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ON GENERATORS OF POSITIVE C -SEMIGROUPS AND A NOTE ON COMPACT C-SEMIGROUPS

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Let X be a Banach space and $T(t), 0 \leq t < \infty$, be a one parameter C-semigroup of bounded linear operators on X. In this paper, we give a characterization for the generator of an exponentially bounded positive contractive C-semigroup. Further, sufficient conditions for a C-semigroup to be compact are presented.

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Пусть X — банахово пространство и $T(t), 0 \leq t < \infty$, — однопараметрическая C-полугруппа ограниченных линейных операторов на X. В статье дается характеризация генератора экспоненциально ограниченной положительной сжимающей C-полугруппы. Приводятся достаточные условия компактности C-полугруппы.

Introduction. Let X be a Banach space, $L(X)$ the space of bounded linear operators on X, and C an injective bounded linear operator on X. A strongly continuous family, $T(t),\ 0 \leq t < \infty$, of bounded linear operators on X is called a *C-semigroup*, if $T(0) = C$ and $CT(s+t) = T(s)T(t)$ for all $0 \leq s, t < \infty$. If $C = I$, the identity operator on X, then the C-semigroup is just a strongly continuous semigroup in the ordinary sense. $T(t)$ is called *exponentially bounded* if there exist $M < \infty$ and $\omega \in R$ such that $\|T(t)\| \leq Me^{\omega t}$. The operator A defined by

$$Ax = C^{-1}\Big(\lim_{t\downarrow 0}\frac{T(t)x - Cx}{t}\Big)$$

with

$$D(A) = \Big\{x \in X : \lim_{t\downarrow 0}\frac{T(t)x - Cx}{t} \text{ exists and is in } R(C)\Big\}$$

is called the *generator* of $T(t)$. The complex number λ is in $\rho_C(A)$, the *C-resolvent* of A, if $(\lambda - A)$ is injective and $R(C) \subseteq R(\lambda - A)$. C-semigroups have been given more attention lately. We refer to [1], [2], [4], [5] and [6] for generators and basic structure of C-semigroups.

Let X be an ordered real Banach space with generating closed positive convex cone X^+. An operator $B \in L(X)$ is called *positive* if $B(X^+) \subseteq X^+$. A C-semigroup $T(t)$, $t \in [0, \infty)$, is called *positive* if $T(t)$ is positive for all $t \in [0, \infty)$. A C-semigroup $T(t)$ is called *compact*, if $T(t)$ is a compact operator on X for all $t \in (0, \infty)$.

The main object of this paper is to characterize the generator of a positive contractive C-semigroup. Sufficient and necessary conditions for a C-semigroup to be compact are presented.

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Throughout this paper, X is a Banach space and $L(X)$ is the space of bounded linear operators on X. For a densely defined linear operator A on X, we let $\sigma(A)$ denote the spectrum of A, while $\rho(A)$ denotes the resolvent set of A.

I. Positive C-Semigroups. Let X be an ordered real Banach space with generating closed positive convex cone X^+ and " $\leq$ " as the ordered relation. An element $x \in X$ is called positive if $0 \leq x$. Each element $x \in X$ admits a decomposition $x = u - v$ with $u, v \in X^+$. We refer to [3] and [8] for basic structure of ordered Banach spaces.

Now we introduce the concept of *C-dispersive* operators.

Definition. Let X be an ordered real Banach space with generating closed cone X^+, and C be a positive bounded linear operator on X. An operator A with domain and range in X is said to be *C-dispersive* if for every $\lambda > 0$ and, $x \in D(A)$, the inequality $\inf\{\|C(\lambda - A)x + u\|,\ u \in X^+\} \geq \lambda \inf\{\|Cx + u\|,\ u \in X^+\}$ holds.

Example 1.1. Let $X = C(K)$, the space of continuous functions on the compact metric space K with $\|f\| = \sup\limits_{t\in K}|f(t)|$ for all $f \in C(K)$. The closed generating cone for X is $X^+ = \{f \in C(K) : f(x) \geq 0\ ,\ x \in K\}$. Each $f \in X$ admits a decomposition $f = f^+ - f^-$, where $f^+(x) = \sup\{f(x);\ 0\}$ and $f^-(x) = (-f(x))^+$. It is clear that both f^+ and $f^- \in X^+$ and $\|f\| = \sup\limits_{t\in K}|f(t)| = \sup\limits_{t\in K}\big(f^+(t) + f^-(t)\big) = \sup\limits_{t\in K}|f^+(t) + f^-(t)| = \|f^+ + f^-\|$.

A C-semigroup $T(t)$, $0 \leq t < \infty$, is of **contractions** if $\|T(t)x\| \leq \|Cx\|$ for all $t \geq 0$ and $x \in X$. In this section, we give a characterization of the generator of contraction positive C-semigroup.

Now, we state and prove the main result of this paper.

Theorem 1.2. *Let X be an ordered real Banach space with generating closed cone X^+. Suppose that for all $x \in X$, $\|x\| = \inf\{\|u + v\| : x = u - v,\ u, v \in X^+\}$ and $\|Cx\| = \inf\{\|Cu + Cv\| : x = u - v,\ u, v \in X^+\}$. If A is a densely defined linear operator with domain and range in X such that $(0, \infty) \subseteq \rho(A)$ and C is a positive bounded linear operator on X, then the operator A generates a strongly continuous positive contraction C-semigroup if and only if A satisfies: (i) $CA \subseteq AC$; (ii) $C(D(A))$ is dense in $R(C)$; (iii) A is C-dispersive.*

Proof. Suppose that A generates a positive contraction C-semigroup $T(t)$. Then by Theorem 4.3, ([5]), (i) and (ii) are satisfied.

To prove (iii). Let $x \in X$ and $\lambda > 0$. Define, $R(\lambda)x = \int_0^\infty e^{-\lambda s}T(s)x\, ds$. Since $T(t)$ is positive for all $t > 0$, we see that $R(\lambda)$ is positive.

Since $T(t)$ is a contraction C-semigroup, then for $\lambda > 0$ we have,

$$\|\lambda R(\lambda)y\| = \|\lambda \int_0^\infty e^{-\lambda s}T(s)y\, ds\| \leq \int_0^\infty e^{-\lambda s}\|T(s)y\|\, ds \leq \|Cy\| \tag{1}$$

But by Lemma 2.8([4]), we have $R(\lambda)x = (\lambda - A)^{-1}Cx$, so, for $u \in X^+$ and $x \in D(CA)$ we have $\lambda\|Cx + R(\lambda)u\| = \lambda\|C(\lambda - A)^{-1}(\lambda - A)x + R(\lambda)u\| = \|\lambda\big(R(\lambda)(\lambda - A)x + R(\lambda)u\big)\| = \|\lambda R(\lambda)\big((\lambda - A)x + u\big)\|$. Using (1) we get $\lambda\|Cx + R(\lambda)u\| \leq \|C(\lambda - A)x + Cu\| = \|C(\lambda - A)x + v\|$,where, $v = Cu$ is positive. But $R(\lambda)$ is a positive operator. It follows that

$R(\lambda)u$ is positive, and so $\lambda\|Cx+w\| \le \|C(\lambda-A)x+v\|$, where, $w,v \in X^+$. This implies $\inf\{\|C(\lambda-A)x+v\|,\ v\in X^+\} \ge \lambda \inf\{\|Cx+u\|,\ u\in X^+\}$. Thus A is C-dispersive.
Conversely. Define another norm on $R(C)$, $\|.\|_1$ as follows

$$\|Cx\|_1 = \max\big(\inf\{\|Cx+u\|,\ u\in X^+\}, \inf\{\|-Cx+u\|,\ u\in X^+\}\big).$$

It is known, [1], that $p(y)=\inf\{\|y+u\|, u\in X^+\}$ is the canonical half-norm on X and $\max\big(p(y),p(-y)\big)$ is a norm on $R(C)$ satisfying $\|Cx\|_1 \le \|Cx\|$ for all $x\in X$.

Now, if (Cx_n) is a Cauchy sequence in $(X,\|.\|_1)$, then there exist double sequences $(u_{n,m})$, $(v_{n,m})$ in X^+ such that $\lim\limits_{n,m\to\infty}\|Cx_n-Cx_m+u_{n,m}\|=0$, and $\lim\limits_{n,m\to\infty}\|Cx_m-Cx_n+v_{n,m}\|=0$. Hence

$$\lim_{n,m\to\infty}\|u_{n,m}+v_{n,m}\| \le \lim_{n,m\to\infty}\|Cx_n-Cx_m+u_{n,m}\| + \lim_{n,m\to\infty}\|Cx_m-Cx_n+v_{n,m}\|=0.$$

Since $\|u-v\|=\inf\{\|a+b\|:\ u-v=a-b,\ a,b\in X^+\}$, we get $\lim\limits_{n,m\to\infty}\|u_{n,m}-v_{n,m}\|=0$. Thus $\lim\limits_{n,m\to\infty}\|u_{n,m}\| = \lim\limits_{n,m\to\infty}\|v_{n,m}\|=0$. Further $\lim\limits_{n,m\to\infty}\|Cx_n-Cx_m\| = \lim\limits_{n,m\to\infty}\|Cx_n-Cx_m+u_{n,m}-u_{n,m}\| \le \lim\limits_{n,m\to\infty}\|Cx_n-Cx_m+u_{n,m}\|+\|u_{n,m}\|=0$. Thus (Cx_n) is a Cauchy sequence in $(X,\|.\|)$. But X is complete. It follows that $(\overline{R(C)},\|.\|_1)$ is complete. By the Open Mapping Theorem there exists $\delta>0$ such that $\delta\|Cx\|\le\|Cx\|_1$. Since A is C-dispersive,

$$\inf\{\|C(\lambda-A)x+u\|,\ u\in X^+\} \ge \lambda\inf\{\|Cx+u\|,\ u\in X^+\}.$$

Thus $\lambda\|Cx\|_1 \le \|C(\lambda-A)x\|$ and so, $\delta\lambda\|Cx\|\le\|C(\lambda-A)x\|$, and so $C(\lambda-A)$ is one-one.

Since $D(A)$ is dense, it follows that the operator A is closable. To see that: let $y\in X$ and x_n be a sequence in $D(A)$ such that $\lim\limits_{n\to\infty}x_n=0$, $\lim\limits_{n\to\infty}Ax_n=y$. Density of $D(A)$ in X gives a sequence y_n in $D(A)$ such that $\|y_n-y\|\le\frac{1}{n}$. But, since $\delta\lambda\|Cx\|\le\|C(\lambda-A)x\|$, we have $\|C(\lambda x_n+y_m)-AC(x_n+\lambda^{-1}y_m)\| \ge \delta\lambda\|Cx_n+C\lambda^{-1}y_m\| = \delta\|C\lambda x_n+Cy_m\|$. As $n\to\infty$, we have, $\|Cy_m-y-\lambda^{-1}ACy_m)\|\ge\delta\|Cy_m\|$. As $\lambda\to\infty$, we have, $\|Cy_m-Cy\|\ge\delta\|Cy_m\|$. But, $\delta\|Cy_m\|\le\|Cy_m-Cy\|\le\|C\|\|y_m-y\|\le\dfrac{\|C\|}{m}$. Hence, $\|y\|=\lim\limits_{m\to\infty}\|y_m\|=0$, and $y=0$. Consequently, A is closable and the inequality $\delta\lambda\|Cx\|\le\|C(\lambda-\overline{A})x\|$ holds true. Further, $(\lambda-\overline{A})$ is one-one.

Since $(0,\infty)\subseteq\rho(A)$, it follows that $\overline{(\lambda-A)D(A)}=X$ for all $\lambda>0$. Hence, if $y\in X$, there exists a sequence $x_n\in D(A)$ such that $y_n=(\lambda-A)x_n$ converges to y. But $R(\lambda,A)$ is bounded . Thus $x_n=R(\lambda,A)y_n$ is convergent, say to x. Since $\overline{A}$ is closed, it follows that $x\in D(\overline{A})$ and $(\lambda-\overline{A})x=y$. Hence $(\lambda-\overline{A})D(\overline{A})=X$.

Now, let $x\in D(\overline{A})$ and $\varepsilon>0$. Then, by the property of the space and the positivity of the linear operator C, there exists $u,v\in X^+$ such that $\lambda x-\overline{A}x=u-v$. So $\lambda Cx-\overline{A}Cx=Cu-Cv$ and $\|\lambda Cx-\overline{A}Cx\|\ge\|Cu+Cv\|-\varepsilon$. Since $(\lambda-\overline{A})D(\overline{A})=X$, then there exist $g,h\in D(\overline{A})$ such that $u=\lambda g-\overline{A}g$ and $v=\lambda h-\overline{A}h$. Since A is C-dispersive, its closure $\overline{A}$ is C-dispersive too. Consequently, $\inf\{\|C(\lambda-\overline{A})g+b\|,\ b\in X^+\}\ge\lambda\inf\{\|Cg+z\|,\ z\in X^+\}$. But $C(\lambda-\overline{A})g\in X^+$. It follows that $\inf\{\|C(\lambda-\overline{A})g+b\|,\ b\in X^+\}=0$ and so $\inf\{\|Cg+z\|,\ z\in X^+\}=0$ which implies that $Cg\in X^+$. Similarly $Ch\in X^+$. Since

$$(\lambda-\overline{A})Cx=Cu-Cv=(\lambda-\overline{A})Cg-(\lambda-\overline{A})Ch=(\lambda-\overline{A})C(g-h),$$

and $\lambda-\overline{A}$ is one-one, it follows that $Cx=C(g-h)$.

Now $\|\lambda Cx-\overline{A}Cx\|\ge\|Cu+Cv\|-\varepsilon=\|(\lambda-\overline{A})Cg+(\lambda-\overline{A})Ch\|-\varepsilon=\|(\lambda-\overline{A})C(g+h)\|-\varepsilon\ge\lambda\inf\{\|C(g+h)+2b\|,\ b\in X^+\}-\varepsilon\ge\lambda\inf\{\|Cg+b+Ch+b\|,\ b\in X^+\}-\varepsilon$. Since $\|z+b\|\ge\|z-b\|$, $z,b\in X^+$, it follows that $\|\lambda Cx-\overline{A}Cx\|\ge\lambda\|Cg-Ch\|-\varepsilon=\lambda\|Cx\|-\varepsilon$.

Since ε is arbitrary, we conclude that $\|\lambda Cx - \overline{A}Cx\| \geq \lambda\|Cx\|$.

Now for $\lambda > 0$, we have

$$\|Cx\| = \|(\lambda - \overline{A})(\lambda - \overline{A})^{-1}Cx\| = \|C(\lambda - \overline{A})(\lambda - \overline{A})^{-1}x\| \geq \lambda\|(\lambda - \overline{A})^{-1}Cx\|,$$

which implies by Theorem 4.3 ([5]), that $\overline{A}$ generates a contraction C-semigroup.

For $x \in X^+$, we have $\|Cx\| = \|C(\lambda - \overline{A})(\lambda - \overline{A})^{-1}x\|$. But $\overline{A}$ is C-dispersive. Then, $\inf\{\|Cx + b\|,\ b \in X^+\} = \inf\{\|C(\lambda - \overline{A})(\lambda - \overline{A})^{-1}x + b\|,\ b \in X^+\} \geq \lambda \inf\{\|(\lambda - \overline{A})^{-1}Cx + z\|,\ z \in X^+\}$. Positivity of Cx implies that $\inf\{\|Cx + b\|,\ b \in X^+\} = 0$ and so $\inf\{\|(\lambda - \overline{A})^{-1}Cx + z\|,\ z \in X^+\} = 0$. Hence $(\lambda - \overline{A})^{-1}Cx$ is positive and so $(\lambda - \overline{A})^{-1}$ leaves X^+ invariant. Now, $\inf\{\|Cx + b\|,\ b \in X^+\} = \inf\{\|C(\lambda - \overline{A})^2(\lambda - \overline{A})^{-2}x + b\|,\ b \in X^+\}$ $\geq \lambda \inf\{\|(\lambda - \overline{A})(\lambda - \overline{A})^{-2}Cx + z\|,\ z \in X^+\} \geq \lambda^2 \inf\{\|(\lambda - \overline{A})^{-2}Cx + z\|,\ z \in X^+\}$. Since Cx is positive, it follows that $(\lambda - \overline{A})^{-2}$ leaves CX^+ invariant. Using induction, one gets $(\lambda - \overline{A})^{-n}\, CX^+ \subseteq CX^+$. But by Theorem 3.2 ([6]), $T(t)x = \lim_{m\to\infty}(I - \frac{t}{m}\overline{A})^{-m}Cx$. It follows that $T(t)$ is a positive C-semigroup. □

Remark 1.3. One can prove Theorem 1.3 by replacing the condition $\|Cx\| = \inf\{\|Cu + Cv\| : x = u - v,\ u, v \in X^+\}$ by the assumption $\overline{CX^+} = X^+$.

The proof goes as follows.

Proof. Now: let $x \in D(\overline{A})$ and $\varepsilon > 0$. Then by the property of the space and the positivity of the linear operator C, there exists $u, v \in X^+$ such that $\lambda Cx - \overline{A}Cx = u - v$ and $\|\lambda Cx - \overline{A}Cx\| \geq \|u + v\| - \varepsilon$. Since $\overline{CX^+} = X^+$, there exist sequences $(x_n), (y_n)$ in X^+ such that $\lim_{n\to\infty} Cx_n = u$ and $\lim_{n\to\infty} Cy_n = v$. Hence $\lim_{n\to\infty} C(x_n + y_n) = u + v$ and $\lim_{n\to\infty} C(x_n - y_n) = u - v$. Continuity of the norm implies that $\lim_{n\to\infty}\|C(x_n + y_n)\| = \|u + v\|$ and $\lim_{n\to\infty}\|C(x_n - y_n)\| = \|u - v\|$.

Now for $\varepsilon > 0$ there exist N such that for all $n > N$ we have $-\varepsilon < \|C(x_n + y_n)\| - \|u + v\| < \varepsilon$ or $-\varepsilon + \|u + v\| < \|C(x_n + y_n)\| < \|u + v\| + \varepsilon$. Since $(\lambda - \overline{A})D(\overline{A}) = X$, for each $n > N$, there exist $g_n, h_n \in D(\overline{A})$ such that $x_n = \lambda g_n - \overline{A}g_n$ and $y_n = \lambda h_n - \overline{A}h_n$. Since A is C-dispersive, its closure $\overline{A}$ is C-dispersive too, and so for each $n > N$, we have

$$\inf\{\|C(\lambda - \overline{A})g_n + b\|,\ b \in X^+\} \geq \lambda \inf\{\|Cg_n + z\|,\ z \in X^+\}.$$

Since $C(\lambda - \overline{A})g_n \in X^+$, it follows that $\inf\{\|C(\lambda - \overline{A})g_n + b\|,\ b \in X^+\} = 0$ and so $\inf\{\|Cg_n + z\|,\ z \in X^+\} = 0$ which implies that $Cg_n \in X^+$. Similarly $Ch_n \in X^+$. Now

$$(\lambda - \overline{A})Cx = u - v = \lim_{n\to\infty} C(x_n - y_n) = \lim_{n\to\infty}(\lambda - \overline{A})C(g_n - h_n).$$

Since $\lambda - \overline{A}$ is a closed operator, one has

$$(\lambda - \overline{A})Cx = \lim_{n\to\infty}(\lambda - \overline{A})C(g_n - h_n) = (\lambda - \overline{A})\lim_{n\to\infty} C(g_n - h_n).$$

But $\lambda - \overline{A}$ is one-one. It follows that $Cx = \lim_{n\to\infty} C(g_n - h_n)$.

Now $\|\lambda Cx - \overline{A}Cx\| \geq \|u + v\| - \varepsilon \geq \|(\lambda - \overline{A})Cg_n + (\lambda - \overline{A})Ch_n\| - 2\varepsilon = \|(\lambda - \overline{A})C(g_n + h_n)\| - 2\varepsilon \geq \lambda \inf\{\|C(g_n + h_n) + 2b\|,\ b \in X^+\} - 2\varepsilon \geq \lambda \inf\{\|Cg_n + b + Ch_n + b\|,\ b \in X^+\} - 2\varepsilon$. Since $\|z + b\| \geq \|z - b\|,\ z, b \in X^+$, it follows that $\|\lambda Cx - \overline{A}Cx\| \geq \lambda\|Cg_n - Ch_n\| - 2\varepsilon$, and as n goes to infinity we have $\|\lambda Cx - \overline{A}Cx\| \geq \lambda\|Cx\| - 2\varepsilon$. Since ε is arbitrary we conclude that $\|\lambda Cx - \overline{A}Cx\| \geq \lambda\|Cx\|$. Further $\|Cx\| = \|(\lambda - \overline{A})(\lambda - \overline{A})^{-1}Cx\| = \|C(\lambda - \overline{A})(\lambda - \overline{A})^{-1}x\| \geq \lambda\|(\lambda - \overline{A})^{-1}Cx\|$. Using Theorem 4.3 ([5]), we obtain that $\overline{A}$ generates a contraction C-semigroup. The rest of the proof goes as the proof of the theorem. □

II. Compact C-Semigroups. In this section we present necessary conditions and sufficient conditions for a C-semigroup to be compact.

Proposition 2.1. *Let $T(t)$ be a strongly continuous C-semigroup of bounded linear operators on a Banach space X. If $T(t)$ is compact for some $t_0 > 0$, then $CT(t)$ is compact for all $t > t_0$.*

Proof. Suppose $T(t_0)$ is compact for some $t_0 > 0$. Then, $CT(t) = CT(t - t_0 + t_0) = T(t - t_0)T(t_0)$. Thus $CT(t)$ is compact for all $t > t_0$. □

Theorem 2.2. *Let $T(t)$ be a strongly continuous exponentially bounded C-semigroup of bounded linear operators on a Banach space X. If $T(t)$ is compact for all $t > t_0$ for some $t_0 > 0$, then the map $t \to CT(t)$ is continuous from the big in the uniform operator topology for all $t > t_0$.*

Proof. Let $\|T(s)\| \leq M$ for $0 \leq s \leq 1$. Then for $t > t_0$, the set $U_t = \{T(t)x : \|x\| \leq 1\}$ is relatively compact. Now: for $\varepsilon > 0$, the collection $W = \{B\big(T(t)x, \frac{\varepsilon}{2(M+\|C\|)}\big) : \|x\| \leq 1\}$ is an open cover for the compact set $\overline{U_t}$. So W has a finite subcover. Thus there exist $x_1, x_2, ..., x_N$ in X such that $\bigcup_{i=1}^{N} B\big(T(t)x, \frac{\epsilon}{2(M+\|C\|)}\big)$ covers U_t. That means for each $x \in X$; $\|x\| \leq 1$ there exists j (j depends on x), $1 \leq j \leq N$ such that $\|T(t)x - T(t)x_j\| \leq \dfrac{\epsilon}{2(M + \|C\|)}$. But $T(t)$ is strongly continuous. So there exists h_0 , $0 \leq h_0 \leq 1$ such that $\|T(t+h)x_j - T(t)x_j\| \leq \varepsilon/(2\|C\|)$ for $0 \leq h \leq h_0$ and all j, $1 \leq j \leq N$.

Now, for $0 \leq h \leq h_0$, and $x \in X$; $\|x\| \leq 1$, set $J = \|CT(t+h)x - CT(t)x\|$. Then:

$$
\begin{aligned}
J &= \big\|CT(t+h)x - CT(t+h)x_j + CT(t+h)x_j - CT(t)x_j + CT(t)x_j - CT(t)x\big\| \\
&\leq \big\|CT(t+h)x - CT(t+h)x_j\big\| + \big\|CT(t+h)x_j - CT(t)x_j\big\| + \big\|CT(t)x_j - CT(t)x\big\| \\
&\leq \big\|T(t)T(h)x - T(t)T(h)x_j\big\| + \big\|C\big\|\big\|T(t+h)x_j - T(t)x_j\big\| + \big\|C\big\|\big\|T(t)x_j - T(t)x\big\| \\
&\leq \big\|T(h)\big\|\big\|T(t)x - T(t)x_j\big\| + \big\|C\big\|\big\|T(t+h)x_j - T(t)x_j\big\| + \big\|C\big\|\big\|T(t)x_j - T(t)x\big\| \\
&\leq M\frac{\varepsilon}{2(M + \|C\|)} + \big\|C\big\|\frac{\varepsilon}{2\|C\|} + \frac{\|C\|\varepsilon}{2(M + \|C\|)} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus $CT(t)$ is continuous in the uniform operator topology. □

Theorem 2.3. *Let $T(t)$ be a strongly continuous exponentially bounded C-semigroup of bounded linear operators on a Banach space X. If $T(t)$ is compact for some $t_0 > 0$, then the map $t \to C^2T(t)$ is continuous from the big in the uniform operator topology for all $t > t_0$.*

Proof. Since $T(t_0)$ is compact, by Proposition 2.1 it follows that $CT(t)$ is compact for all $t > t_0$. The rest of the proof is similar to the proof of Theorem 2.2 by taking $U_t = \{CT(t)x : \|x\| \leq 1\}$ and $W = \{B\big(CT(t)x, \frac{\varepsilon}{2(M+\|C\|)}\big) : \|x\| \leq 1\}$. □

Corollary 2.4. *Let $T(t)$ be a strongly continuous exponentially bounded C-semigroup such that $T(t)$ is compact for some $t_0 > 0$. If Rang(C) is closed then the map $t \to T(t)$ is continuous from the big in the uniform operator topology.*

Proof. Since $T(t)$ is compact for some $t_0 > 0$, by Theorem 2.3, it follows that $C^2T(t)$ is continuous in the uniform operator topology. Further, since C is a bounded one-one linear operator with a closed range, the Open Mapping Theorem implies that C^{-1} is bounded. Hence $T(t)$ is continuous in the uniform operator topology. □

Theorem 2.5. *Let $T(t)$ be a strongly continuous C-semigroup of bounded linear operators on a Banach space X with generator A such that $\|T(t)\| \leq Me^{\omega t}$. If $T(t)$ is compact for all $t > 0$, then $C^2R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$.*

Proof. Let $\lambda \in \rho(A)$, $\lambda \in R$ and $\lambda > \omega$. Then by Theorem 3.3, ([4]) we have

$$CR(\lambda, A)x = R(\lambda, A)Cx = \int_0^\infty e^{-\lambda s}T(s)x\,ds.$$

Define $R_t(\lambda, A)x = C\int_t^\infty e^{-\lambda s}T(s)xds = \int_t^\infty e^{-\lambda s}CT(s-t+t)xds = \int_t^\infty e^{-\lambda s}T(s-t)T(t)xds$ $= T(t)\int_t^\infty e^{-\lambda s}T(s-t)xds$. Since $T(t)$ is compact and the operator P, $P(x) = \int_t^\infty e^{-\lambda s}T(s-t)x\,ds$ is a bounded operator in $L(X)$ for $\lambda > \omega$ and all $t > 0$, the operators $R_t(\lambda, A)$ is compact for all $t > 0$ and all $\lambda \in R$, $\lambda > \omega \geq 0$. Further,
$\|R_t(\lambda, A) - C^2R(\lambda, A)\| = \|C\int_t^\infty e^{-\lambda s}T(s)\,ds - C\int_0^\infty e^{-\lambda s}T(s)\,ds\| \leq \|\int_0^t e^{-\lambda s}CT(s)\,ds\| \leq M\|C\|\int_0^t e^{(\omega-\lambda)s}\,ds$. Since $\lim\limits_{t\to 0^+} M\|C\|\int_0^t e^{(\omega-\lambda)s}\,ds = 0$, $R_t(\lambda, A)$ is compact for all $t > 0$, it follows that $C^2R(\lambda, A)$ is compact for all $\lambda \in R$, $\lambda > \omega \geq 0$.

Since $C^2R(\lambda, A)$ is compact for $\lambda \in \rho(A)$ and $\lambda > \omega$, the resolvent identity

$$C^2R(\mu, A) = C^2R(\lambda, A) + (\lambda - \mu)C^2R(\lambda, A)R(\mu, A)$$

implies $C^2R(\mu, A)$ is compact that for any $\mu \in \rho(A)$. □

Theorem 2.6. *Let $(T(t))_{t\geq 0}$ be a differentiable strongly continuous C-semigroup on a Banach space X with generator A such that $\|T(t)\| \leq Me^{\omega t}$. If: (i) $R(C)$ is dense in X, (ii) there exists $\lambda_0 \in \rho(A)$ such that $R(\lambda_0, A)$ is compact, (iii) $T(t)$ is uniformly continuous, then $T(t)$ is compact for all $t > 0$.*

Proof. Let $\lambda_0 \in \rho(A)$ and $\lambda_0 = 0$. Define $B(t)x = \int_0^t T(s)x\,ds$. Then $B \in L(X)$ and $AB(t)x = A\int_0^t T(s)x\,ds = T(t)x - Cx = (T(t) - C)x$ for all $x \in D(C)$, Lemma 2.7 ([4]). Hence $-AB(t)x = (0 - A)B(t)x = (C - T(t))x$. So $B(t)x = R(0, A)(C - T(t))x$.
Since $D(A)$ and $R(C)$ are both dense in X, $B(t) = R(0, A)(C-T(t))$. But $R(0, A)$ is compact. So $B(t)$ is compact for all $t > 0$.

Now, since $T(t)$ is uniformly continuous, $B'(t)$ exists and
$B'(t) = \lim\limits_{h\to 0}\frac{1}{h}(B(t + h) - B(t)) = \lim\limits_{n\to\infty} n\big(B\big(t+\frac{1}{n}\big) - B(t)\big) = \lim\limits_{n\to\infty} n\big(R(0, A)\big(C - T\big(t+\frac{1}{n}\big)\big) - R(0, A)(C - T(t))\big) = \lim\limits_{n\to\infty} nR(0, A)\big(T(t) - T\big(t + \frac{1}{n}\big)\big)$.

Define: $D_n(t) = nR(0, A)\big(T(t) - T(t + \frac{1}{n})\big)$. Since $R(0, A)$ is compact, it follows that $D_n(t)$ is compact for all $t > 0$ and all $n \in N$. But $B'(t) = \dfrac{d}{dt}\displaystyle\int_0^t T(s)\,ds = T(t)$, since $T(t)$ is uniformly continuous. Thus $T(t)$ is compact for all $t > 0$.

For $\lambda_0 > 0$, define $S(t) = e^{-\lambda_0 t}T(t)$. Then if A is the generator of $T(t)$, then $A - \lambda_0$ is the generator of $e^{-\lambda_0 t}T(t)$. So if $\lambda_0 \in \rho(A - \lambda_0)$, then $0 \in \rho(A)$. □

Theorem 2.7. *Let $T(t)$ be a strongly continuous C-semigroup of bounded linear operators on a Banach space X with generator A such that $\|T(t)\| \le Me^{\omega t}$. If $R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$ and $T(t)$ is uniformly continuous, then $T(t)$ is compact for all $t > 0$.*

Proof. Since $R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$ and $T(t) \in L(X)$ for all $t > 0$, it follows that $\lambda R(\lambda, A)T(t)$ is compact. But, for $\lambda \in R$ and $\lambda > \omega$, we have $CR(\lambda, A)x = R(\lambda, A)Cx = \int_0^\infty e^{-\lambda s}T(s)x\,ds$. Theorem 3.3 ([4]) now implies (noting that C is injective) $R(\lambda, A) = C^{-1}\int_0^\infty e^{-\lambda s}T(s)\,ds$. Let $J = \|\lambda R(\lambda, A)T(t) - T(t)\|$. Then

$$J = \|\lambda C^{-1}\int_0^\infty e^{-\lambda s}T(s)T(t)\,ds - \lambda\int_0^\infty e^{-\lambda s}T(t)\,ds\| = \|\lambda C^{-1}\int_0^\infty e^{-\lambda s}CT(s+t)\,ds - \lambda\int_0^\infty e^{-\lambda s}T(t)\,ds\|$$

$$= \|\lambda C^{-1}C\int_0^\infty e^{-\lambda s}T(s+t)\,ds - \lambda\int_0^\infty e^{-\lambda s}T(t)\,ds\| = \|\lambda\int_0^\infty e^{-\lambda s}(T(s+t) - T(t))\,ds\|$$

$$\le \lambda\int_0^\infty e^{-\lambda s}\|(T(s+t) - T(t))\|ds = \int_0^b e^{-\lambda s}\|(T(s+t) - T(t))\|ds + \int_b^\infty e^{-\lambda s}\|(T(s+t) - T(t))\|ds$$

$$\le \sup_{0\le s\le b}\|(T(s+t) - T(t))\|\int_0^b e^{-\lambda s}ds + \int_b^\infty e^{-\lambda s}M(e^{\omega(s+t)} + e^{\omega t})ds \le \varepsilon + Me^{\omega t}\left(\frac{e^{(\omega-\lambda)b}}{\lambda-\omega} - \frac{e^{-\lambda b}}{\lambda}\right),$$

which implies $\lim_{\lambda\to\infty}\sup\|\lambda R(\lambda, A)T(t) - T(t)\| \le \varepsilon$ for every $b > 0$. Since b is arbitrary, we get $\lim_{\lambda\to\infty}\|\lambda R(\lambda, A)T(t) - T(t)\| = 0$. Thus $T(t)$ is compact. □

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