigma) = e^\sigma$ and α be a slowly increasing to $+\infty$ continuously differentiable function such that $\alpha(\lambda_{n_k+1}) \le \sqrt{\lambda_{n_k+1}/\lambda_{n_k}}$ for all $k \ge 1$, $\frac{x\alpha'(x)}{\alpha(x)}\ln x = O(1)$, $x \to +\infty$, and the function $\Phi_1(\sigma) = \alpha(e^\sigma)e^\sigma$ belongs to $\Omega(+\infty)$. We choose a_n such that $\ln|a_n| = -\lambda_n\Psi(\varphi(\lambda_n))$, where φ_1 and Ψ_1 correspond to Φ_1.

Then $\Phi \in \Omega_1(+\infty)$, $\Phi_1'(\sigma) = e^\sigma(\alpha'(e^\sigma)e^\sigma + \alpha(e^\sigma))$ and in order to find an asymptotic of φ_1 we need to solve equation $e^\sigma\alpha(e^\sigma)(1+o(1)) = x$ $(x \to +\infty)$, i. e. $\sigma + \ln\alpha(e^\sigma) + o(1) = \ln x$ $(x \to +\infty)$. Since $\ln\alpha(e^\sigma) = o(\sigma)$, $\sigma \to +\infty$, we find a solution of the previous asymptotical equation in the form $\sigma = \ln x - \beta$, $\quad \beta = \beta(x) = o(\ln x)$ $(x \to +\infty)$. Hence, we obtain for some $\xi \in (\ln x - \beta, \ln x)$

$$\beta = \ln\alpha(e^{\ln x - \beta}) + o(1) = \ln\alpha(e^{\ln x}) - \frac{e^\xi\alpha'(e^\xi)}{\alpha(e^\xi)}o(\xi) + o(1) = \ln\alpha(x) + o(1)\ (x \to +\infty),$$

i. e.,

$$\varphi_1(x) = \ln x - \ln\alpha(x) + o(1), \quad x \to +\infty. \tag{4}$$

Since $\Psi(\sigma) = \sigma - 1$ and $\Psi_1(\sigma) = \sigma - 1 + o(1)$, $\sigma \to +\infty$, using (4) we have

$$k_\Phi = \varliminf_{n\to\infty} \lambda_n \exp\left\{-\frac{1}{\lambda_n}\ln\frac{1}{|a_n|} - 1\right\} = \varliminf_{n\to\infty} \lambda_n \exp\{-\Psi_1(\varphi_1(\lambda_n)) - 1\} =$$
$$\varliminf_{n\to\infty} \lambda_n \exp\{-\ln\lambda_n + \ln\alpha(\lambda_n) + o(1)\} = \varliminf_{n\to\infty} \alpha(\lambda_n) = +\infty,$$

that is $F \in S(\Lambda, +\infty, k_\Phi = +\infty) \subset S(\Lambda, +\infty, k_\Phi > 0)$.

On the other hand, if we put $\varkappa = \frac{1}{\lambda_{n+1}-\lambda_n}\int\limits_{\lambda_n}^{\lambda_{n+1}} \varphi_1(t)dt$ then ([2-3])

$$\ln\mu(\varkappa_n, F) = G_1(\lambda_n, \lambda_{n+1}, \Phi_1) = \frac{\lambda_{n+1}\lambda_n}{\lambda_{n+1}-\lambda_n}\Psi_1(\varphi_1(x))\Big|_{\lambda_n}^{\lambda_{n+1}}$$

and

$$\ln\mu(\varkappa_{n_k}, F) = \frac{\lambda_{n_k+1}\lambda_{n_k}}{\lambda_{n_k+1}-\lambda_{n_k}}(\ln\lambda_{n_k+1} - \ln\lambda_{n_k} - \ln\alpha(\lambda_{n_k+1}) + \ln\alpha(\lambda_{n_k}) + o(1)) \le$$

$$\le \frac{\lambda_{n_k+1}\lambda_{n_k}}{\lambda_{n_k+1}-\lambda_{n_k}}\ln\frac{\lambda_{n_k+1}}{\lambda_{n_k}}e^{o(1)} = (1+o(1))\lambda_{n_k}\ln\frac{\lambda_{n_k+1}}{\lambda_{n_k}}, \quad k \to \infty. \tag{5}$$

In view of (4)

$$\varkappa_{n_k} \ge \frac{1}{\lambda_{n_k+1}-\lambda_{n_k}}\int\limits_{\lambda_{n_k}}^{\lambda_{n_k+1}} \ln t\,dt - \ln\alpha(\lambda_{n_k+1}) + o(1) = \ln\lambda_{n_k+1} - \ln\alpha(\lambda_{n_k}) + o(1)\ (k \to \infty),$$

and, therefore,

$$\exp\{\varkappa_{n_k}\} \ge \frac{\lambda_{n_k+1}}{\alpha(\lambda_{n_k+1})}\exp\{1+o(1)\}, \quad k \to \infty. \tag{6}$$

From (5) and (6) it follows that $\tau_\Phi \le \varliminf_{k\to\infty}\frac{\ln\mu(\varkappa_{n_k},F)}{\exp\{\varkappa_{n_k}\}} \le \varliminf_{k\to\infty}\alpha(\lambda_{n_k+1})\frac{\lambda_{n_k}}{e\lambda_{n_k+1}}\ln\frac{\lambda_{n_k+1}}{\lambda_{n_k}} = 0$. The necessity of the condition $\lambda_{n+1} = O(\lambda_n)$, $n \to \infty$ is proved in the case $A = +\infty$.

Now, we prove the necessity of the condition $\lambda_{n+1} = O(\lambda_n)$, $n \to \infty$, in the case $A < +\infty$. Without loss of generality, we may assume that $A = 0$. If there exists a sequence (λ_{n_k}) such that $\lambda_{n_k+1}/\lambda_{n_k} \to \infty$, $k \to \infty$, let $\Phi(\sigma) = 1/|\sigma|$ and α be a slowly increasing to $+\infty$ continuosly differentiable function such that $\alpha(\lambda_{n_k+1}) \le \sqrt[4]{\lambda_{n_k+1}/\lambda_{n_k}}$, $\frac{x\alpha'(x)}{\alpha(x)}\ln x = O(1)$, $x \to +\infty$, and the function $\Phi_1(\sigma) = \alpha\left(\frac{1}{|\sigma|}\right)\frac{1}{|\sigma|}$ belongs to $\Omega(0)$. We choose a_n such that $\ln|a_n| = -\lambda_n\Psi_1(\varphi_1(\lambda_n))$. Then $\Psi(\sigma) = 2\sigma$, $\Phi_1'(\sigma) = \frac{1+o(1)}{|\sigma|^2}\alpha\left(\frac{1}{|\sigma|}\right)$ and $\Psi_1(\sigma) = 2(1+o(1))\sigma$ as $\sigma \uparrow 0$, and, as above, using the condition $\frac{x\alpha'(x)}{\alpha(x)}\ln x = O(1)$, $x \to +\infty$, we obtain $\varphi_1(x) = -(1+o(1))\sqrt{\alpha(\sqrt{x})/x}$ $(x \to \infty)$. Hence, $k_\Phi =$

$\underline{\lim}_{n\to\infty}\lambda_n\left|\frac{1}{2\lambda_n}\ln\frac{1}{|a_n|}\right|^2 = \underline{\lim}_{n\to\infty}\frac{1}{4}\lambda_n|\Psi_1(\varphi_1(\lambda_n))|^2 = \underline{\lim}_{n\to\infty}\alpha(\sqrt{\lambda_n}) = +\infty$, that is $F\in S(\Lambda,0,k_{\Phi=+\infty})\subset S(\Lambda,0,k_\Phi>0)$. Similarly, as above, we obtain also

$$|\varkappa_{n_k}| = \frac{1+o(1)}{\lambda_{n_k+1}-\lambda_{n_k}}\int\limits_{\lambda_{n_k}}^{\lambda_{n_k+1}}\frac{\sqrt{\alpha(\sqrt{x})}}{\sqrt{x}}dx \le$$

$$\le 2(1+o(1))\sqrt{\alpha(\sqrt{\lambda_{n_k+1}})}\frac{\sqrt{\lambda_{n_k+1}}-\sqrt{\lambda_{n_k}}}{\lambda_{n_k+1}-\lambda_{n_k}} = \frac{2(1+o(1))\sqrt{\alpha(\sqrt{\lambda_{n_k+1}})}}{\sqrt{\lambda_{n_k+1}}}\ (k\to\infty),$$

$$\ln\mu(\varkappa_{n_k},F) = -\frac{\lambda_{n_k+1}\lambda_{n_k}}{\lambda_{n_k+1}-\lambda_{n_k}}2(1+o(1))\frac{\sqrt{\alpha(\sqrt{x})}}{\sqrt{x}}\Bigg|_{\lambda_{n_k}}^{\lambda_{n_k+1}} =$$

$$= 2(1+o(1))\left(\frac{\sqrt{\alpha(\sqrt{\lambda_{n_k}})}}{\sqrt{\lambda_{n_k}}} - \frac{\sqrt{\alpha(\sqrt{\lambda_{n_k+1}})}}{\sqrt{\lambda_{n_k+1}}}\right) \le 2(1+o(1))\sqrt{\lambda_{n_k}\alpha(\sqrt{\lambda_{n_k}})}\ (k\to\infty),$$

and $\tau_\Phi \le \underline{\lim}_{k\to\infty}|\varkappa_{n_k}|\ln\mu(\varkappa,F) \le 4\,\underline{\lim}_{k\to\infty}\frac{\sqrt{\alpha(\sqrt{\lambda_{n_k+1}})}}{\sqrt{\lambda_{n_k+1}}}\sqrt{\lambda_{n_k}\alpha(\sqrt{\lambda_{n_k}})} \le$

$$\le 4\,\underline{\lim}_{k\to\infty}\sqrt{\frac{\lambda_{n_k}}{\lambda_{n_k+1}}}\alpha(\sqrt{\lambda_{n_k+1}}) \le 4\,\underline{\lim}_{k\to\infty}\sqrt{\frac{\lambda_{n_k}}{\lambda_{n_k+1}}}\sqrt[4]{\frac{\lambda_{n_k+1}}{\lambda_{n_k}}} = 0.$$

□

4. Remark. The class $\Omega_1(0)$ contains all function $\Phi(\sigma)=B\left(\frac{1}{|\sigma|}\right)$ such that $\frac{B(x)}{x^2B'(x)}\searrow\omega\ge0$ as $x\to+\infty$ (in particular, the function $\ln(1/|\sigma|)$, $|\sigma|^{-p}$, $\exp\{1/|\sigma|\}$ belong to $\Omega_1(0)$). The class $\Omega_1(+\infty)$ contains, for example, the functions $Te^{\varrho\sigma}$, $\exp_k\{\sigma\}$. The function $\Phi(\sigma)=T\sigma^p$ $(\sigma\ge\sigma_0)$ with $p>1$ not belongs to $\Omega_1(+\infty)$. But we may generalize the theorem, replacing the condition $\Phi\in\Omega_1(A)$ by the condition $\Phi\in\Omega_\alpha(A)$, $\alpha>1$, where $\Omega_\alpha(A)$ is a the class of functions $\Phi\in\Omega(A)$ such that $\Phi/(\Phi')^\alpha$ is a nonincreasing function.

Indeed, if $\Phi\in\Omega_\alpha(A)$ then

$$G_1(\lambda_n,\lambda_{n+1},\Phi) = \frac{\lambda_{n+1}\lambda_n}{\lambda_{n+1}-\lambda_n}\int\limits_{\lambda_n}^{\lambda_{n+1}}\frac{\Phi(\varphi(t))}{t^\alpha}t^{\alpha-2}dt \ge \frac{\lambda_{n+1}\lambda_n}{\lambda_{n+1}-\lambda_n}\frac{\Phi(\varphi(\lambda_{n+1}))}{\lambda_{n+1}^\alpha}\frac{\lambda_{n+1}^{\alpha-1}-\lambda_n^{\alpha-1}}{\alpha-1}$$

and instead of (2) and (3) we have

$$\ln\mu(\sigma,F) \ge K\Phi(\sigma)\frac{\lambda_{n+1}^{1-\alpha}\lambda_n}{\lambda_{n+1}-\lambda_n}\frac{\lambda_{n+1}^{\alpha-1}-\lambda_n^{\alpha-1}}{\alpha-1} \ge K\Phi(\sigma)\frac{1-L^{1-\alpha}}{(L-1)(\alpha-1)},$$

that is the sufficiency of the condition $\lambda_{n+1}=O(\lambda_n)$, $n\to\infty$, is proved. The necessity it also proved, because $\Omega_1(A)\subset\Omega_\alpha(A)$.

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