УДК 517.95

M. M. Bokalo, O. V. Domanska

ON WELL-POSEDNESS OF BOUNDARY PROBLEMS FOR ELLIPTIC EQUATIONS IN GENERAL ANISOTROPIC LEBESGUE-SOBOLEV SPACES

M. M. Bokalo, O. V. Domanska. *On well-posedness of boundary problems for elliptic equations in general anisotropic Lebesgue-Sobolev spaces*, Matematychni Studii, **28** (2007) 77–91.

We establish the well-posedness of the boundary-value problems for some class of nonlinear elliptic equations having exponential nonlinearities in unbounded domains. We consider mixed boundary conditions and varying exponents of nonlinearity which are different with respect to various derivatives and we seek for the weak solutions in the corresponding general anisotropic Lebesgue-Sobolev spaces.

Н. М. Бокало, А. В. Доманская. *О корректности краевых задач для эллиптических уравнений в обобщенных анизотропных пространствах Лебега-Соболева* // Математичні Студії. – 2007. – Т.28, №1. – С.77–91.

Исследуется корректность краевых задач для некоторого класса нелинейных эллиптических уравнений со степенными нелинейностями в неограниченных областях. Рассматриваются смешанные граничные условия, когда показатели нелинейности — переменные и различные относительно различных производных, и ищутся слабые решения из соответсвующих обобщенных анизотропных пространств Лебега-Соболева.

Introduction. The study of boundary problems in anisotropic spaces is motivated by their applications to physics and mechanics. In particular, such problems describe flows of electrorheological fluids, processes of image restoration, filtration through inhomogeneous media, motion of nonideal electrons in crystalline solid ([1]–[2]). They also appear in anisotropic generalization of the special relativity theory that use nonlinear differential operators action on functions from anisotropic spaces.

Boundary problems for static equations were studied in many mathematical publications (see, for instance, [3]–[10] and the literature cited therein). In particular, H. Brezis presented in [5] the first example

$$-\Delta u + |u|^{p-2}u = f(x), \quad x \in \mathbb{R}^n, \quad p > 2,$$

of an elliptic equation given in unbounded domain such that the corresponding boundary problems have unique solution without restrictions on its behavior and increasing of initial data at infinity. Later on, the class of such equations and systems was extended ([7]–[10]).

In this paper we supplement this class with equations having exponential nonlinearities that vary at x and are different with respect to various derivatives. A typical example of such an equations is

$$-\sum_{i=1}^{n}\Big(|u_{x_i}|^{p_i(x)-2}u_{x_i}\Big)_{x_i} + |u|^{p_0(x)-2}u = f(x), \quad x \in \Omega, \tag{1}$$

2000 *Mathematics Subject Classification:* 35J55.

where Ω is an unbounded domain, $p_0, \ldots, p_n$ are measurable functions satisfying the conditions: $1 < p_i \leq 2$, $i \in \{1, \ldots, n\}$, $p_0 \geq 2$ for a.e. $x \in \Omega$. We consider mixed boundary conditions: the Dirichlet boundary condition on one part the of the boundary of the domain Ω and the Neumann boundary condition on the other part. Equations of type (1) but with constant exponents of nonlinearities were considered in [9]–[10]. In addition to one-valued solvability of the problem in the class of functions with arbitrary behavior at infinity we consider the question of continuous dependence of the solution on initial data.

1. Preliminaries. Let n be a natural number. We denote by $\mathbb{R}^n$ the arithmetical space of n ordered arrays of real numbers, i.e. a linear space consisting of elements $x = (x_1, \ldots, x_n)$, where $x_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$, with the norm $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$.

All functions considered here are given in the corresponding subsets of the spaces $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ and take the values in $\mathbb{R}$. If $v(z)$, $z \in \widetilde{D}$, is a given function, then $v|_D$ denotes its restriction to a set $D \subset \widetilde{D}$.

Let Ω be an unbounded domain in $\mathbb{R}^n$ with piece wise regular boundary $\Gamma \stackrel{\text{def}}{=} \partial\Omega$; $\Gamma = \overline{\Gamma}_1 \bigcup \overline{\Gamma}_2$, where Γ_1, Γ_2 are open sets on $\partial\Omega$ (one of them can be empty), $\Gamma_1 \bigcap \Gamma_2 = \varnothing$; ν is a unit exterior normal vector to $\partial\Omega$. Without loss of generality we will suppose that $0 \in \Omega$. For all $R > 0$ we denote by Ω_R the connected component of the set $\Omega \cap \{x : |x| < R\}$ such that $0 \in \Omega_R$. Let $S_R = \partial\Omega_R \cap \Omega$, $\Gamma_{k,R} \stackrel{\text{def}}{=} \Gamma_k \bigcap \partial\Omega_R$, $k \in \{1, 2\}$, $R > 0$.

Let $r \in L_{\infty,\text{loc}}(\Omega)$ and $r(x) \geq 1$ for a.e. $x \in \Omega$. On the space $C(\overline{\Omega}_R)$ of continuous functions on $\overline{\Omega}_R$, where $R > 0$ is an arbitrary number, we introduce the norm
$\|v\|_{L_{r(\cdot)}(\Omega_R)} \stackrel{\text{def}}{=} \inf\{\lambda > 0 : \rho_{r,R}(v/\lambda) \leq 1\}$, where $\rho_{r,R}(v) \stackrel{\text{def}}{=} \int\limits_{\Omega_R} |v(x)|^{r(x)}\, dx$.
Let $L_{r(\cdot)}(\Omega_R)$ denote the completion of the linear space $C(\overline{\Omega}_R)$ by this norm (see [12]). The set $L_{r(\cdot)}(\Omega_R)$ is a linear subspace of the space $L_1(\Omega_R)$ and is called the *general Lebesgue space.*

Let $L_{r(\cdot),\text{loc}}(\overline{\Omega})$ denote the closure of the space $C(\overline{\Omega})$ in the topology generated by the system of semi-norms $\|\cdot\|_{L_{r(\cdot)}(\Omega_R)}$, $R > 0$. Put

$$L_{r(\cdot)}(\Omega) = \big\{v \in L_{r(\cdot),\text{loc}}(\overline{\Omega}) : \sup_{R>0} \|v|_{\Omega_R}\|_{L_{r(\cdot)}(\Omega_R)} < \infty\big\}.$$

Let $p_i \in L_{\infty,\text{loc}}(\Omega)$, $i \in \{0, \ldots, n\}$, with $p_i(x) \geq 1$, $i \in \{0, \ldots, n\}$, for a.e. $x \in \Omega$. Denote $p \stackrel{\text{def}}{=} (p_0, p_1, \ldots, p_n)$. For all $R > 0$ we define $W^1_{p(\cdot)}(\Omega_R)$ to be the Banach space obtained as the completion of the space $C^1(\overline{\Omega}_R)$ by the norm

$$\|v\|_{W^1_{p(\cdot)}(\Omega_R)} \stackrel{\text{def}}{=} \|v\|_{L_{p_0(\cdot)}(\Omega_R)} + \sum_{i=1}^{n} \|v_{x_i}\|_{L_{p_i(\cdot)}(\Omega_R)}.$$

It is obvious that $W^1_{p(\cdot)}(\Omega_R)$ is a subspace of the space $\big\{v(x),\ x \in \Omega_R :\ v \in L_{p_0(\cdot)}(\Omega_R),\ v_{x_i} \in L_{p_i(\cdot)}(\Omega_R),\ i \in \{1, \ldots, n\}\big\}$.

On the space $C^1(\overline{\Omega})$, consider a locally convex linear topology generated by the system of semi-norms: $\|\cdot\|_{W^1_{p(\cdot)}(\Omega_R)}$, $R > 0$, and let $W^1_{p(\cdot),\text{loc}}(\overline{\Omega})$ be the completion of $C^1(\overline{\Omega})$ in this topology. It is evident that a sequence $\{v_k\}_{k=1}^{\infty}$ is convergent to v in this space if $\|v_k - v\|_{W^1_{p(\cdot)}(\Omega_R)} \underset{k\to\infty}{\longrightarrow} 0$ for all $R > 0$. Note that $v\big|_{\Omega_R} \in W^1_{p(\cdot)}(\Omega_R)$ for all $R > 0$ provided $v \in W^1_{p(\cdot),\text{loc}}(\overline{\Omega})$.

Let $C^1_{\text{c}}(\overline{\Omega})$ be the subspace of the space $C^1(\overline{\Omega})$ consisting of the functions with supports in $\overline{\Omega}$, and $C^{1,+}_{\text{c}}(\overline{\Omega})$ be the subspace of the space $C^1_{\text{c}}(\overline{\Omega})$ consisting of the nonnegative functions.

2. The statement of the problem and main results. We denote by $\mathbb{P}$ the set of vector-functions $p=(p_0,p_1,\dots,p_n)$ such that $p_i\in L_{\infty,\mathrm{loc}}(\overline{\Omega})$ and $p_i(x)>1$, $i\in\{0,\dots,n\}$, for a.e. $x\in\Omega$. For a function $p\in\mathbb{P}$, by $p^*=(p_0^*,p_1^*,\dots,p_n^*)$ we denote the vector-function such that $\frac{1}{p_i(x)}+\frac{1}{p_i^*(x)}=1$, $i\in\{0,\dots,n\}$, for a.e. $x\in\Omega$ (it is obvious that $p^*\in\mathbb{P}$).

For $p\in\mathbb{P}$, we define $\mathbb{A}_p$ to be the set of ordered arrays $a=(a_0,a_1,\dots,a_n)$ of $n+1$ real-valued functions defined on $\Omega\times\mathbb{R}$ and satisfying the following conditions:
1) for all $i\in\{0,\dots,n\}$ $a_i(x,\xi)$, $(x,\xi)\in\Omega\times\mathbb{R}$ is a Caratheodory function, i.e. for a.e. $x\in\Omega$ the function $a_i(x,\cdot):\mathbb{R}\to\mathbb{R}$ is continuous and for all $\xi\in\mathbb{R}$ the function $a_i(\cdot,\xi):\Omega\to\mathbb{R}$ is Lebesgue measurable.
1′) $a_i(x,0)=0$, $i\in\{0,\dots,n\}$, for a.e. $x\in\overline{\Omega}$;
2) for a.e. $x\in\Omega$ there exist the derivatives $\frac{\partial a_i(x,\xi)}{\partial\xi}$, $\xi\neq 0$, $i\in\{1,\dots,n\}$, and the following inequalities hold:

$$K_i|\xi|^{p_i(x)-2}\le\frac{\partial a_i(x,\xi)}{\partial\xi}\le\widetilde{K}_i(1+|x|)^{\sigma_i}|\xi|^{p_i(x)-2},\qquad \xi\neq 0,\qquad i\in\{1,\dots,n\},$$

where $K_i>0$, $\widetilde{K}_i>0$, $\sigma_i\ge 0$, $i\in\{1,\dots,n\}$, are some constants;
3) for a.e. $x\in\Omega$ there exists $\frac{\partial a_0(x,\xi)}{\partial\xi}$, $\xi\neq 0$, and the following inequalities are satisfied

$$\frac{\partial a_0(x,\xi)}{\partial\xi}\ge K_0|\xi|^{p_0(x)-2},\quad \xi\neq 0,\quad\text{and}\quad |a_0(x,\xi)|\le\widetilde{K}_0|\xi|^{p_0(x)-1}+h(x),\quad \xi\in\mathbb{R},$$

where K_0, $\widetilde{K}_0$ are some positive constants and h is a function from $L_{p_0^*(\cdot),\mathrm{loc}}(\overline{\Omega})$.

Remark 1. The set $\mathbb{A}_p$ contains the array of functions

$$(a_0(x)\,|\xi|^{p_0(x)-2}\,\xi,\ a_1(x)\,|\xi|^{p_1(x)-2}\,\xi,\ \dots,\ a_n(x)\,|\xi|^{p_n(x)-2}\,\xi),$$

where a_i, $i\in\{0,\dots,n\}$, satisfy the condition: $|a_i(x)|\le K_i^*(1+|x|)^{\sigma_i}$, where $K_i^*>0$ is a constant.

For all $p\in\mathbb{P}$, put $\mathbb{F}_p\stackrel{\text{def}}{=}L_{p_0^*(\cdot),\mathrm{loc}}(\overline{\Omega})\times L_{p_1^*(\cdot),\mathrm{loc}}(\overline{\Omega})\times\dots\times L_{p_n^*(\cdot),\mathrm{loc}}(\overline{\Omega})$. On $\mathbb{F}_p$ we introduce the Cartesian product locally convex topology.

On the space $W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$ define an equivalence relation such that two elements v_1 and v_2 are equivalent if $v_1=v_2$ on Γ_1 in the sense of traces. We denote by $\mathbb{V}_p$ the quotient-space of $W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$ by this equivalence relation. The space $\mathbb{V}_p$ is a locally convex space with respect to the set of semi-norms: $pn_R(\Phi)=\inf\limits_{\varphi\in\Phi}\|\varphi\|_{W^1_{p(\cdot)}(\Omega_R)}$, $\Phi\in\mathbb{V}_p$, $R>0$. It is easy to see that a sequence $\{\Phi_k\}_{k=1}^{\infty}$ is convergent to Φ in $\mathbb{V}_p$ if and only if there exists φ and a sequence $\{\varphi_k\}_{k=1}^{\infty}$ such that $\varphi\in\Phi$, $\varphi_k\in\Phi_k$, $k\in\mathbb{N}$, and $\varphi_k\underset{k\to\infty}{\longrightarrow}\varphi$ in $W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$.

For $p\in\mathbb{P}$, we denote by $\mathbb{U}_p$ the linear space $W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$ with the following convergence: a sequence of elements $\{v_k\}_{k=1}^{\infty}$ is convergent to v in $\mathbb{U}_p$ if

$$\int\limits_{\Omega}\Big\{\sum_{i=1}^{n}\big(|v_{k,x_i}|^{p_i(x)-2}v_{k,x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\big)\big(v_{k,x_i}-v_{x_i}\big)+|v_k-v|^{p_0(x)}\Big\}\,dx\underset{k\to\infty}{\longrightarrow}0.$$

Remark 2. The choice of such a convergence on $\mathbb{U}_p$ was motivated by the fact that

$$(|r_k|^{p-2}r_k-|r|^{p-2}r)(r_k-r)\underset{k\to\infty}{\longrightarrow}0\qquad\text{if and only if}\qquad r_k\underset{k\to\infty}{\longrightarrow}r$$

for all $r\in\mathbb{R}$, r_k, $k\in\mathbb{N}$, and arbitrary $p>1$.

Now we formulate the investigated *problem*. Let $\widetilde{\mathbb{P}}\subset\mathbb{P}$ and $\widetilde{\mathbb{A}}_p\subset\mathbb{A}_p$, $\widetilde{\mathbb{F}}_p\subset\mathbb{F}_p$, $\widetilde{\mathbb{V}}_p\subset\mathbb{V}_p$, $\widetilde{\mathbb{U}}_p\subset\mathbb{U}_p$ for $p\in\widetilde{\mathbb{P}}$. The main problem $\mathbf{PA}\big(\widetilde{\mathbb{A}}_p,\widetilde{\mathbb{F}}_p,\widetilde{\mathbb{V}}_p,\widetilde{\mathbb{U}}_p:p\in\widetilde{\mathbb{P}}\big)$ (**P**roblem in **A**nisotropic spaces) is to find for every $p\in\widetilde{\mathbb{P}}$ and $a\in\widetilde{\mathbb{A}}_p$, $f\in\widetilde{\mathbb{F}}_p$, $\Phi\in\widetilde{\mathbb{V}}_p$ the set $\mathbf{SPA}\big(a,f,\Phi\big)$ (**S**olutions of **P**roblem in **A**nisotropic spaces) of functions $u\in\widetilde{\mathbb{U}}_p$ such that $u\in\Phi$ and the equality

$$\int\limits_{\Omega}\Big\{\textstyle\sum_{i=1}^{n}a_i(x,u_{x_i})v_{x_i}+a_0(x,u)v\Big\}\,dx=\int\limits_{\Omega}\Big\{f_0v+\textstyle\sum_{i=1}^{n}f_iv_{x_i}\Big\}\,dx \tag{2}$$

holds for all $v\in W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$, $\ v|_{\Gamma_1}=0$, $\ \operatorname{supp}v$ is compact.

Remark 3. It is seen from the statement of the investigated problem that the restricting condition **1**$'$) is not essential. Otherwise, we can introduce new functions

$$\widetilde{a}_i(x,\xi) \overset{\text{def}}{=} a_i(x,\xi) - a_i(x,0), \qquad \widetilde{f}_i(x) \overset{\text{def}}{=} f_i(x) - a_i(x,0), \quad i \in \{0,\dots,n\},$$

for a.e. $x \in \Omega$ and rewrite identity (2) with $\widetilde{a}_i(x,\xi)$, $\widetilde{f}_i(x)$, $i \in \{0,\dots,n\}$, instead of $a_i(x,\xi), f_i(x)$, $i \in \{0,\dots,n\}$, respectively, where functions $a_i(x,\xi)$, $i \in \{0,\dots,n\}$, satisfy condition **1**$'$).

Let us say that $\mathbf{PA}\big(\widetilde{\mathbb{A}}_p,\, \widetilde{\mathbb{F}}_p,\, \widetilde{\mathbb{V}}_p,\, \widetilde{\mathbb{U}}_p : p \in \widetilde{\mathbb{P}}\big)$ is *solvable* (*unique*, *uniquely solvable*) problem, if for every $p \in \widetilde{\mathbb{P}}$ and arbitrary $a \in \widetilde{\mathbb{A}}_p$, $f \in \widetilde{\mathbb{F}}_p$ and $\Phi \in \widetilde{\mathbb{V}}_p$ the set $\mathbf{SPA}\big(a,f,\Phi\big) \subset \widetilde{\mathbb{U}}_p$ is *non-empty* (*contains at most one element*, *has exactly one element*).

Let us say that $\mathbf{PA}\big(\widetilde{\mathbb{A}}_p,\widetilde{\mathbb{F}}_p,\widetilde{\mathbb{V}}_p,\widetilde{\mathbb{U}}_p : p \in \widetilde{\mathbb{P}}\big)$ *is a weakly well-posed* problem, if it is uniquely solvable and for all $p \in \widetilde{\mathbb{P}}$ and arbitrary elements $a \in \widetilde{\mathbb{A}}_p$, $f \in \widetilde{\mathbb{F}}_p$, $\Phi \in \widetilde{\mathbb{V}}_p$ and a sequence $\{f^k\}_{k=1}^{\infty} \subset \widetilde{\mathbb{F}}_p$ such that $f^k \underset{k\to\infty}{\longrightarrow} f$ in $\widetilde{\mathbb{F}}_p$, we have $u_k \underset{k\to\infty}{\longrightarrow} u$ in $\mathbb{U}_p$, where $u_k \in \mathbf{SPA}\big(a,f^k,\Phi\big)$, $k \in \mathbb{N}$, $u \in \mathbf{SPA}\big(a,f,\Phi\big)$.

It is obvious that problem $\mathbf{PA}\big(\widetilde{\mathbb{A}}_p,\, \widetilde{\mathbb{F}}_p,\, \widetilde{\mathbb{V}}_p,\, \widetilde{\mathbb{U}}_p : p \in \widetilde{\mathbb{P}}\big)$ can be formally interpreted as the boundary value problem for the equation

$$-\sum_{i=1}^{n} \frac{d}{dx_i} a_i(x,u_{x_i}) + a_0(x,u) = f_0(x) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x), \quad x \in \Omega,$$

with boundary conditions

$$u(x) = \varphi(x), \qquad x \in \Gamma_1,$$

$$\tfrac{\partial u}{\partial \nu_a} \equiv \sum_{i=1}^{n} a_i(x,u_{x_i}) \cos(\nu, x_i) = 0, \quad x \in \Gamma_2,$$

where $a \in \widetilde{\mathbb{A}}_p$, $f \in \widetilde{\mathbb{F}}_p$ for $p \in \widetilde{\mathbb{P}}$, φ is an arbitrary element from $\Phi \in \widetilde{\mathbb{V}}_p$.

We seek for the set $\widetilde{\mathbb{P}}$ and spaces $\widetilde{\mathbb{A}}_p$, $\widetilde{\mathbb{F}}_p$, $\widetilde{\mathbb{V}}_p$, $p \in \widetilde{\mathbb{P}}$ such that problem $\mathbf{PA}\big(\widetilde{\mathbb{A}}_p,\, \widetilde{\mathbb{F}}_p,\, \widetilde{\mathbb{V}}_p,\, \mathbb{U}_p : p \in \widetilde{\mathbb{P}}\big)$ is unique, uniquely solvable or weakly well-posed. Note that we do not want to impose any restrictions on increasing of the elements of the sets $\widetilde{\mathbb{F}}_p$, $\widetilde{\mathbb{V}}_p$ $(p \in \widetilde{\mathbb{P}})$ at infinity.

Here we make the following choice of the indicated sets.

Let $\mathbb{P}^*$ be a set of elements $p = (p_0, p_1, \dots, p_n) \in \mathbb{P}$ such that

$$p_0^- \overset{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} p_0(x) \ge 2, \qquad p_0^+ \overset{\text{def}}{=} \operatorname*{ess\,sup}_{x\in\Omega} p_0(x) < \infty,$$

$$p_i^- \overset{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} p_i(x) > 1, \quad p_i^+ \overset{\text{def}}{=} \operatorname*{ess\,sup}_{x\in\Omega} p_i(x) \le 2, \quad i \in \{1,\dots,n\},$$

the functions $q_i(x) \overset{\text{def}}{=} \frac{p_0(x)p_i(x)}{p_0(x)-p_i(x)}$, $\quad x \in \Omega$, $\quad i \in \{1,\dots,n\}$, belong to the space $L_\infty(\Omega)$,

$$n - q_i^- < 0, \quad \text{where} \quad q_i^- \overset{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} q_i(x), \quad i \in \{1,\dots,n\}.$$

For all $p \in \mathbb{P}^*$, define $\mathbb{A}_p^*$ as the set of functions arrays $a \in \mathbb{A}_p$ that satisfy the additional condition:

4) Constants $\sigma_1, \dots, \sigma_n$ in condition **2**) are such that

$$n - q_i^- + \sigma_i \frac{q_i^+}{p_i^-} < 0, \quad \text{where} \quad q_i^+ \overset{\text{def}}{=} \operatorname*{ess\,sup}_{x\in\Omega} q_i(x), \qquad i \in \{1,\dots,n\}.$$

Denote by $\mathbb{F}_p^*$ the subset of $\mathbb{F}_p$ with elements $(f_0, 0, \dots, 0)$, i.e. if $f \in \mathbb{F}_p^*$, then $f_i = 0$, $i \in \{1,\dots,n\}$, $f_0 \in L_{p_0^*(\cdot),\mathrm{loc}}(\overline{\Omega})$.

Theorem 1. *The following propositions are valid.*

1) The problem $\mathbf{PA}\big(\mathbb{A}_p^*,\, \mathbb{F}_p,\, \mathbb{V}_p,\, \mathbb{U}_p : p \in \mathbb{P}^*\big)$ *is uniquely solvable and for all* $p \in \mathbb{P}^*$ *and*

$a \in \mathbb{A}_p^*$, $f \in \mathbb{F}_p$, $\Phi \in \mathbb{V}_p$ the (unique) function $u \in \mathbf{SPA}\big(a, f, \Phi\big)$ for every $R_0 > 0$, $R \geq 1$, $R_0 < R$, satisfies the inequality

$$\int\limits_{\Omega_{R_0}} \Big[\sum_{i=1}^n |u_{x_i}(x)|^{p_i(x)} + |u(x)|^{p_0(x)} \Big] dx \leq$$

$$\leq \left(\tfrac{R}{R-R_0}\right)^s \Big\{ C_1 R^{n-\gamma} + C_2 \int\limits_{\Omega_R} \Big[|f_0(x)|^{p_0^*(x)} + \sum_{i=1}^n |f_i(x)|^{p_i^*(x)} \Big] dx +$$

$$+ C_3 \int\limits_{\Omega_R} \big[|\varphi(x)|^{p_0(x)} + \sum_{i=1}^n R^{\sigma_i} |\varphi_{x_i}(x)|^{p_i(x)} \big] \, dx + C_4 \int\limits_{\Omega_R} |h(x)|^{p_0^*(x)} \, dx \Big\}, \qquad (3)$$

where $\gamma = \min\limits_{1\leq i\leq n} (q_i^- - \sigma_i q_i^+/p_i^-)$, $s > \max\limits_{1\leq i\leq n} q_i^+$ is an arbitrary number, C_1, C_2, C_3, C_4 are some positive constants depending only on $n, s, p_i^-, p_i^+ (i \in \{0, \ldots, n\})$, q_i^-, q_i^+ $(i \in \{1, \ldots, n\})$.
2) The problem $\mathbf{PA}\big(\mathbb{A}_p^*, \mathbb{F}_p^*, \mathbb{V}_p, \mathbb{U}_p : p \in \mathbb{P}^*\big)$ is weakly well-posed and its solution satisfies estimate (3) with $f_i = 0$, $i \in \{1, \ldots, n\}$.

3. Auxiliary statements. It is easy to establish that the following proposition is valid (see [13], p. 312).

Proposition 1. *Let $R > 0$ be an arbitrary number, $r \in L_\infty(\Omega_R)$, $r^- \stackrel{\mathrm{def}}{=} \operatorname*{ess\,inf}\limits_{x\in\Omega_R} r(x) > 1$, $r^+ \stackrel{\mathrm{def}}{=} \operatorname*{ess\,sup}\limits_{x\in\Omega_R} r(x) < +\infty$. Then for every function $v \in L_{r(\,\cdot\,)}(\Omega_R)$ the following inequalities hold*

$$\min\Big\{ (\rho_{r,R}(v))^{1/r^-}, (\rho_{r,R}(v))^{1/r^+} \Big\} \leq \|v\|_{L_{r(\,\cdot\,)}(\Omega_R)} \leq \max\Big\{ (\rho_{r,R}(v))^{1/r^-}, (\rho_{r,R}(v))^{1/r^+} \Big\},$$

$$\min\Big\{ \|v\|^{r^-}_{L_{r(\,\cdot\,)}(\Omega_R)}, \|v\|^{r^+}_{L_{r(\,\cdot\,)}(\Omega_R)} \Big\} \leq \rho_{r,R}(v) \leq \max\Big\{ \|v\|^{r^-}_{L_{r(\,\cdot\,)}(\Omega_R)}, \|v\|^{r^+}_{L_{r(\,\cdot\,)}(\Omega_R)} \Big\}.$$

Remark 4. For all $a \geq 0, b \geq 0, \varepsilon > 0, \nu > 1$ Young's inequality [4] ($a\,b \leq \frac{a^\nu}{\nu} + \frac{a^{\nu^*}}{\nu^*}$, $\nu^* = \frac{\nu}{\nu-1}$) implies the inequality: $a\,b \leq \varepsilon a^\nu + \frac{\varepsilon^{1-\nu^*}}{\nu^*} \left(\frac{1}{\nu}\right)^{\frac{\nu^*}{\nu}} b^{\nu^*}$. Hence taking into account that $\left(\frac{1}{\nu}\right)^{\frac{\nu^*}{\nu}} < 1$ for all $\nu > 1$, we obtain

$$a\,b \leq \varepsilon a^\nu + \tfrac{\varepsilon^{1-\nu^*}}{\nu^*} b^{\nu^*}. \qquad (4)$$

Remark 5. Young's inequality [4] ($a\,b\,c \leq \frac{a^{\nu_1}}{\nu_1} + \frac{a^{\nu_2}}{\nu_2} + \frac{a^{\nu_3}}{\nu_3}$, $a \geq 0, b \geq 0, c \geq 0, \varepsilon > 0, \nu_1 > 1, \nu_2 > 1, \nu_3 > 1$, $\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} = 1$) simply implies the inequality $a\,b\,c \leq \varepsilon a^{\nu_1} + \varepsilon b^{\nu_2} + \frac{\varepsilon^{1-\nu_3}}{\nu_3} \left(\frac{1}{\nu_1}\right)^{\frac{\nu_3}{\nu_1}} \left(\frac{1}{\nu_2}\right)^{\frac{\nu_3}{\nu_2}} c^{\nu_3}$. Thus, reasoning just as in the previous Remark, we derive

$$a\,b\,c \leq \varepsilon a^{\nu_1} + \varepsilon b^{\nu_2} + \tfrac{\varepsilon^{1-\nu_3}}{\nu_3} c^{\nu_3}. \qquad (5)$$

Lemma 1. *For every $t, s \in \mathbb{R}$ the following inequalities hold:*

$$(|s|^{q-2}s - |t|^{q-2}t)(s-t) \geq 2^{2-q}|s-t|^q, \quad \text{if} \quad q \geq 2, \qquad (6)$$

$$0 \leq (|s|^{q-2}s - |t|^{q-2}t)(s-t) \leq 2^{2-q}|s-t|^q, \quad \text{if} \quad 1 < q \leq 2. \qquad (7)$$

Proof. Inequality (6) is proved in [11], and inequality (7) can be easily proved by a similar argument. □

Lemma 2. *Let $a \in \mathbb{A}_p$, where $p \in \mathbb{P}^*$. For a.e. $x \in \Omega$ and arbitrary ξ_1, ξ_2 from $\mathbb{R}$ the following inequalities are valid:*

$$(a_0(x,\xi_1) - a_0(x,\xi_2))(\xi_1 - \xi_2) \geq K_0^- |\xi_1 - \xi_2|^{p_0(x)}, \tag{8}$$

$$(a_i(x,\xi_1) - a_i(x,\xi_2))(\xi_1 - \xi_2) \geq K_i^- \big(|\xi_1|^{p_i(x)-2}\xi_1 - |\xi_2|^{p_i(x)-2}\xi_2\big)\big(\xi_1 - \xi_2\big), \quad i \in \{1,\dots,n\}, \tag{9}$$

$$(a_i(x,\xi_1) - a_i(x,\xi_2))(\xi_1 - \xi_2) \leq K_i^+ (1+|x|)^{\sigma_i} |\xi_1 - \xi_2|^{p_i(x)}, \quad i \in \{1,\dots,n\}, \tag{10}$$

where K_i^- $(i \in \{0,\dots,n\})$, K_j^+ $(j \in \{1,\dots,n\})$ are some positive constants.

Proof. First of all let us prove inequality (8). Using first of the inequalities from condition **3**) and Lemma 1, we obtain for a.e. $x \in \Omega$ and arbitrary $\xi_1, \xi_2 \in \mathbb{R}$

$$(a_0(x,\xi_1) - a_0(x,\xi_2))(\xi_1 - \xi_2) = \left(\int_0^1 \frac{d\,a_0(x,\tau\xi_1 + (1-\tau)\xi_2)}{d\tau}\,d\tau\right)(\xi_1 - \xi_2) =$$
$$= \left(\int_0^1 \frac{\partial\,a_0(x,\tau\xi_1+(1-\tau)\xi_2)}{\partial\xi}(\xi_1 - \xi_2)\,d\tau\right)(\xi_1 - \xi_2) \geq$$
$$\geq K_0 \left(\int_0^1 |\tau\xi_1 + (1-\tau)\xi_2|^{p_0(x)-2}(\xi_1 - \xi_2)\,d\tau\right)(\xi_1 - \xi_2) = \Big[s = \tau\xi_1 + (1-\tau)\xi_2\Big] =$$
$$= K_0 \left(\int_{\xi_2}^{\xi_1} |s|^{p_0(x)-2}\,ds\right)(\xi_1 - \xi_2) = K_0 \frac{|s|^{p_0(x)-2}s}{p_0(x)-1}\Bigg|_{s=\xi_2}^{s=\xi_1} (\xi_1 - \xi_2) =$$
$$= \frac{K_0}{p_0(x)-1}(|\xi_1|^{p_0(x)-2}\xi_1 - |\xi_2|^{p_0(x)-2}\xi_2)(\xi_1 - \xi_2) \geq \frac{K_0 2^{2-p_0(x)}}{p_0(x)-1}|\xi_1 - \xi_2|^{p_0(x)} \geq$$
$$\geq \frac{K_0 2^{2-p_0^+}}{p_0^+ - 1}|\xi_1 - \xi_2|^{p_0(x)} = K_0^- |\xi_1 - \xi_2|^{p_0(x)},$$

where $K_0^- = (K_0 2^{2-p_0^+})/(p_0^+ - 1)$. Inequalities (9) and (10) can be proved in the same way. □

Lemma 3. *Let $p \in \mathbb{P}^*$ and $a \in \mathbb{A}_p^*$, $f_i \in L_{p_i^*(\cdot),\,loc}(\overline{\Omega})$, $i \in \{1,\dots,n\}$, and for every $l \in \{1,2\}$ functions $f_0^l \in L_{p_0^*(\cdot),\,loc}(\overline{\Omega})$, $u_l \in \mathbb{U}_p$ are such that $u_1 = u_2$ on Γ_{1,R_*} and*

$$\int_{\Omega_{R_*}} \Big\{ \sum_{i=1}^n a_i(x, u_{l,x_i}) v_{x_i}(x) + a_0(x,u_l)v(x) - f_0^l(x)v(x) - \sum_{i=1}^n f_i(x) v_{x_i}(x) \Big\}\,dx = 0 \tag{11}$$

for all $v \in W^1_{p(\cdot),\,loc}(\overline{\Omega})$, $v|_{\Gamma_{1,R_*}} = 0$, $\operatorname{supp} v$ is a compact in $\overline{\Omega}_{R_*}$, where $R_* > 1$ is some number.

Then for every $R_0 > 0, R \geq 1$, $R_0 < R \leq R_*$, the inequality holds:

$$\int_{\Omega_{R_0}} \Big[\sum_{i=1}^n (|u_{1,x_i}|^{p_i(x)-2} u_{1,x_i} - |u_{2,x_i}|^{p_i(x)-2} u_{2,x_i})(u_{1,x_i}(x) - u_{2,x_i}(x)) + |u_1(x) - u_2(x)|^{p_0(x)} \Big]\,dx \leq$$

$$\leq \left(\frac{R}{R-R_0}\right)^s \Big[C_5 R^{n-\gamma} + C_6 \int_{\Omega_R} |f_0^1(x) - f_0^2(x)|^{p_0^*(x)}\,dx \Big], \tag{12}$$

where s and γ are the same as in Theorem 1, and C_5, C_6 are positive constants which do not depend on u_l, f_0^l $(l \in \{1,2\})$.

Proof. Let us put $w \overset{\text{def}}{=} u_1 - u_2$. Using integral identities derived from (11) for $l = 1$ and $l = 2$ respectively, we get

$$\int\limits_{\Omega_{R_*}} \sum_{i=1}^{n} (a_i(x, u_{1,x_i}) - a_i(x, u_{2,x_i}))v_{x_i} + (a_0(x, u_1) - a_0(x, u_2))v - (f_0^1 - f_0^2)v\Big\}\, d\,x = 0 \quad (13)$$

for arbitrary $v \in W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$, $v|_{\Gamma_{1,R_*}} = 0$, $\operatorname{supp} v$ is a compact set in $\overline{\Omega}_{R_*}$.

Let R be any number in the interval $[1; R_*]$. Let us set in (13) $v = w\zeta^s$, where $\zeta(x) = \frac{1}{R}(R^2 - |x|^2)$ for $|x| < R$ and $\zeta(x) = 0$ for $|x| \geq R$, $s > 1$ is a sufficiently large number (value of s will be defined more precisely later). Hence we derive the equality

$$\int\limits_{\Omega_R} \sum_{i=1}^{n} \Big[(a_i(x, u_{1,x_i}) - a_i(x, u_{2,x_i}))\, w_{x_i}\, \zeta^s + (a_0(x, u_1) - a_0(x, u_2))\, w\, \zeta^s\Big]\, d\,x =$$

$$= \int\limits_{\Omega_R} (f_0^1 - f_0^2)\, w\, \zeta^s\, d\,x - s \int\limits_{\Omega_R} \sum_{i=1}^{n} (a_i(x, u_{1,x_i}) - a_i(x, u_{2,x_i}))\, w\, \zeta^{s-1} \zeta_{x_i}\, d\,x. \quad (14)$$

Let us estimate each term of (14). By virtue of inequality (8) we have

$$\int\limits_{\Omega_R} (a_0(x, u_1) - a_0(x, u_2))\, w(x)\, \zeta^s(x)\, d\,x \geq K_0^- \int\limits_{\Omega_R} |w(x)|^{p_0(x)}\, \zeta^s(x)\, d\,x. \quad (15)$$

Using inequality (4) we deduce

$$\int\limits_{\Omega_R} (f_0^1(x) - f_0^2(x))\, w(x)\, \zeta^s(x)\, d\,x \leq$$

$$\leq \eta_1 \int\limits_{\Omega_R} |w(x)|^{p_0(x)}\, \zeta^s(x)\, d\,x + C_7(\eta_1) \int\limits_{\Omega_R} |f_0^1(x) - f_0^2(x)|^{p_0^*(x)}\, \zeta^s(x)\, d\,x, \quad (16)$$

where η_1 is an arbitrary number out of $(0; 1)$, $C_7(\eta_1) = \frac{\eta_1^{1-p_0^+}}{p_0^-}$.

Remark that

$$\frac{1}{p_i^*(x)} + \frac{1}{p_0(x)} < 1, \quad i \in \{1, \ldots, n\}, \quad \text{for a.e.} \quad x \in \Omega. \quad (17)$$

Indeed, as $q_i^+ > 1$, $i \in \{1, \ldots, n\}$, then

$$\frac{1}{p_i^*(x)} + \frac{1}{p_0(x)} = \frac{p_0(x)p_i(x) - p_0(x) + p_i(x)}{p_0(x)p_i(x)} = 1 - \frac{1}{q_i(x)} \leq 1 - \frac{1}{q_i^+} < 1.$$

Let $x \in \Omega_R$ be any point such that $w(x)$, $p_i^- \leq p_i(x) \leq p_i^+$ ($i \in \{0, \ldots, n\}$), $a_j(x, u_{l,x_j}(x))$ ($l \in \{1, 2\}$, $j \in \{1, \ldots, n\}$) are defined and (17) is valid. Fix $i \in \{1, \ldots, n\}$. Putting in inequality (5) $\nu_1 = p_i^*(x)$, $\nu_2 = p_0(x)$, $\nu_3 = q_i(x)$,
$a = |a_i(x, u_{1,x_i}(x)) - a_i(x, u_{2,x_i}(x))|(K_i^+(1 + |x|)^{\sigma_i})^{-\frac{1}{p_i(x)}} \zeta^{s/\nu_1}(x)$, $b = |w(x)|\zeta^{s/\nu_2}(x)$,
$c = (K_i^+(1 + |x|)^{\sigma_i})^{\frac{1}{p_i(x)}}\ |\zeta_{x_i}(x)|\zeta^{s/\nu_3 - 1}(x)$, $\varepsilon = \eta_2 \in (0; 1)$, we obtain

$$|a_i(x, u_{1,x_i}(x)) - a_i(x, u_{2,x_i}(x))|\, |w(x)|\, |\zeta_{x_i}(x)|\, \zeta^{s-1}(x) \leq$$

$$\leq \eta_2 |a_i(x, u_{1,x_i}(x)) - a_i(x, u_{2,x_i}(x))|^{p_i^*(x)} (K_i^+(1 + |x|)^{\sigma_i})^{-\frac{p_i^*(x)}{p_i(x)}}\, \zeta^s(x) + \eta_2 |w(x)|^{p_0(x)} \zeta^s(x) +$$

$$+ C_8(\eta_2)(K_i^+(1 + |x|)^{\sigma_i})^{\frac{q_i(x)}{p_i(x)}} \zeta^{\,s - q_i(x)}(x), \quad C_8(\eta_2) = \max_{i \in \{1,\ldots,n\}} \frac{\eta_2^{1-q_i^+}}{q_i^-} 2^{q_i^+}. \quad (18)$$

In virtue of inequality (10) in Lemma 2, reasoning just as in [9, p. 163], we deduce

$$|a_i(x,u_{1,x_i})-a_i(x,u_{2,x_i})|^{p_i^*(x)}(K_i^+(1+|x|)^{\sigma_i})^{-\frac{p_i^*(x)}{p_i(x)}}\zeta^s(x)\le$$
$$\le\Big(a_i(x,u_{1,x_i})-a_i(x,u_{2,x_i})\Big)\Big(u_{1,x_i}(x)-u_{2,x_i}(x)\Big)\zeta^s(x). \qquad (19)$$

Taking account of $|\zeta(x)|\le R$, $x\in\mathbb{R}^n$ and $R\ge 1$ we obtain for $s>q_i(x)$

$$(K_i^1(1+|x|)^{\sigma_i})^{\frac{q_i(x)}{p_i(x)}}\zeta^{\,s-q_i(x)}(x)\le C_9\sum_{i=1}^n R^{s-q_i(x)+\sigma_i\frac{q_i(x)}{p_i(x)}}, \qquad (20)$$

where C_9 is some positive constant.

Inequalities ()–(20) hold for arbitrary $i\in\{1,\dots,n\}$ and a.e. $x\in\Omega_R$. Consequently assuming that $s>\max\limits_{1\le i\le n}q_i^+$, from (14) according to (15)–(20) with sufficiently small values η_1, η_2, we get

$$\int\limits_{\Omega_R}\Big\{\sum_{i=1}^n(a_i(x,u_{1,x_i})-a_i(x,u_{2,x_i}))(u_{1,x_i}(x)-u_{2,x_i}(x))+|w(x)|^{p_0(x)}\Big\}\zeta^s(x)\,dx\le$$
$$\le C_{10}\int\limits_{\Omega_R}\sum_{i=1}^n R^{s-q_i(x)+\sigma_i\frac{q_i(x)}{p_i(x)}}\,dx+C_{11}\int\limits_{\Omega_R}|f_0^1(x)-f_0^2(x)|^{p_0^*(x)}\zeta^s(x)\,dx, \qquad (21)$$

where $s>\max\limits_{1\le i\le n}q_i^+$ is an arbitrary constant, C_{10}, C_{11} are some positive constants.

Note that $s-q_i(x)+\sigma_i\frac{q_i(x)}{p_i(x)}\le s-q_i^-+\sigma_i\frac{q_i^+}{p_i^-}$ for a.e. $x\in\Omega$, $i\in\{1,\dots,n\}$. It is easy to verify that $0\le\zeta(x)\le R$ when $x\in\mathbb{R}^n$, and $\zeta(x)\ge R-R_0$ if $|x|\le R_0$, where $R_0\in(0,R)$ is any number. Taking into account stated above and, in particular, that $R\ge 1$, in virtue of inequality (9) from (21) we conclude

$$\int\limits_{\Omega_{R_0}}\Big[\sum_{i=1}^n(|u_{1,x_i}|^{p_i(x)-2}u_{1,x_i}-|u_{2,x_i}|^{p_i(x)-2}u_{2,x_i})(u_{1,x_i}(x)-u_{2,x_i}(x))+|w(x)|^{p_0(x)}\Big]\,dx\le$$
$$\le\Big(\frac{R}{R-R_0}\Big)^s\Big[C_{12}\sum_{i=1}^n R^{n-q_i^-+\sigma_i\frac{q_i^+}{p_i^-}}+C_{13}\int\limits_{\Omega_R}|f_0^1(x)-f_0^2(x)|^{p_0^*(x)}\,dx\Big], \qquad (22)$$

where C_{12}, C_{13} are positive constants depending only on $n,s,p_i^-,p_i^+(i\in\{0,\dots,n\}),q_i^-,q_i^+$ $(i\in\{1,\dots,n\})$.

Observing in (22) that $n-q_i^-+\sigma_i\frac{q_i^+}{p_i^-}\le n-\gamma$ $(i\in\{1,\dots,n\})$, where $\gamma=\min\limits_{1\le i\le n}\Big\{q_i^--\sigma_i\frac{q_i^+}{p_i^-}\Big\}$, we obtain inequality (12). □

Lemma 4. *Let $p\in\mathbb{P}^*$ and $a\in\mathbb{A}_p^*$, $f\in\mathbb{F}_p$, $u\in\mathbb{U}_p$ are such that*

$$\int\limits_{\Omega_{R_*}}\Big\{\sum_{i=1}^n a_i(x,u_{x_i})v_{x_i}(x)+a_0(x,u)v(x)-f_0(x)v(x)-\sum_{i=1}^n f_i(x)v_{x_i}(x)\Big\}\,dx=0 \qquad (23)$$

for arbitrary $v\in W^1_{p(\cdot),loc}(\overline{\Omega})$, $v|_{\Gamma_{1,R_*}}=0$, $\operatorname{supp}v$ is a compact set in $\overline{\Omega}_{R_*}$, where $R_*>1$ is some number.

Then for every numbers $R_0>0, R\ge 1$, $R_0<R\le R_*$, estimate (3) holds, where φ is any function from $\mathbb{U}_p$ such that $u=\varphi$ on Γ_{1,R_*}.

Proof. Let R be any number in the interval $[1;R_*]$. Put in (23) $v=(u-\varphi)\zeta^s$, where ζ is defined in Lemma 3. After simple transformations we get

$$\int\limits_{\Omega_R}\Big[\sum_{i=1}^n a_i(x,u_{x_i})u_{x_i}+a_0(x,u)u\Big]\zeta^s\,dx=\int\limits_{\Omega_R}\Big[f_0(u-\varphi)+\sum_{i=1}^n f_i(u-\varphi)_{x_i}\Big]\zeta^s\,dx+$$
$$+\int\limits_{\Omega_R}\Big[\sum_{i=1}^n a_i(x,u_{x_i})\varphi_{x_i}+a_0(x,u)\varphi\Big]\zeta^s\,dx+$$
$$+s\int\limits_{\Omega_R}\sum_{i=1}^n f_i(u-\varphi)\,\zeta^{s-1}\zeta_{x_i}\,dx-s\int\limits_{\Omega_R}\sum_{i=1}^n a_i(x,u_{x_i})(u-\varphi)\,\zeta^{s-1}\zeta_{x_i}\,dx. \tag{24}$$

Now estimate each term of (24). Applying first inequality (4) and using condition **1′**), by a reasoning similar to that from the proof of (19), we deduce

$$\int\limits_{\Omega_R}\sum_{i=1}^n a_i(x,u_{x_i})\varphi_{x_i}\,\zeta^s\,dx\le\varepsilon_1\int\limits_{\Omega_R}\sum_{i=1}^n|a_i(x,u_{x_i})|^{p_i^*(x)}(K_i^+(1+|x|)^{\sigma_i})^{-\frac{p_i^*(x)}{p_i(x)}}\zeta^s\,dx+$$
$$+C_{14}(\varepsilon_1)\int\limits_{\Omega_R}\sum_{i=1}^n|\varphi_{x_i}(x)|^{p_i(x)}(K_i^+(1+|x|)^{\sigma_i})^{\frac{p_i(x)}{p_i(x)}}\zeta^s\,dx\le$$
$$\le\varepsilon_1\int\limits_{\Omega_R}\sum_{i=1}^n a_i(x,u_{x_i})u_{x_i}\,\zeta^s\,dx+C_{15}(\varepsilon_1)\int\limits_{\Omega_R}\sum_{i=1}^n R^{\sigma_i}|\varphi_{x_i}(x)|^{p_i(x)}\,\zeta^s\,dx, \tag{25}$$

where $\varepsilon_1\in(0;1)$ is an arbitrary number and $C_{14}(\varepsilon_1),C_{15}(\varepsilon_1)$ are some positive constants.

Next use the inequality

$$|a\pm b|^\nu\le C_\nu(a^\nu+b^\nu),\qquad a,b\ge0,\quad \nu>1, \tag{26}$$

where C_ν is a constant depending only on ν.

Combining inequality (4), condition **3**) and inequality (26) we get

$$\int\limits_{\Omega_R}a_0(x,u)\varphi\,\zeta^s\,dx\le\varepsilon_2\int\limits_{\Omega_R}|a_0(x,u)|^{p_0^*(x)}\,\zeta^s\,dx+C_{16}(\varepsilon_2)\int\limits_{\Omega_R}|\varphi(x)|^{p_0(x)}\,\zeta^s\,dx\le$$
$$\le\varepsilon_2C_{17}\int\limits_{\Omega_R}|u(x)|^{p_0(x)}\,\zeta^s\,dx+\varepsilon_2C_{18}\int\limits_{\Omega_R}|h(x)|^{p_0^*(x)}\,\zeta^s\,dx+C_{16}(\varepsilon_2)\int\limits_{\Omega_R}|\varphi(x)|^{p_0(x)}\,\zeta^s\,dx, \tag{27}$$

where $\varepsilon_2\in(0;1)$ is an arbitrary number, and $C_{16}(\varepsilon_2),C_{17},C_{18}$ are some positive constants. Reasoning just as when obtaining inequality (16) and applying inequality (26), we conclude

$$\int\limits_{\Omega_R}f_0(x)(u-\varphi)\,\zeta^s\,dx\le\varepsilon_3C_{19}\int\limits_{\Omega_R}|u(x)|^{p_0(x)}\,\zeta^s\,dx+\varepsilon_3C_{19}\int\limits_{\Omega_R}|\varphi(x)|^{p_0(x)}\,\zeta^s\,dx+$$
$$+C_{20}(\varepsilon_3)\int\limits_{\Omega_R}|f_0(x)|^{p_0^*(x)}\,\zeta^s\,dx, \tag{28}$$

$$\int\limits_{\Omega_R}\sum_{i=1}^n f_i(x)(u-\varphi)_{x_i}\,\zeta^s\,dx\le\varepsilon_4C_{21}\int\limits_{\Omega_R}\sum_{i=1}^n|u_{x_i}(x)|^{p_i(x)}\,\zeta^s\,dx+$$
$$+\varepsilon_4C_{21}\int\limits_{\Omega_R}\sum_{i=1}^n|\varphi_{x_i}(x)|^{p_i(x)}\,\zeta^s\,dx+C_{22}(\varepsilon_4)\int\limits_{\Omega_R}\sum_{i=1}^n|f_i(x)|^{p_i^*(x)}\,\zeta^s\,dx, \tag{29}$$

where $\varepsilon_3,\varepsilon_4\in(0;1)$ are arbitrary numbers, $C_{19},C_{20}(\varepsilon_3),C_{21},C_{22}(\varepsilon_4)$ are some positive constants.

Arguing the same way as in proof of Lemma 3 (see ()–(20)) and applying inequality (26) we derive

$$s\int\limits_{\Omega_R}\sum_{i=1}^n f_i(u-\varphi)\,\zeta^{s-1}\zeta_{x_i}\,dx\le\varepsilon_5C_{23}\int\limits_{\Omega_R}|u(x)|^{p_0(x)}\,\zeta^s\,dx+\varepsilon_5C_{23}\int\limits_{\Omega_R}|\varphi(x)|^{p_0(x)}\,\zeta^s\,dx+$$
$$+\varepsilon_5\int\limits_{\Omega_R}\sum_{i=1}^n|f_i(x)|^{p_i^*(x)}\,\zeta^s\,dx+C_{24}(\varepsilon_5)\int\limits_{\Omega_R}\sum_{i=1}^n|\zeta_{x_i}(x)|^{q_i(x)}\,\zeta^{s-q_i(x)}\,dx, \tag{30}$$

$$-s\int\limits_{\Omega_R}\sum_{i=1}^n a_i(x,u_{x_i})(u-\varphi)\,\zeta^{s-1}\,\zeta_{x_i}d\,x\le\varepsilon_6C_{25}\int\limits_{\Omega_R}\left[|u(x)|^{p_0(x)}+|\varphi(x)|^{p_0(x)}\right]\zeta^s\,d\,x+$$

$$+\varepsilon_6\int\limits_{\Omega_R}\sum_{i=1}^n a_i(x,u_{x_i})u_{x_i}\zeta^s d\,x+C_{26}(\varepsilon_6)\int\limits_{\Omega_R}\sum_{i=1}^n|\zeta_{x_i}(x)|^{q_i(x)}(K_i^1(1+|x|)^{\sigma_i})^{\frac{q_i(x)}{p_i(x)}}\,\zeta^{s-q_i(x)}\,d\,x,\quad(31)$$

where $\varepsilon_5,\varepsilon_6\in(0;1)$ are the arbitrary numbers, $C_{23},C_{24}(\varepsilon_5),C_{25},C_{26}(\varepsilon_6)$ are some positive constants.

It is obvious that

$$\int\limits_{\Omega_R}\sum_{i=1}^n R^{s-q_i(x)}d\,x\le\int\limits_{\Omega_R}\sum_{i=1}^n R^{s-q_i(x)+\sigma_i\frac{q_i(x)}{p_i(x)}}d\,x\le\widetilde{C}_1\sum_{i=1}^n R^{n+s-q_i^-+\sigma_i\frac{q_i^+}{p_i^-}}\,d\,x,\qquad(32)$$

where $\widetilde{C}_1$ is some positive constant.

From (24) on the basis of (25)–(32) with sufficiently small values $\varepsilon_1,\ldots,\varepsilon_6$ we have

$$\int\limits_{\Omega_R}\left\{\sum_{i=1}^n|u_{x_i}(x)|^{p_i(x)}+|u(x)|^{p_0(x)}\right\}\zeta^s(x)\,d\,x\le$$

$$\le\widehat{C}_1\int\limits_{\Omega_R}\sum_{i=1}^n R^{s-q_i(x)+\sigma_i\frac{q_i(x)}{p_i(x)}}\,d\,x+\widehat{C}_2\int\limits_{\Omega_R}\left[|f_0(x)|^{p_0^*(x)}+\sum_{i=1}^n|f_i(x)|^{p_i^*(x)}\right]\zeta^s(x)d\,x+$$

$$+\widehat{C}_3\int\limits_{\Omega_R}\left[|\varphi(x)|^{p_0(x)}+\sum_{i=1}^n(1+|x|)^{\sigma_i}|\varphi_{x_i}(x)|^{p_i(x)}\right]\zeta^s(x)d\,x+\widehat{C}_4\int\limits_{\Omega_R}|h(x)|^{p_0^*(x)}\zeta^s(x)d\,x,\quad(33)$$

where $s>\max\limits_{1\le i\le n}q_i^+$ is an arbitrary number; $\widehat{C}_1$, $\widehat{C}_2$, $\widehat{C}_3$, $\widehat{C}_4$ are some positive constants depending only on $n,s,p_i^-,p_i^+(i\in\{0,\ldots,n\}),q_i^-,q_i^+,(i\in\{1,\ldots,n\})$.

Proceeding just as in the proof of Lemma 3 (see (22)), we obtain the estimate (3). □

4. Proof of Theorem 1.

Solvability of problem PA$(\mathbb{A}_p^*,\ \mathbb{F}_p,\ \mathbb{V}_p,\ \mathbb{U}_p:p\in\mathbb{P}^*)$. Let $a\in\mathbb{A}_p^*$, $f\in\mathbb{F}_p$, $\Phi\in\mathbb{V}_p$ for some $p\in\mathbb{P}^*$, $\varphi\in\Phi$ and k is an arbitrary natural number. Put $f_{i,k}\stackrel{\text{def}}{=}f_i\chi_k$, $i\in\{0,\ldots,n\}$, $\varphi_k\stackrel{\text{def}}{=}\varphi\chi_k$, where $\chi_k\in C^\infty(\overline{\Omega})$, $0\le\chi_k\le1$ on $\overline{\Omega}$, $\chi_k\equiv1$ on $\Omega_{k-3/4}$, $\chi_k\equiv0$ on $\Omega\backslash\Omega_{k-1/2}$.

Define U_k as the subspace of the space $W^1_{p(\cdot)}(\Omega_k)$ consist of functions satisfying the condition $v|_{\Gamma_{1,k}\cup S_k}=0$ in a sense of trace. Let U_k^* denote the adjoint to U_k space and $\langle\cdot,\cdot\rangle_k$ denotes the inner product of $U_k^*\times U_k$.

Define the operator $L_k:U_k\longrightarrow U_k^*$ as follows:

$$\langle L_kw,v\rangle_k\stackrel{\text{def}}{=}\int\limits_{\Omega_k}\left\{\sum_{i=1}^n a_i(x,(w+\varphi_k)_{x_i})v_{x_i}+a_0(x,w+\varphi_k)v\right\}d\,x,\quad w,v\in U_k.$$

It is easy to verify that the operator $L_k:U_k\longrightarrow U_k^*$ is strictly monotone, bounded, coercive and hemi-continuous. This fact can be proved by analogy to the case of constant exponent of nonlinearity with the aid of inequalities in Proposition 1.

We seek for a function $w_k\in U_k$ satisfying the inequality

$$\langle L_kw_k,v\rangle_k=\int\limits_{\Omega_k}\left\{f_{0,k}(x)v(x)+\sum_{i=1}^n f_{i,k}(x)v_{x_i}(x)\right\}d\,x\qquad(34)$$

for all $v\in U_k$. The existence of a function $w_k\in U_k$ satisfying identity (34) can be proved by Galerkin's method (see, for instance, [3, p. 22]). Uniqueness of a function w_k follows from strictly monotonicity of the operator L_k.

Given functions w_k for all $k \in \mathbb{N}$, define the function $u_k = w_k + \varphi_k$ and extend it by zero on Ω. Keep the notation u_k for this extension.

We claim that the sequence $\{u_k\}_{k=1}^{\infty}$ contains the subsequence converging to $u \in \mathbf{SPA}\big(a, f, \Phi\big)$ in some sense. Indeed, let k and l be arbitrary natural numbers and $1 < k < l$; R_0, R are arbitrary real numbers such that $0 < R_0 < R \le k-1$, $R \ge 1$. Take into account that $f_{i,k} = f_{i,l}$, $i \in \{0, \dots, n\}$, on Ω_{k-1}. Then in virtue of Lemma 3, taking $R_* = k-1$, we get

$$\int\limits_{\Omega_{R_0}} \Big[\sum_{i=1}^{n} (|u_{k,x_i}|^{p_i(x)-2} u_{k,x_i} - |u_{l,x_i}|^{p_i(x)-2} u_{l,x_i})(u_{k,x_i}(x) - u_{l,x_i}(x)) + |u_k(x) - u_l(x)|^{p_0(x)} \Big] \, dx \le$$

$$\le C_5 \left(\tfrac{R}{R-R_0}\right)^s R^{n-\gamma}, \tag{35}$$

where $C_5 > 0$, $s > 0$ are constants not depending on k, l, R_0 and R, γ is such that $n - \gamma < 0$ (it can be assigned in such a way on the basis of Theorem 1 assertion).

Let $\varepsilon > 0$ be an arbitrary number. Fix any value of $R_0 > 0$ and take $R > \max\{1; R_0\}$ large enough to make right-hand side of inequality (35) be less than ε. Then for every $k \ge R+1$ and $l > k$ the left-hand side of inequality (35) is less than ε. It means that the sequence $\big\{u_k\big|_{\Omega_{R_0}}\big\}_{k=1}^{\infty}$ is fundamental in $L_{p_0(\cdot)}(\Omega_{R_0})$. Since R_0 is an arbitrary positive number, there exists a function $u \in L_{p_0(\cdot),\mathrm{loc}}(\overline{\Omega})$ such that

$$u_k \underset{k\to\infty}{\longrightarrow} u \quad \text{strongly} \quad \text{in} \quad L_{p_0(\cdot),\mathrm{loc}}(\overline{\Omega}). \tag{36}$$

Show that the sequences $\big\{u_k\big\}_{k=1}^{\infty}$, $\big\{a_0(\cdot, u_k(\cdot))\big\}_{k=1}^{\infty}$, $\big\{a_i(\cdot, u_{k,x_i}(\cdot))\big\}_{k=1}^{\infty}$, $i \in \{1, \dots, n\}$, are bounded in $W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$, $L_{p_0^*(\cdot),\mathrm{loc}}(\overline{\Omega})$, $L_{p_i^*(\cdot),\mathrm{loc}}(\overline{\Omega})$, $i \in \{1, \dots, n\}$ respectively. Indeed, let R_0, R be some real numbers such that $0 < R_0 < R$, $R \ge 1$. According to Lemma 4 for every $k > R+1$ we conclude

$$\int\limits_{\Omega_{R_0}} \Big[\sum_{i=1}^{n} |u_{k,x_i}(x)|^{p_i(x)} + |u_k(x)|^{p_0(x)} \Big] dx \le \left(\tfrac{R}{R-R_0}\right)^s \Big\{ C_1 R^{n-\gamma} +$$

$$+ C_2 \int\limits_{\Omega_R} \Big[|f_{0,k}(x)|^{p_0^*(x)} + \sum_{i=1}^{n} |f_{i,k}(x)|^{p_i^*(x)} \Big] dx +$$

$$+ C_3 \int\limits_{\Omega_R} \big[|\varphi_k(x)|^{p_0(x)} + \sum_{i=1}^{n} R^{\sigma_i} |\varphi_{k,x_i}(x)|^{p_i(x)} \big] \, dx + C_4 \int\limits_{\Omega_R} |h(x)|^{p_0^*(x)} \, dx \Big\}. \tag{37}$$

Taking into account condition **2**), inequality (10) in Lemma 2, Remark 1 and estimation (37) we have

$$\int\limits_{\Omega_{R_0}} |a_0\big(x, u_k(x)\big)|^{p_0^*(x)} \, dx \le C_{14} \int\limits_{\Omega_{R_0}} |u_k(x)|^{p_0(x)} dx + C_{15} \int\limits_{\Omega_{R_0}} |h(x)|^{p_0^*(x)} dx \le \widetilde{C}_{15}(R_0), \tag{38}$$

$$\int\limits_{\Omega_{R_0}} |a_i\big(x, u_{k,x_i}(x)\big)|^{p_i^*(x)} dx \le \int\limits_{\Omega_{R_0}} \big(K_i^+ (1 + |R_0|^{\sigma_i})\big)^{p_i^*(x)} |u_{k,x_i}(x)|^{p_i(x)} dx \le C_{16}(R_0), \tag{39}$$

$i \in \{1, \dots, n\}$, where $\widetilde{C}_{15}(R_0) > 0, C_{16}(R_0) > 0$ are constants not depending on k but probably depending on R_0.

Condition **1**), (36)–(39), the reflexivity of the spaces $L_{p_0^*(\cdot)}(\Omega_{R_0})$ and $L_{p_i(\cdot)}(\Omega_{R_0})$, $i \in \{1, \dots, n\}$, $R_0 > 0$ yield the existence of a subsequence $\big\{u_{k_j}\big\}_{j=1}^{\infty}$ of the sequence $\big\{u_k\big\}_{k=1}^{\infty}$ and functions $v \in W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$, $\chi_0 \in L_{p_0^*(\cdot),\mathrm{loc}}(\overline{\Omega})$, $\chi_i \in L_{p_i^*(\cdot),\mathrm{loc}}(\overline{\Omega})$, $i \in \{1, \dots, n\}$, such that

$$u_{k_j} \underset{j\to\infty}{\longrightarrow} v \quad \text{weakly in} \quad W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega}), \tag{40}$$

$$u_{k_j} \underset{j\to\infty}{\longrightarrow} u \quad \text{a.e. on} \quad \Omega, \tag{41}$$

$$a_0\big(\cdot, u_{k_j}(\,\cdot\,)\big) \underset{j\to\infty}{\longrightarrow} \chi_0(\,\cdot\,) \ \text{ weakly in } \ L_{p_0^*(\,\cdot\,),\mathrm{loc}}(\overline{\Omega}), \tag{42}$$

$$a_0\big(x, u_{k_j}(x)\big) \underset{j\to\infty}{\longrightarrow} a_0\big(x, u(x)\big) \quad \text{a.e. on} \quad x \in \Omega. \tag{43}$$

$$a_i\big(\cdot, u_{k_j,x_i}(\,\cdot\,)\big) \underset{j\to\infty}{\longrightarrow} \chi_i(\,\cdot\,) \quad \text{weakly in} \quad L_{p_i^*(\,\cdot\,),\mathrm{loc}}(\overline{\Omega}), \quad i \in \{1,\dots,n\}. \tag{44}$$

From (40)–(43) and Lemma 1.3 in [3, p.25] we deduce that

$$v = u, \tag{45}$$

$$\chi_0(\,\cdot\,) = a_0\big(\cdot, u(\,\cdot\,)\big). \tag{46}$$

Show that

$$\chi_i(\,\cdot\,) = a_i\big(\cdot, u_{x_i}(\,\cdot\,)\big), \qquad i \in \{1,\dots,n\}. \tag{47}$$

By virtue of inequality (9) in Lemma 2 we derive

$$\int\limits_{\Omega} \sum_{i=1}^{n} \big(a_i(x, u_{k_j,x_i}) - a_i(x, w_{x_i})\big)\big(u_{k_j,x_i} - w_{x_i}\big)\, \psi \, d\,x \ge 0 \tag{48}$$

for all $j \in \mathbb{N}$, $w \in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $\psi \in C_c^{1,+}(\overline{\Omega})$.

Take into account that for every $j \in \mathbb{N}$ the equality

$$\int\limits_{\Omega_{k_j}} \Big\{ \textstyle\sum_{i=1}^{n} a_i(x, u_{k_j,x_i}) v_{x_i} + a_0(x, u_{k_j}) v - f_{0,k_j} v - \sum_{i=1}^{n} f_{i,k_j} v_{x_i} \Big\}\, d\,x = 0 \tag{49}$$

holds for all $v \in W^1_{p(\,\cdot\,),\mathrm{c}}(\overline{\Omega})$, $v|_{\Gamma_{1,k_j} \cup S_{k_j}} = 0$, $\operatorname{supp} v$ is a compact set in $\overline{\Omega}_{k_j}$. Let us take $v = (u_{k_j} - \varphi_{k_j})\,\psi$, where $\psi \in C_c^{1,+}(\overline{\Omega})$. Combining the obtained equality and (48), we conclude

$$\int\limits_{\Omega} \Big\{ a_0(x, u_{k_j})(u_{k_j} - \varphi_{k_j})\,\psi - f_{0,k_j}\,(u_{k_j} - \varphi_{k_j})\,\psi - \textstyle\sum_{i=1}^{n} f_{i,k_j}\,(u_{k_j} - \varphi_{k_j})_{x_i}\,\psi -$$
$$- \sum_{i=1}^{n} a_i(x, u_{k_j,x_i}) \varphi_{k_j,x_i} \psi + \textstyle\sum_{i=1}^{n} a_i(x, u_{k_j,x_i})(u_{k_j} - \varphi_{k_j})\psi_{x_i} - \sum_{i=1}^{n} f_{i,k_j}(x)(u_{k_j} - \varphi_{k_j})\psi_{x_i} \Big\} d\,x +$$

$$+ \int\limits_{\Omega} \sum_{i=1}^{n} \Big(a_i(x, u_{k_j,x_i}) w_{x_i} + a_i(x, w_{x_i})(u_{k_j,x_i} - w_{x_i})\Big)\ \psi\, d\,x \le 0 \tag{50}$$

for arbitrary $w \in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $\psi \in C_c^{1,+}(\overline{\Omega})$.

Passing to the limit in (50) and keeping in mind the definition of φ_{k_j}, f_{i,k_j}, (36), (40) and (45), (42), (46) and (44), we derive

$$\int\limits_{\Omega} \Big\{ a_0(x, u)(u - \varphi)\,\psi - f_0\,(u - \varphi)\,\psi - \textstyle\sum_{i=1}^{n} f_i\,(u - \varphi)_{x_i}\,\psi \, - \sum_{i=1}^{n} \chi_i\,\varphi_{x_i}\,\psi + \sum_{i=1}^{n} \chi_i\,(u - \varphi)\,\psi_{x_i} -$$

$$- \textstyle\sum_{i=1}^{n} f_i\,(u - \varphi)\,\psi_{x_i} \Big\}\, d\,x + \displaystyle\int\limits_{\Omega} \sum_{i=1}^{n} \big(\chi_i w_{x_i} + a_i(x, w_{x_i})(u_{x_i} - w_{x_i})\big)\ \psi\, d\,x \le 0 \tag{51}$$

for all $w \in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $\psi \in C_c^{1,+}(\overline{\Omega})$.

Let $\psi \in C_c^{1,+}(\overline{\Omega})$ be an arbitrary function and $j_0 \in \mathbb{N}$ be such that $\operatorname{supp}\psi \subset \Omega_{k_{j_0}}$. Putting in equality (49) $v = (u - \varphi)\,\psi$ and passing to the limit as $j \to \infty$, we conclude

$$-\int\limits_{\Omega} \sum_{i=1}^{n} \chi_i\, u_{x_i}\, \psi\, d\,x = \int\limits_{\Omega} \Big\{ a_0(x,u)(u-\varphi)\,\psi - f_0\,(u-\varphi)\,\psi - \sum_{i=1}^{n} f_i\,(u-\varphi)_{x_i}\,\psi -$$

$$- \sum_{i=1}^{n} \chi_i\,\varphi_{x_i}\,\psi + \sum_{i=1}^{n} \chi_i\,(u-\varphi)\,\psi_{x_i} - \sum_{i=1}^{n} f_i\,(u-\varphi)\,\psi_{x_i} \Big\}\, d\,x. \tag{52}$$

From (51) and (52) it follows that

$$\int\limits_{\Omega} \sum_{i=1}^{n} (a_i(x, w_{x_i}) - \chi_i)\,(u_{x_i} - w_{x_i})\;\psi\, d\,x \le 0 \tag{53}$$

for all $w \in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $\psi \in C_c^{1,+}(\overline{\Omega})$. Taking in (53) $w = u - \lambda g$, $\lambda > 0$, $g \in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, we deduce

$$\int\limits_{\Omega} \sum_{i=1}^{n} (a_i(x, (u-\lambda g)_{x_i}) - \chi_i)\, g_{x_i}\, \psi\, d\,x \le 0 \qquad \forall g \in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega}).$$

Let us tend λ to 0, keeping in mind that the operator L_k is hemi-continuous, we get

$$\int\limits_{\Omega} \sum_{i=1}^{n} (a_i(x, u_{x_i}) - \chi_i)\, g_{x_i}\, \psi\, d\,x \le 0 \quad \forall g \in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega}). \tag{54}$$

Since (54) holds for any $g \in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, assigning first $g(x) = x_l$, $l \in \{1, \ldots, n\}$, then $g(x) = -x_l$, $l \in \{1, \ldots, n\}$, we obtain

$$\chi_i(\,\cdot\,) = a_i(\cdot, u_{x_i}(\,\cdot\,)), \quad i \in \{1, \ldots, n\}. \tag{55}$$

From (55) and (44) we have (47).

Let $v \in W^1_{p(\,\cdot\,),\mathrm{loc}}(\overline{\Omega})$, $v|_{\Gamma_1} = 0$, and $\operatorname{supp} v$ is a compact set in $\overline{\Omega}$. For all $j \ge j_0$, where $j_0 \in \mathbb{N}$ is such that $\operatorname{supp} v \subset \overline{\Omega}_{k_{j_0}}$, according to definition of u_{k_j} we deduce

$$\int\limits_{\Omega_{k_j}} \Big\{ \sum_{i=1}^{n} a_i(x, u_{k_j,x_i}) v_{x_i}(x) + a_0(x, u_{k_j}) v(x) - f_0(x) v(x) - \sum_{i=1}^{n} f_i(x) v_{x_i}(x) \Big\}\, d\,x = 0. \tag{56}$$

Let us pass to the limit in (56) as $j \to +\infty$ and take into account (42) and (46), (44) and (47). As a result we obtain (2) for the given function v. As v is an arbitrary function and $0 = u_{k_j} - \varphi_{k_j} \to u - \varphi$ on Γ_1, we proved that $u \in \mathbf{SPA}\big(a, f, \Phi\big)$.

Uniqueness of problem $\mathbf{PA}\big(\mathbb{A}_p^*, \mathbb{F}_p, \mathbb{V}_p, \mathbb{U}_p : p \in \mathbb{P}^*\big)$. Let $a \in \mathbb{A}_p^*$, $f \in \mathbb{F}_p$, $\Phi \in \mathbb{V}_p$ for some $p \in \mathbb{P}^*$. We claim that the set $\mathbf{SPA}\big(a, f, \Phi\big)$ contains at most one element. Arguing by contradiction, we assume that there are two (different) elements u_1, u_2 from $\mathbf{SPA}\big(a, f, \Phi\big)$. By Lemma 3 ($R_*$ is an arbitrary number) we conclude

$$\int\limits_{\Omega_{R_0}} \Big[\sum_{i=1}^{n} (|u_{1,x_i}(x)|^{p_i(x)-2} u_{1,x_i}(x) - |u_{2,x_i}(x)|^{p_i(x)-2} u_{2,x_i}(x))(u_{1,x_i}(x) - u_{2,x_i}(x)) +$$

$$+ |u_1(x) - u_2(x)|^{p_0(x)} \Big]\, d\,x \le C_5 \left(\frac{R}{R - R_0}\right)^{s} R^{n-\gamma}, \tag{57}$$

where R_0, R are some constants such that $0 < R_0 < R$, $R \ge 1$, $\gamma > 0$ is such that $n - \gamma < 0$, and $C_5 > 0$, s are the constants not depending on R_0 and R. Fix $R_0 > 0$ and pass to the limit in (57) as $R \to +\infty$. As a result we obtain that $u_1 = u_2$ on Ω_{R_0}. Since $R_0 > 0$ is an arbitrary number, $u_1 = u_2$ a.e. on Ω.

Weakly well-posedness of problem $\mathbf{PA}\big(\mathbb{A}_p^*,\mathbb{F}_p^*,\mathbb{V}_p,\mathbb{U}_p : p \in \mathbb{P}^*\big)$.
Problem $\mathbf{PA}\big(\mathbb{A}_p^*,\mathbb{F}_p^*,\mathbb{V}_p,\mathbb{U}_p : p \in \mathbb{P}^*\big)$ is a particular case of problem $\mathbf{PA}\big(\mathbb{A}_p^*,\mathbb{F}_p,\mathbb{V}_p, \mathbb{U}_p : p \in \mathbb{P}^*\big)$, therefore its unique solvability follows from unique solvability of problem $\mathbf{PA}\big(\mathbb{A}_p^*,\mathbb{F}_p,\mathbb{V}_p,\mathbb{U}_p : p \in \mathbb{P}^*\big)$.

Let us finish the proof of weak well-posedness of problem $\mathbf{PA}\big(\mathbb{A}_p^*,\mathbb{F}_p^*,\mathbb{V}_p,\mathbb{U}_p : p \in \mathbb{P}^*\big)$. Let $a \in \mathbb{A}_p^*$, $f^k \underset{k\to\infty}{\longrightarrow} f$ in $\mathbb{F}_p^*$, $\Phi \in \mathbb{V}_p$ and $u \in \mathbf{SPA}\big(a, f, \Phi\big)$, $u_k \in \mathbf{SPA}\big(a, f^k, \Phi\big)$, $k \in \mathbb{N}$. On the basis of definition of functions u and u_k, $k \in \mathbb{N}$, it is valid that

$$\int\limits_{\Omega} \Big\{ \textstyle\sum_{i=1}^n a_i(x, u_{x_i}) v_{x_i} + a_0(x, u)v - f_0 v \Big\}\, dx = 0, \tag{58}$$

$$\int\limits_{\Omega_k} \Big\{ \textstyle\sum_{i=1}^n a_i(x, u_{k,x_i}) v_{x_i} + a_0(x, u_k)v - f_{0,k} v \Big\}\, dx = 0, \tag{59}$$

where $v \in W^1_{p(\cdot),\mathrm{loc}}(\overline{\Omega})$, $v|_{\Gamma_1} = 0$, $\operatorname{supp} v$ is a compact set in $\overline{\Omega}$. Let $k_0 \ge 2$ be some fixed natural number, R_0 and R be arbitrary constants such that $0 < R_0 < R$, $R \ge 1$. From (58) and (59) by virtue of Lemma 3, taking $R_* = k_0 - 1$, we deduce for arbitrary $k > k_0$

$$\int\limits_{\Omega_{R_0}} \Big[\textstyle\sum_{i=1}^n (|u_{k,x_i}(x)|^{p_i(x)-2} u_{k,x_i}(x) - |u_{x_i}(x)|^{p_i(x)-2} u_{x_i}(x))(u_{k,x_i}(x) - u_{x_i}(x)) +$$

$$+ |u_k(x) - u(x)|^{p_0(x)} \Big]\, dx \le \Big(\tfrac{R}{R-R_0}\Big)^s \Big[C_5 R^{n-\gamma} + C_6 \int\limits_{\Omega_R} |f_{0,k}(x) - f_0(x)|^{p_0^*(x)}\, dx \Big]. \tag{60}$$

Let $\varepsilon > 0$ be an arbitrary however small number. Fix arbitrary selected $R_0 > 0$ and pick $R \ge \max\{1; 2R_0\}$ so large that

$$C_5 \Big(\tfrac{R}{R-R_0}\Big)^s R^{n-\gamma} < \tfrac{\varepsilon}{2}, \tag{61}$$

and fix this value.

Observing that $\big\|f_{0,k} - f_0\big\|_{L_{p_0^*(\cdot)}(\Omega_R)} \underset{k\to\infty}{\longrightarrow} 0$, derive that the left-hand side of (60) tends to zero when $k \to \infty$. Because of $\frac{R}{R-R_0} \le 1 + \frac{R_0}{R-R_0} \le 2$, all said above yields the existence of a natural number $k_1 > k_0$ such that

$$C_6 \Big(\tfrac{R}{R-R_0}\Big)^s \int\limits_{\Omega_R} |f_{0,k}(x) - f_0(x)|^{p_0^*(x)}\, dx < \tfrac{\varepsilon}{2} \tag{62}$$

for all $k \ge k_1$. Taking into account (61) and (62) from (60) we obtain

$$\int\limits_{\Omega_{R_0}} \Big[\textstyle\sum_{i=1}^n (|u_{k,x_i}(x)|^{p_i(x)-2} u_{k,x_i}(x) - |u_{x_i}(x)|^{p_i(x)-2} u_{x_i}(x))(u_{k,x_i}(x) - u_{x_i}(x)) +$$

$$+ |u_k(x) - u(x)|^{p_0(x)} \Big]\, dx \le \varepsilon$$

for all $k \ge k_1$. Hence it follows that $u_k \underset{k\to\infty}{\longrightarrow} u$ in $\mathbb{U}_p$. Thus we have proved the well-posedness of the problem $\mathbf{PA}\big(\mathbb{A}_p^*,\mathbb{F}_p^*,\mathbb{V}_p,\mathbb{U}_p : p \in \mathbb{P}^*\big)$. □

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Faculty of Mechanics and Mathematics,
Lviv Ivan Franko National University

Received 28.11.2006