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EXISTENCE RESULTS FOR PERTURBED NEUTRAL FUNCTIONAL EVOLUTION INCLUSIONS

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In this paper, we shall establish sufficient conditions for the existence of extremal mild solutions for perturbed neutral functional evolution inclusions in Banach spaces.

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В этой статье мы установим достаточные условия существования экстремальных умеренных решений для возмущенных нейтральных функциональных эволюционных включений в банаховых пространствах.

Introduction. In this paper, we are concerned with the existence of extremal mild solutions for first and second order neutral functional evolution inclusions in a separable Banach space $(X,|\cdot|)$. In Section "First Order Neutral Functional Evolution Inclusions" we consider the following class of neutral functional evolution inclusions

$$\frac{d}{dt}[y(t)-h(t,y_t)]\in Ay(t)+F(t,y_t)+G(t,y_t),\quad t\in J\colon =[0,T] \tag{1}$$

$$y(t)=\phi(t),\quad t\in[-r,0], \tag{2}$$

where $h\colon J\times C([-r,0],X)\to X$, $F,G\colon J\times C([-r,0],X)\to\mathcal{P}(X)$ are multivalued maps, $\mathcal{P}(X)$ is the family of all nonempty subsets of X, $A\colon D(A)\subset X\to X$ is the infinitesimal generator of a strongly continuous semigroup $S(t),\ t\geq 0$, and $\phi\colon[-r,0]\to X$ a given continuous function. For any function y defined on $[-r,T]$, and any $t\in J$, we denote by y_t the element of $C([-r,0],X)$ defined by $y_t(\theta)=y(t+\theta),\ \ \theta\in[-r,0]$. Here $y_t(\cdot)$ represents the history of the state from $t-r$, up to the present time t.

We note that $C([-r,0],X)$ is a Banach space with the norm $\|\phi\|_\infty=\sup\{|\phi(\theta)|\colon\ \theta\in[-r,0]\}$.

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In Section "Second Order Neutral Functional Evolution Inclusions", we consider second order neutral functional evolution inclusions of the form

$$\frac{d}{dt}[y'(t) - h(t, y_t)] \in Ay(t) + F(t, y_t) + G(t, y_t), \quad t \in J: \ = [0, T] \tag{3}$$

$$y_0 = \phi, \quad y'(0) = \eta \tag{4}$$

where h, F, G are as in problem (1)–(2), $\eta \in X$ and A is the infinitesimal generator of a strongly continuous cosine operators.

Recently Dhage et all. in [2], obtained existence results for extremal solutions for perturbed semilinear functional differential equations. Here we extend the results to perturbed neutral functional evolution inclusions. Our approach is based on the theory of analytic semigroups and fixed point theorems.

Preliminaries. In this section, we introduce notations and preliminary facts that are used throughout this paper. $C(J, X)$ is the Banach space of all continuous functions from J into X with the norm $\|y\|_\infty = \sup\{|y(t)| : t \in J\}$, and $B(X)$ denotes the Banach space of bounded linear operators from X into X, with norm $\|N\| = \sup\{|N(y)| : |y| = 1\}$. $L^1(J, X)$ denotes the Banach space of measurable functions $y: J \longrightarrow X$ which are Bochner integrable normed by $\|y\|_1 = \int_0^b |y(t)|dt$.

Definition 1. A *semigroup of class* (C_0) is a one parameter family $\{S(t) \mid t \geq 0\} \subset B(X)$ satisfying the conditions:

(i) $S(t) \circ S(s) = S(t + s)$, for $t, s \geq 0$,

(ii) $S(0) = I$, (the identity operator in X),

(iii) the map $t \to S(t)(x)$ is strongly continuous, for each $x \in X$, i.e;

$$\lim_{t \to 0} S(t)x = x, \ \forall x \in X.$$

Definition 2. Let $S(t)$ be a semigroup of class (C_0) defined on X. The *infinitesimal generator* A of $S(t)$ is the linear operator defined by

$$A(x) = \lim_{h \to 0} \frac{S(h)(x) - x}{h}, \quad \text{for } x \in D(A),$$

where $D(A) = \left\{ x \in X \,\middle|\, \lim_{h \to 0} \dfrac{S(h)(x) - x}{h} \text{ exists in } X \right\}.$

Throughout this paper, $A: D(A) \to X$ will be the infinitesimal generator of an analytic semigroup, $S(t)$, $t \geq 0$, of bounded linear operators on X. For the theory of strongly continuous semigroup, refer to Pazy [12]. If $S(t)$, $t \geq 0$, is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fraction power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression $|x|_\alpha =$

$|(-A)^\alpha x|, \quad x \in D(-A)^\alpha$ defines a norm on $D(-A)^\alpha$. Hereafter we denote by X_α the Banach space $D(-A)^\alpha$ normed with $|\cdot|_\alpha$. Then for each $0 < \alpha \le 1$, X_α is a Banach space, and $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta \le \alpha \le 1$ and the imbedding is compact whenever the resolvent operator of A is compact. Also for every $0 < \alpha \le 1$ there exists $C_\alpha > 0$ such that

$$\|(-A)^\alpha S(t)\| \le \frac{C_\alpha}{t^\alpha}, \quad 0 < t \le T. \tag{5}$$

We say that a family $\{C(t) \mid t \in \mathbb{R}\}$ of operators in $B(X)$ is a *strongly continuous cosine family* if

(i) $C(0) = I$,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $s, t \in \mathbb{R}$,

(iii) the map $t \mapsto C(t)(x)$ is strongly continuous, for each $x \in X$.

The strongly continuous sine family $\{S(t) \mid t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$, is defined by

$$S(t)(x) = \int_0^t C(s)(x)\, ds, \quad x \in X,\ t \in \mathbb{R}. \tag{6}$$

The infinitesimal generator $A\colon X \to X$ of a cosine family $\{C(t) \mid t \in \mathbb{R}\}$ is defined by

$$A(x) = \frac{d^2}{dt^2} C(t)(x)\Big|_{t=0}.$$

Proposition 1 ([13]). Let $C(t), t \in \mathbb{R}$ be a strongly continuous cosine family in X. Then:

(i) there exist constants $M_1 \ge 1$ and $\omega \ge 0$ such that $\|C(t)\| \le M_1 e^{\omega|t|}$ for all $t \in \mathbb{R}$;

(ii) $\|S(t_1) - S(t_2)\| \le M_1 \left| \int_{t_2}^{t_1} e^{\omega|s|} ds \right|$ for all $t_1, t_2 \in \mathbb{R}$.

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [5], Heikkila and Lakshmikantham [8] and Fattorini [4] and the papers [13] and [14].

Let (X, d) be a metric space. We use the notations:

$$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ closed}\}, \mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ bounded}\},$$

$$\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ convex}\}, \mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X)\colon Y \text{ compact}\}.$$

Consider $H_d\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\Big\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $\quad d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{bd,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

A multivalued map $N\colon J \to \mathcal{P}_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $Y\colon J \to \mathbb{R}$ defined by $Y(t) = d(x, N(t)) = \inf\{d(x,z)\colon z \in N(t)\}$, is measurable.

Definition 3. A measurable multivalued function $F\colon J \to \mathcal{P}_{bd,cl}(X)$ is said to be *integrably bounded* if there exists a function $w \in L^1(J, \mathbb{R}^+)$ such that $\|v\| \leq w(t)$ a.e. $t \in J$ for all $v \in F(t)$.

A multivalued map $G\colon X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ i.e. $\sup_{x\in B}\{\sup\{|y|\colon y \in G(x)\}\} < \infty$. G is called upper semi-continuous (u.s.c for short) on X if for each $x_0 \in X$ the set $G(x_0)$ is nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $G(\mathcal{V}) \subseteq \mathcal{U}$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$. If the multivalued map G is completely continuous with nonempty compact valued, then G is u.s.c if and only if G has closed graph i.e. $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

For more details on multivalued maps and the proofs of the known results cited in this section we refer interested reader to the books of Deimling [3], Górniewicz [6] and Hu and Papageorgiou [10].

Definition 4. A nonempty closed subset C of a Banach space $(X, |\cdot|)$ is said to be *a cone* if

(i) $C + C \subset C$,

(ii) $\lambda C \subset C$, for $\lambda \geq 0$,

(iii) $\{-C\} \cap \{C\} = \{0\}$.

A cone C is called normal if the norm $|\cdot|$ is semi-monotone on C, i.e., there exists a constant $N > 0$ such that $|x| \leq N|y|$, whenever $x \leq y$. We equip the space X with the order relation $\leq$ induced by a cone C in X, that is for all $y, \overline{y} \in X\colon\ y \leq \overline{y}$ if and only if $\overline{y}(t) - y(t) \in C,\ \ \forall t \in J$. In what follows will assume that the cone C is normal. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. Cones and their properties are detailed in [7, 8]. Let $a, b \in X$ be such that $a \leq b$. Then, by an order interval $[a,b]$ we mean a set of points in X given by $[a,b] = \{x \in X \mid a \leq x \leq b\}$.

In the space $C(J,X)$ with the norm $\|\cdot\|_\infty$ we define an order relation $\leq$ in $C(J,X)$ by $x \leq y \Leftrightarrow x(t) \leq y(t)$ for all $t \in J$. Here the cone C in $C(J,X)$ is defined by $C = \{x \in C(J,X)\colon x(t) \geq 0\}$, which is obviously normal.

Definition 5. A mapping $T\colon [a,b] \to X$ is said to be *nondecreasing* or *monotone increasing* if $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in [a,b]$.

Our main result is based upon the following form of the fixed point theorem of Dhage ([1, 2]).

Theorem 1. *Let $[a, b]$ be an order interval in an ordered Banach space X. Let $\mathcal{A} : [a, b] \to X$ and $\mathcal{B}, \mathcal{C} \colon X \to \mathcal{P}_{cp,cv}(X)$ be three operators satisfying:*

(a) $\mathcal{A}$ is a single-valued contraction with contraction constant $k < 1/2$,

(b) $\mathcal{B}$ is completely continuous,

(c) $\mathcal{C}$ is compact and right monotone increasing, and

(d) $\mathcal{A}x + \mathcal{B}y + \mathcal{C}z \subset [a, b]$ for all $x, y, z \in [a, b]$.

Further if the cone C in X is normal, then the operator inclusion $x \in \mathcal{A}x + \mathcal{B}x + \mathcal{C}x$ has a solution in $[a, b]$.

We also need the following definitions in the sequel.

Definition 6. A multivalued map $\beta \colon J \times C([-r, 0], X) \to \mathcal{P}(X)$ is said to be *Carathéodory* if

(i) $t \longmapsto \beta(t, x)$ is measurable for each $x \in C([-r, 0], X)$, and

(ii) $x \longmapsto \beta(t, x)$ is u.s.c. for almost all $t \in J$.

Furthermore, a Carathéodory map β is said to be L^1-Carathéodory if

(iii) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(J, \mathbb{R}_+)$ such that

$$\|\beta(t, x)\|_{\mathcal{P}(X)} \colon = \sup\{|v| \colon \quad v \in \beta(t, x)\} \leq h_\rho(t),$$

for a.e. $t \in J$, and for all $\|x\|_\infty \leq \rho$.

Definition 7. A mapping $\beta \colon J \times C([-r, 0], X) \to \mathcal{P}_{cp}(X)$ is said to be *Chandrabhan* if

(i) $t \longmapsto \beta(t, x)$ is measurable for each $x \in C([-r, 0], X)$, and

(ii) $x \longmapsto \beta(t, x)$ is right monotone increasing almost everywhere for $t \in J$.

Furthermore, a Chandrabhan function β is said to be L^1-*Chandrabhan* if

(iii) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(J, \mathbb{R}_+)$ such that

$$|\beta(t, x)| \leq h_\rho(t),$$

for a.e. $t \in J$, and for all $\|x\|_\infty \leq \rho$.

For each $y \in C(J, X)$ let the set $S_{F,y}$, known as the set of selectors from F, defined by

$$S_{F,y} = \{v \in L^1(J, X) \colon \quad v(t) \in F(t, y_t) \;\; a.e. \;\; t \in J\}.$$

We have the following lemma due to Lazota and Opial [11].

Lemma 1. *Let E be a Banach space, and β be an L^1-Carathéodory multivalued map with compact convex values, and let $\Gamma\colon L^1(J,X) \to C(J,X)$ be a linear continuous mapping. Then the operator*

$$\Gamma \circ S_\beta\colon C(J,X) \to \mathcal{P}_{cp,cv}(C(J,X))$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

First Order Neutral Functional Evolution Inclusions. In this section we give our main existence result for problem (1)-(2). Before stating and proving this result, we give the definition of its mild solution.

Definition 8 ([9]). We say that a function $y \in C([-r,T],X) \to X$ is a *mild solution of problem (1)–(2)* if there exist functions $v, w \in L^1(J,X)$ such that $v(t) \in F(t,y_t), w(t) \in G(t,y_t)$, a.e. on J, $y(t) = \phi(t)$, $t \in [-r,0]$, the restriction of $y(\cdot)$ to the interval $[0,T)$ is continuous, and for each $0 \le t < T$ the function $AS(t-s)h(s,y_s), s \in [0,t)$, is integrable and y satisfies the integral equation

$$\begin{aligned} y(t) &= S(t)[\phi(0) - h(0,\phi(0))] + h(t,y_t) + \int_0^t AS(t-s)h(s,y_s)ds \\ &\quad + \int_0^t S(t-s)v(s)ds + \int_0^t S(t-s)w(s)ds, \quad t \in J. \end{aligned}$$

Definition 9. We say that a function $a \in C([-r,T],X) \to X$ is a *lower mild solution of problem (1)–(2)* if there exist functions $v, w \in L^1(J,X)$ such that $v(t) \in F(t,y_t), w(t) \in G(t,y_t)$, a.e. on J, $a(t) \le \phi(t)$, $t \in [-r,0]$, the restriction of $a(\cdot)$ to the interval $[0,T)$ is continuous, and for each $0 \le t < T$ the function $AS(t-s)h(s,a_s), s \in [0,t)$, is integrable and a satisfies the integral inequality

$$\begin{aligned} a(t) &\le S(t)[\phi(0) - h(0,\phi(0))] + h(t,a_t) + \int_0^t AS(t-s)h(s,a_s)ds \\ &\quad + \int_0^t S(t-s)v(s)ds + \int_0^t S(t-s)w(s)ds, \quad t \in J. \end{aligned}$$

Similarly, an upper mild solution b of the problem (1)–(2) is defined. Finally, a function $x \in C([-r,T],X)$ is a solution of the problem (1)–(2) on $[-r,T]$ if it is a lower mild as well an upper mild solution of the problem (1)–(2) on $[-r,T]$.

Definition 10. A solution x_M of the problem (1)–(2) is said to be *maximal* if for any other solution x to the problem (1)–(2) one has $x(t) \le x_M(t)$ for all $t \in [-r,T]$. Again, a solution x_m of the problem (1)–(2) is said to be minimal if $x_m(t) \le x(t)$ for all $t \in [-r,T]$, where x is any solution for the problem (1)–(2) on $[-r,T]$.

Theorem 2. *Assume that the following conditions are satisfied:*

1. *$A\colon D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, of bounded linear operators on X. Assume that $0 \in \rho(A)$, $S(t)$ is compact for $t > 0$, and there exist constants $M \geq 1$ and $C_{1-\beta}$ such that*
$$\|S(t)\| \leq M \quad \textit{and} \quad \|(-A)^{1-\beta}S(t)\| \leq \frac{C_{1-\beta}}{t^{1-\beta}}, \quad \textit{for all } t > 0;$$
2. *$S(t)$ is order preserving, that is $S(t)v \geq 0$ whenever $v \geq 0$;*
3. *there exist constants $0 < \beta < 1$ and L_f such that h is X_β-valued, $(-A)^\beta h$ is continuous, and*
$$|(-A)^\beta h(t, x_1) - (-A)^\beta h(t, x_2)| \leq L_f \|x_1 - x_2\|_\infty, \ (t, x_i) \in J \times C([-r, 0], X),$$
$i = 1, 2$, with $L_f \left\{ \|(-A)^{-\beta}\| + \dfrac{C_{1-\beta}T^\beta}{\beta} \right\} < \dfrac{1}{2}$;
4. *$F\colon J \times X \to \mathcal{P}_{cv,cp}(X)$ is L^1-Carathéodory multimap;*
5. *$G(t, x)$ is right monotone increasing in x almost everywhere for $t \in J$; that is if $x \leq y$ then $S_G(x) \leq S_G(y)$ for all $x, y \in X$;*
6. *$G\colon J \times X \to \mathcal{P}_{cv,cp}(X)$ is L^1-Chandrabhan;*
7. *the problem (1)–(2) has a lower mild solution a and an upper mild solution b with $a \leq b$.*

Then the problem (1)–(2) has a minimal and a maximal solution in $[a, b]$ defined on $[-r, T]$.

Proof. Define an order interval in $C([-r, T], X)$ which does exist in view of hypothesis (2.7). Transform the problem (1)–(2) into a fixed point problem. Consider the operators:
$$\mathcal{A}\colon [a, b] \to C([-r, T], X) \quad \text{and} \quad \mathcal{B}, \mathcal{C}\colon C([-r, T], X) \to \mathcal{P}(C([-r, T], X))$$
defined by
$$\mathcal{A}(y)(t)\colon = \begin{cases} 0, & \text{if } t \in [-r, 0]; \\ \left\{ -S(t)h(0, \phi) + h(t, y_t) + \displaystyle\int_0^t AS(t-s)h(s, y_s)ds \right\}, & \text{if } t \in J, \end{cases}$$
$$\mathcal{B}(y)\colon = \left\{ h \in C([-r, T], X)\colon \ h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0]; \\ S(t)\phi(0) + \\ + \displaystyle\int_0^t S(t-s)v(s)ds, & \text{if } t \in J, \end{cases} \right\}$$

and

$$\mathcal{C}(y)\colon = \left\{ h \in C([-r,T],X)\colon \quad h(t) = \begin{cases} 0, & \text{if } t \in [-r,0]; \\ \displaystyle\int_0^t S(t-s)w(s)ds, & \text{if } t \in J, \end{cases} \right\}$$

where $v \in S_{F,y}$ and $w \in S_{G,y}$.

Then the problem of finding the solution of (1)–(2) is reduced to finding the solution of the operator inclusion $y(t) \in \mathcal{A}(y)(t) + \mathcal{B}(y)(t) + \mathcal{C}(y)(t)$, $t \in [-r,T]$. We shall show that the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfies all the conditions of Theorem 1. The proof will be given in several steps.

Step I: We show that $\mathcal{A}$ is a contraction. Let $x, y \in [a,b]$. Then

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq |h(t,x_t) - h(t,y_t)| + \left| \int_0^t AS(t-s)[h(s,x_s) - h(s,y_s)]\, ds \right| \\ &\leq \|(-A)^{-\beta}\| L_f \|x_t - y_t\|_\infty + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\, ds\, L_f \|x_t - y_t\|_\infty \\ &\leq L_f \left\{ \|(-A)^{-\beta}\| + \frac{C_{1-\beta}T^\beta}{\beta} \right\} \|x_t - y_t\|_\infty. \end{aligned}$$

Taking supremum over t,

$$\|\mathcal{A}x - \mathcal{A}y\|_\infty \leq L_0 \|x - y\|\|_\infty, \quad L_0\colon = L_f \left\{ \|(-A)^{-\beta}\| + \frac{C_{1-\beta}T^\beta}{\beta} \right\}.$$

This shows that A is a multi-valued contraction, since $L_0 < 1/2$.

Step 2: We shall show that the operator B has closed convex values and it is completely continuous. As a result B will be compact valued. This will be given in several claims.

Claim 1: First we show that $B(y)$ *is closed for each* $y \in C([-r,T],X)$. Observe that the operator B is equivalent to the composition $\mathcal{L} \circ S_{F,y}$ of two operators on $L^1(J,X)$, where $\mathcal{L}\colon L^1(J,X) \to C(J,X)$ is the continuous operator defined by

$$\mathcal{L}v(t) = S(t)\phi(0) + \int_0^t S(t-s)v(s)ds.$$

It then suffices to show that the operator $S_{F,y}$ has closed values on $L^1(J,X)$. Let $y \in C([-r,T],X)$ and let $v \in \overline{S_{F,y}}$. Then there exists a sequence Kv_n in $S_{F,y}$ with K$v_n \to v$. Then, by the definition of $S_{F,y}$, $v_n(t) \in F(t,y)$ a.e. $t \in J$. Then $v_n \to v$ in measure so there exists a subsequence v_{n_k} Kof v_n with v_{n_k} converging to v a.e.. Now since the values of F are closed we have $v \in F(t,y)$ a.e. Therefore B is a closed valued multivalued operator.

Claim 2: $\mathcal{B}$ maps bounded sets into bounded sets in $C([-r,T],X)$.

Let B a bounded set in $C([-r,T],X)$. There exists a real number $q>0$ such that $\|y\|_\infty \le q$ for any $y\in B$. Now for each $h\in\mathcal{B}(y)$, there exists $f\in S_{F,y}$ such that

$$h(t)=S(t)\phi(0)+\int_0^t S(t-s)f(s)ds.$$

Then,

$$|h(t)|\le M|\phi(0)|+M\int_0^t h_q(s)ds\le M|\phi(0)|+M\|h_q\|_1;$$

this further implies that$\|h\|_\infty \le M|\phi(0)|+M\|h_q\|_1$ for all $h\in\mathcal{B}(y)\subset\mathcal{B}(B)=\bigcup_{y\in B}\mathcal{B}(y)$. Hence $\mathcal{B}(B)$ is bounded.

Claim 3: $\mathcal{B}$ maps bounded sets into equicontinuous sets.

Let $\tau_1,\tau_2\in J,\ \ \tau_1<\tau_2$ and B be, as above, a bounded set and $h\in\mathcal{B}(y)$ for some $y\in B$. Then, there exists $v\in S_{F,y}$ such that

$$h(t)=S(t)\phi(0)+\int_0^t S(t-s)v(s)ds.$$

Thus if $\epsilon>0$ and $\epsilon\le\tau_1<\tau_2$ we have

$$\begin{aligned}|h(\tau_2)-h(\tau_1)|&\le|[S(\tau_2)-S(\tau_1)]\phi(0)|+\int_0^{\tau_1-\epsilon}\|S(\tau_2-s)-S(\tau_1-s)\||v(s)|ds\\
&+\int_{\tau_1-\epsilon}^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\||v(s)|ds+\int_{\tau_1}^{\tau_2}\|S(\tau_2-s)\|ds\le|[S(\tau_2)-S(\tau_1)]\phi(0)|\\
&+\int_0^{\tau_1-\epsilon}\|S(\tau_2-s)-S(\tau_1-s)\|h_q(s)ds+\int_{\tau_1-\epsilon}^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|h_q(s)ds\\
&+M\int_{\tau_1}^{\tau_2}h_q(s)ds.\end{aligned}$$

As $\tau_1\longrightarrow\tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero, since $S(t)$ is a strongly continuous operator and the compactness of $S(t)$ for $t>0$ implies the continuity in the uniform operator topology. As a consequence of Claims 1 to 3 together with Arzelá-Ascoli theorem it suffices to show that $\mathcal{B}$ maps B into a precompact set in X.

Let $0<t<T$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $y\in B$ we define

$$h_\epsilon(t)=S(t)\phi(0)+S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)f(s)ds,$$

where $f\in S_{F,y}$. Since $S(t)$ is a compact operator, the set $H_\epsilon(t)=\{h_\epsilon(t)\colon h_\epsilon\in\mathcal{B}(y)\}$ is precompact in X for every $\epsilon,\ \ 0<\epsilon<t$. Moreover, for every $h\in\mathcal{B}(y)$ we have

$$|h(t)-h_\epsilon(t)|\le\int_{t-\epsilon}^t\|S(t-s)\||v(s)|ds\le M\int_{t-\epsilon}^t h_q(s)ds.$$

Therefore, there are precompact sets arbitrarily close to the set $H(t) = \{h(t)\colon h \in \mathcal{B}(y)\}$. Hence the set $H(t) = \{h(t)\colon h \in \mathcal{B}(B)\}$ is precompact in E. Hence the operator $\mathcal{B}$ is totally bounded.

CLAIM 4 : $\mathcal{B}$ has closed graph.

This step is obvious and follows directly by Lemma 1. We omit the details.

CLAIM 5: $\mathcal{B}(y)$ is convex for each $y \in C([-r,T],X)$. Let $h_1, h_2 \in \mathcal{B}(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in J$ we have

$$h_i(t) = S(t)\phi(0) + \int_0^t S(t-s)g_i(s)ds, \quad i = 1,2.$$

Let $0 \le \delta \le 1$. Then, for each $t \in J$, we have

$$(\delta h_1 + (1-\delta)h_2)(t) = S(t)\phi(0) + \int_0^t S(t-s)[\delta g_1(s) + (1-\delta)g_2(s)]ds.$$

Since $F(t,y)$ has convex values, one has $\delta h_1 + (1-\delta)h_2 \in \mathcal{B}(y)$.

Step III: We shall show that the operator $\mathcal{C}$ is compact and convex valued and it is right monotone increasing. The proof that the operator $\mathcal{C}$ is compact and convex valued is similar to that of Step II, and is omitted. Let $u_1 \in \mathcal{C}x$. Then there is a $v_1 \in S_{G,y}$ such that

$$u_1(t) = \int_0^t S(t-s)v_1(s)\,ds, \quad t \in J.$$

Since G is right monotone increasing in x, we have $S_G(x) \le S_G(y)$ and so there is a $v_2 \in S_G(y)$ such that $v_1 \le v_2$ on J. Therefore, we have

$$u_1(t) = \int_0^t S(t-s)v_1(s)\,ds \le \int_0^t S(t-s)v_2(s)\,ds = u_2(t),$$

for all $t \in J$, where $u_2 \in \mathcal{C}y$. Thus $\mathcal{C}$ is right monotone increasing on $[a,b]$.

Thus the operators $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ satisfy all conditions of Theorem 1 and hence the operator inclusion $x \in \mathcal{A}x + \mathcal{B}x + \mathcal{C}x$, and consequently, the problem (1)–(2) has a minimal and a maximal solution in $[a,b]$ defined on $[-r,T]$. This completes the proof. □

Second Order Neutral Functional Evolution Inclusions. Consider now the problem (3)–(4) for second order semilinear neutral functional differential inclusions.

Definition 11 ([9]). We say that a function $y \in C([-r,T],X) \to X$ is a *mild solution of problem (3)–(4)* if $y(t) = \phi(t), t \in [-r,0]$, $y'(0) = \eta$ and there exist functions $v, w \in L^1(J,X)$ such that $v(t) \in F(t,x_t), w(t) \in G(t,x_t)$ and

$$y(t) \;=\; C(t)\phi(0) + S(t)[\eta - h(0,\phi(0))] + \int_0^t C(t-s)h(s,y_s)ds$$

$$+\int_0^t S(t-s)v(s)ds + \int_0^t S(t-s)w(s)ds, \quad t \in J.$$

Definition 12. A function $a \in C([-r,T],X) \to X$ is a *mild lower solution of problem (3)-(4)* if $a(t) \leq \phi(t), t \in [-r,0]$, $a'(0) \leq \eta$ and it satisfies the integral equation

$$\begin{aligned} a(t) &= C(t)\phi(0) + S(t)[\eta - h(0,\phi(0))] + \int_0^t C(t-s)h(s,a_s)ds \\ &\quad + \int_0^t S(t-s)v(s)ds + \int_0^t S(t-s)w(s)ds, \quad t \in J. \end{aligned}$$

Similarly, an upper mild solution $b \in C([-r,T],X)$ of the problem (3)–(4) on $[-r,T]$ is defined. Note that a function $x \in C([-r,T],X)$ is a solution of the problem (3)–(4) on J if it is a lower mild as well an upper mild solution of the problem (3)–(4) on $[-r,T]$.

Definition 13. A solution x_M of the problem (3)–(4) is said to be *maximal* if for any other solution x to the problem (3)–(4) one has $x(t) \leq x_M(t)$ for all $t \in [-r,T]$. Again, a solution x_m of the problem (3)–(4) is said to be minimal if $x_m(t) \leq x(t)$ for all $t \in [-r,T]$, where x is any solution for the problem (3)–(4) on $[-r,T]$.

Theorem 3. *Assume that the hypotheses (4), (5) and (6) in theorem 2 hold. Suppose also that:*

1 *there exists a function $\ell \in L^1(J,\mathbb{R}^+)$ such that*

$$|h(t,x) - h(t,y)| \leq \ell(t)\|x-y\|_\infty \text{ a.e. } t \in J$$

for all $x, y \in C([-r,0],X)$;

2 *$A\colon D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in J$, and there exist constants $N_1 \geq 1$, and $N_2 \geq 1$ such that $\|C(t)\| \leq N_1$, $\|S(t)\| \leq N_2$ for all $t \in \mathbb{R}$;*

3 *for each bounded $B \subseteq C([-r,T],X)$ and $t \in J$ the sets*

$$\left\{\int_0^t S(t-s)v(s)ds, y \in B\right\} \text{ and } \left\{\int_0^t S(t-s)w(s)ds, y \in B\right\}$$

are relatively compact in X;

4 *$C(t)$ and $S(t)$ are preserving the order, that is $C(t)v \geq 0, S(t)v \geq 0$ whenever $v \geq 0$;*

5 *the problem (3)–(4) has a lower mild solution a and an upper mild solution b with $a \leq b$.*

Then, if $N_1\|\ell\|_1 < 1/2$, the problem (3)–(4) has a minimal and a maximal solution in $[a,b]$ defined on $[-r,T]$.

Proof. Consider the order interval $[a,b]$ in the Banach space $C([-r,T],X)$ which is well defined in view of hypothesis (5) in theorem 2 Transform the problem (3)–(4) into a fixed point problem. Consider the operators:

$$\mathcal{A}_1\colon [a,b] \to C([-r,T],X) \quad \text{and} \quad \mathcal{B}_1, \mathcal{C}_1\colon C([-r,T],X) \to P(C([-r,T],X))$$

defined by

$$\mathcal{A}_1(y)(t)\colon = \begin{cases} \phi(t), & \text{if } t \in [-r,0]; \\ C(t)\phi(0) - S(t)h(0,\phi) + \displaystyle\int_0^t C(t-s)h(s,y_s)ds, & \text{if } t \in J, \end{cases}$$

$$\mathcal{B}_1(y)\colon = \left\{ h \in C([-r,T],X)\colon \ h(t) = \begin{cases} 0, & \text{if } t \in [-r,0]; \\ S(t)\eta + \\ + \displaystyle\int_0^t S(t-s)v(s)ds, & \text{if } t \in J, \end{cases} \right\}$$

and

$$\mathcal{C}_1(y)\colon = \left\{ h \in C([-r,T],X)\colon \quad h(t) = \begin{cases} 0, & \text{if } t \in [-r,0]; \\ \displaystyle\int_0^t S(t-s)w(s)ds, & \text{if } t \in J, \end{cases} \right\}$$

where $v \in S_{F,y}$ and $w \in S_{G,y}$.

Then the problem of finding the solution of (3)–(4) is reduced to finding the solution of the operator equation $\mathcal{A}_1(y)(t) + \mathcal{B}_1(y)(t) + \mathcal{C}_1(y)(t) = y(t)$, $t \in [-r,T]$. We shall show that the operators $\mathcal{A}_1$, $\mathcal{B}_1$ and $\mathcal{C}_1$ satisfies all the conditions of Theorem 1. The proof will be given in the following steps.

Step I: We show that $\mathcal{A}_1$ is a contraction. Let $x, y \in [a,b]$. Then

$$\begin{aligned} |(\mathcal{A}_1 x)(t) - (\mathcal{A}_1 y)(t)| &= \left| \int_0^t C(t-s)[h(s,x_s) - h(s,y_s)]\, ds \right| \\ &\le N_1 \int_0^t \ell(s) \|x_s - y_s\|_\infty \, ds \le N_1 \|\ell\|_1 \|x_t - y_t\|_\infty. \end{aligned}$$

Taking supremum over $t, \|\mathcal{A}_1 x - \mathcal{A}_1 y\|_\infty \le N_1 \|\ell\|_1 \|x_t - y_t\|_\infty$. This shows that $\mathcal{A}_1$ is a contraction, since $N_1 \|\ell\|_1 < 1/2$.

Step II: As in Theorem 2 we can prove that $\mathcal{B}_1(y)$ has closed convex values and it is completely continuous. As a result $\mathcal{B}_1(y)$ will be compact valued. We show only that $\mathcal{B}_1$ maps bounded sets into bounded set in $C([-r,T],X)$ and that $\mathcal{B}_1$ maps bounded sets into equicontinuous sets. Let B a bounded set in X. There exists a real number $q > 0$ such that $\|y\|_\infty \le q$ for any $y \in B$. Now for each $h \in \mathcal{B}_1(B)$, we have

$$|h(t)| \le N_1 \int_0^t h_q(s)ds \le N_1 \|h_q\|_1;$$

this further implies that $\|h\|_\infty \le N_1\|h_q\|_1$ for all $h \in \mathcal{B}_2(B)$. Hence $\mathcal{B}_1(B)$ is bounded.

Next we show that $\mathcal{B}_1$ maps bounded sets into equicontinuous sets. Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and B be, as above, a bounded set. Thus if $\epsilon > 0$ and $\epsilon \le \tau_1 < \tau_2$ we have using Proposition 1

$$\begin{aligned}|h(\tau_2) - h(\tau_1)| &\le \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\| |v(s)| ds + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\| |v(s)| ds \\ &\le \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\| h_q(s) ds + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\| h_q(s) ds \\ &\le \int_0^{\tau_1} \int_{\tau_1 - s}^{\tau_2 - s} e^{\omega x} dx\, h_q(s)\, ds + N_2 \int_{\tau_1}^{\tau_2} h_q(s)\, ds \le e^{\omega b}(\tau_2 - \tau_1) \int_0^{\tau_1} h_q(s)\, ds + \\ &\quad + N_2 \int_{\tau_1}^{\tau_2} h_q(s)\, ds.\end{aligned}$$

As $\tau_1 \longrightarrow \tau_2$ the right-hand side of the above inequality tends to zero. As a consequence of the Arzelá-Ascoli theorem and second assumption from Theorem 3 $\mathcal{B}_1$ is totally bounded.

Also as in Theorem 2 we can prove that $\mathcal{B}_1$ has closed graph.

Step III: We shall show that the operator $\mathcal{C}_1$ is compact and convex valued and it is right monotone increasing. The proof that the operator $\mathcal{C}_1$ is compact and convex valued is similar to that of Step II, and is omitted. Let $u_1 \in \mathcal{C}_1 x$. Then there is a $v_1 \in S_{G,y}$ such that

$$u_1(t) = \int_0^t S(t - s) v_1(s)\, ds, \quad t \in J.$$

Since G is right monotone increasing in x, we have $S_G(x) \le S_G(y)$ and so there is a $v_2 \in S_G(y)$ such that $v_1 \le v_2$ on J. Therefore, we have

$$u_1(t) = \int_0^t S(t - s) v_1(s)\, ds \le \int_0^t S(t - s) v_2(s)\, ds = u_2(t),$$

for all $t \in J$, where $u_2 \in \mathcal{C}_1 y$. Thus $\mathcal{C}$ is right monotone increasing on $[a, b]$.

Thus the operators $\mathcal{A}_1$, $\mathcal{B}_1$ and $\mathcal{C}_1$ satisfy all the conditions of Theorem 1 and hence the operator equation $\mathcal{A}_1 x + \mathcal{B}_2 x + \mathcal{C}_1 x = x$ and consequently the problem (3)-(4) has a minimal and a maximal solution in $[a, b]$ defined on $[-r, T]$. This completes the proof. □

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