УДК 517.5

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SHARP LOGARITHMIC DERIVATIVE ESTIMATES FOR MEROMORPHIC FUNCTIONS

I. E. Chyzhykov. *Sharp logarithmic derivative estimates for meromorphic functions*, Matematychni Studii, **27** (2007) 120–138.

Using recent estimates of the logarithmic derivative of a polynomial due to J.M.Anderson and V.Ya.Eiderman, we prove sharp estimates of the logarithmic derivative $f'(re^{i\theta})/f(re^{i\theta})$ of a meromorphic function f in the plane or in the unit disk outside sets of finite logarithmic measures of values r.

И. Э. Чижиков. *Точные оценки логарифмических производных мероморфных функций* // Математичні Студії. – 2007. – Т.27, №2. – С.120–138.

Используя полученные недавно Дж. М. Андерсоном и В. Я. Эйдерманом оценки логарифмической производной полинома, доказаны точные оценки логарифмической производной $f'(re^{i\theta})/f(re^{i\theta})$ произвольной мероморфной функции f в плоскости или единичном круге вне множеств конечной логарифмической меры значений r.

1. Introduction and main results. We assume that the reader is familiar with standard notation and fundamental results of the theory of meromorphic functions in $\mathbb{C}$ ([1], [2]). We denote by $\operatorname{mes} E$ the Lebesgue measure of a measurable set $E \subset \mathbb{R}$. The symbol C with indices stands for some positive constants that depend on variables going in braces. We write $a(r) \asymp b(r)$ if $C_1 a(r) \le b(r) \le C_2 a(r)$ for some positive constants C_1 and C_2, and $a(r) \sim b(r)$ if $\lim_{r\to R} a(r)/b(r) = 1$, $0 < R \le +\infty$. We write $\arg z$ for the value of $\operatorname{Arg} z$ from the interval $[-\pi, \pi)$.

Let f be a meromorphic function in a domain $D(0, R)$, $0 < R \le +\infty$, where $D(z, r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$. Estimates of the so-called logarithmic derivative $f'(z)/f(z)$ or, more general, of the expression $f^{(k)}(z)/f(z)$, $k \in \mathbb{N}$, play an important role in the Nevanlinna theory [1, 2] and complex differential equations [3], [7], [8], [9]. A sharp upper estimate for $m(r, f'/f)$ via $T(\rho, f)$ where f is meromorphic in $D(0, R)$, $0 < r < \rho < R$ is proved by A. Gol'dberg and V. Grinshtein in [4] (see also [6]). If we estimate $m(r, f'/f)$ via $T(r, f)$, then an exceptional set can appear. Sharp estimates of $m(r, f'/f)$ outside sets of finite logarithmic measure were obtained by J. Miles [5] for the cases $R = \infty$ and $R = 1$ without any restrictions on the growth of a meromorphic function f in $D(0, R)$.

But in differential equations we need uniform estimates of the logarithmic derivative. There are always exceptional sets, except trivial cases, because the logarithmic derivative of f has simple poles at the zeros and poles of f.

2000 *Mathematics Subject Classification:* 30D35.

Keywords: logarithmic derivative, sharp estimate, meromorphic function, unit disk, finite logarithmic measure
doi:10.30970/ms.27.2.120-138

The following theorem is due to G.Gundersen.

Theorem A ([7]). *Let f be a transcendental meromorphic function in $\mathbb{C}$. Let $\alpha > 1$ be a constant, and k and j be integers satisfying $k > j \geq 0$. Let $n_j(r)$ denote the counting function of all the zeros and poles of $f^{(j)}$. Then the following two statements hold:*

(a) There exists a set $E_1 \subset (1, \infty)$ which has finite logarithmic measure, and a constant $C > 0$, such that for all z satisfying $|z| \notin E_1 \cup [0, 1]$, we have (with $r = |z|$)

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq C\left(\frac{T(\alpha r, f)}{r} + \frac{n_j(\alpha r)}{r}(\log r)^\alpha \log^+ n_j(\alpha r)\right)^{k-j}. \tag{1.1}$$

(b) There exists a set $E_2 \subset [0, 2\pi)$ which has linear measure zero, and a constant $C > 0$, such that if $\theta \in [0, 2\pi) \setminus E_2$, then there is a constant $R = R(\theta) > 0$ such that (1.1) holds for all z satisfying $\arg z = \theta$ *and $|z| \geq R$.*

For meromorphic functions of finite order in $\mathbb{C}$ this theorem gives sharp estimates. In the next result, ρ denotes the order of f as a meromorphic function in $\mathbb{C}$.

Corollary B ([7]). *Let f be a transcendental meromorphic function in $\mathbb{C}$ of finite order ρ. Let $\varepsilon > 0$ be a constant, and k and j be integers satisfying $k > j \geq 0$.*

Then the following two statements hold:

(a) There exists a set $E_1 \subset (1, \infty)$ which has finite logarithmic measure, such that for all z satisfying $|z| \notin E_1 \cup [0, 1]$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{1.2}$$

(b) There exists a set $E_2 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_2$, then there is a constant $R = R(\theta) > 0$ such that (1.2) holds for all z satisfying $\arg z = \theta$ *and $|z| \geq R$.*

For other results that have the same nature as Theorem A and the corollary, see [7]. The gamma function gives an example [7, p. 102] which shows that the inequality (1.2) is sharp for both (a) and (b) in Corollary B, because for both (a) and (b), the constant "$\rho - 1 + \varepsilon$" in (1.2) cannot be replaced by "$\rho - 1$." A counterpart of Theorem A for meromorphic functions in the unit disk is proved in [8] (see also [9]).

In connection with Theorem A it is natural to ask the following question.

Question 1. *Is it possible to replace the expression '$\frac{n_j(\alpha r)}{r}(\log r)^\alpha \log^+ n_j(\alpha r)$' with a smaller one?*

Theorem 1 yields an affirmative answer to Question 1.

Theorem 1. *Let f be a non-constant meromorphic function in $\mathbb{C}$. Let $\alpha > 1$ be a constant, ψ a positive non-decreasing function on $(0, +\infty)$ satisfying $\int_0^\infty \frac{dt}{\psi(t)} < +\infty$. Then there exists a sequence of disks $D_j = D(z_j, r_j)$ ($|z_j| > 0$) such that $\sum_j r_j/|z_j| < +\infty$ and*

$$\left|\frac{f'(z)}{f(z)}\right| \leq C(\alpha)\Big(\frac{T(\alpha r, f)}{r} + \frac{n(\alpha r, 0, \infty, f)}{r}\sqrt{\log^+ n(\alpha r, 0, \infty, f)\psi(\log(\alpha r))}\Big), \ z \notin \bigcup_j D_j, \tag{1.3}$$

where $C(\alpha) = 4\alpha(\sqrt{\alpha} - 1)^{-2}$.

Corollary 1. *Suppose that the assumptions of Theorem 1 hold. There exists a set $E \subset [0, 2\pi)$ which has linear measure zero, and a constant $C > 0$, such that if $\theta \in [0, 2\pi) \setminus E$, then there is a constant $R > 0$ such that (1.3) holds for all z satisfying $\arg z = \theta$ and $|z| \ge R$.*

The proof easily follows from the assertion of Theorem 1 in a standard manner (see [7]).

Both addends in the right-hand side of estimate (1.3) are sharp up to a constant factor. For the non-vanishing entire function $g(z) = \exp\{z^\rho\}$, $\rho \in \mathbb{N}$ we have $T(r, g) \asymp r^\rho$, and $\left|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right| \asymp r^{\rho-1}$ as $r \to \infty$. It shows that the first addend in the right-hand side of estimate (1.3) is the best possible.

Theorem 2. *For an arbitrary $\rho \in (0, +\infty)$ and an arbitrary positive non-decreasing function ψ satisfying $\int_0^\infty \frac{dt}{\psi(t)} = +\infty$ there exists an entire function such that $n(r, 0, g) \asymp T(r, g) \asymp r^\rho$ as $r \to +\infty$, and for any covering $\{D_j\}$, $D_j = D(z_j, r_j)$, $z_j \ne 0$ of the set*

$$\Big\{z \in \mathbb{C} : \Big|\frac{g'(z)}{g(z)}\Big| \ge \frac{n(r, 0, g)}{r}\sqrt{\log^+ n(r, 0, g)\psi(\log^+ r)}\Big\} \tag{1.4}$$

we have $\sum_j r_j/|z_j| = +\infty$.

Theorem 1 yields a 'good' estimate for the logarithmic derivative of a meromorphic functions of finite order. If f has infinite order, then $T(\alpha r, f)$ cannot be compared with $T(r, f)$ as $r \to +\infty$.

Question 2. *What is the best possible upper estimate of the logarithmic derivative in the class of meromorphic functions in $D(0, R)$, $0 < R \le \infty$, if $\alpha = 1$?*

In 1993 Sh. Strelitz [10] obtained some estimates when $\alpha = 1$ and $r \to \infty$ outside exceptional sets of various types. Here we confine by exceptional sets of finite logarithmic measure. Our results improve the mentioned estimates of Strelitz in this case.

Theorems 3 and 5 address Question 2.

In order to formulate the next result let us introduce some classes of positive functions which have "non-oscilating" minorants.

Definition 1. Let $A > 0$. We say that a function ψ: $(0, +\infty) \to (0, +\infty)$ belongs to the class $\Psi(A)$ if there exist a constant $t_0 \ge 1$ and a function $\phi(t)$ such that $\psi(t) \ge \phi(t)$ $(t \ge t_0)$, $\epsilon(x) = \phi(x)/x$ is non-decreasing, $\int^\infty \frac{dt}{\phi(t)} < +\infty$ and $E'(t)E(t) \le A$ $(t \ge t_0)$, where $E(t) = \log \epsilon(e^t)$.

We note that the imposed conditions do not restrict the growth of the function ψ.

Theorem 3. *Let f be a meromorphic function in $\mathbb{C}$, $A > 0$, $\psi \in \Psi(A)$, $l \in \mathbb{N}$. Then there exists a measurable set $F \subset \mathbb{R}_+$ such that $\int_F \frac{dr}{r} < +\infty$ and for $|z| \notin F \cup [0, 1]$ we have*

$$\Big|\frac{f^{(l)}(z)}{f(z)}\Big| \le C(A)\Big(\frac{T(r, f)\varepsilon^2(T(r, f))}{r} + \frac{n(r, 0, \infty, f)\varepsilon(n(r, 0, \infty, f))\sqrt{\log^+ n(r, 0, \infty, f)}}{r}\Big)^l, \tag{1.5}$$

where $\varepsilon(t) = \psi(t)/t$, $C(A)$ is a constant.

Corollary 2. *Supppose that the assumptions of Theorem 3 are satisfied. Then there exists a measurable set $F \subset \mathbb{R}_+$ such that $\int_F \frac{dr}{r} < +\infty$ and for $|z| \notin F \cup [0, 1]$ we have*

$$\Big|\frac{f^{(l)}(z)}{f(z)}\Big| \le C(A)\Big(\frac{T(r, f)\varepsilon^2(T(r, f))\sqrt{\log^+ T(r, f)}}{r}\Big)^l, \tag{1.6}$$

where $\varepsilon(t) = \psi(t)/t$, $C(A)$ is a constant.

Remark 1. We note that the choices $\varepsilon(x) = x^p$, $p > 0$ or $\varepsilon(x) = (\log(x+3))^q$, or $\varepsilon(x) = \log(x+3)(\log\log(x+3))^q$, $q > 1$ is always possible.

Remark 2. Under conditions of Corollary 2 for $l = 1$ the corollary from [10, p.133] yields the estimate

$$\frac{|f'(re^{i\theta})|}{|f(re^{i\theta})|} \le C\frac{T(r,f)\log^{3+\eta} T(r,f)\log r \log^{1+\eta}\log r}{r},$$

for arbitrary $\eta > 0$ outside a set of finite logarithmic measure of values r. Choosing $\varepsilon(x) = (\log(x+3))^{1+\eta}$, $\eta > 0$, by Corollary 2 we obtain the estimate

$$\tfrac{|f'(z)|}{|f(z)|} \le \tfrac{C}{r} T(r,f)\log^{5/2+\eta} T(r,f).$$

It looks like the exponent $5/2$ in the last estimate cannot be improved in this situation.

Remark 3. Using methods similar to that used in the proof of Theorem 2, it is possible to construct an entire function f of infinite order such that $|\frac{f'(r)}{f(r)}| \ge \frac{\psi(n(r,0,f))\sqrt{\log n(r,0,f)}}{r}$ on a set of infinite logarithmic measure, where ψ satisfies $\int_1^\infty \frac{dx}{\psi(x)} = +\infty$. But the author cannot obtain a 'good' estimate for $T(r,f)$ in this case.

Our proofs of the theorems are based on the recent deep results by J.M. Anderson and V.Ya.Eiderman [11], [12] (cf. Lemma 6).

Theorem C ([11, 12]). *Let $\mathcal{Z} = \{z_1, z_2, \ldots, z_N\} \subset \mathbb{C}$, $N > 1$. There is an absolute constant c such that for every $P > 0$ there exists a finite set of disks $D_j = D(w_j, r_j)$ with the properties:*

(1) $\left|\sum_{k=1}^N \frac{1}{z-z_k}\right| < P$, $z \in \mathbb{C}\setminus \bigcup_j D_j$; *(2)* $\sum_j r_j < \frac{c}{P} N\sqrt{\log N}$; *(3)* $(\forall j)$ $D_j \cap \mathcal{Z} \ne \varnothing$.

Remark 4. Property (3) from Theorem C is not included in the formulation of the theorem in [11], but it follows from the next arguments communicated to the author by V.Eiderman.

Proof of property (3). Consider the open set $D = \bigcup_j D_j$. It can be represented as a finite union of pairwise disjoint domains Δ_k, $D = \bigsqcup_k \Delta_k$. Take an arbitrary k, then two cases are possible. If $\Delta_k \cap \mathcal{Z} = \varnothing$, then, by Theorem C, estimate (1) holds on $\partial\Delta_k$ and $F(z) = \sum_{n=1}^N \frac{1}{z-z_n}$ is analytic in Δ_k. By the maximum principle $|F(z)| < P$ for $z \in \Delta_k$. Thus, we can exclude Δ_k from the exceptional set. If $\Delta_k \cap \mathcal{Z} \ne \varnothing$, recall that $\Delta_k = \bigcup_{m=1}^{p_k} D_{j_m}$ is a finite connected union of discs. If $p_k = 1$, there is nothing to prove. Otherwise, consider a pair of discs from this union with non-empty intersection. For simplicity assume that these are D_{j_1} and D_{j_2}. Let $\zeta_1 = w_{j_1}$, $\zeta_2 = w_{j_2}$, $\rho_1 = r_{j_1}$, $\rho_2 = r_{j_2}$, $\omega = \frac{\zeta_2-\zeta_1}{|\zeta_2-\zeta_1|}$. Then $D_{j_1} \cup D_{j_2} \subset D(\zeta^*, \rho_1+\rho_2)$, where $\zeta^* = \frac{1}{2}(\zeta_1+\zeta_2+(\rho_2-\rho_1)\omega)$. In fact, let $w \in D_{j_2}$, i.e. $|w-\zeta_2| < \rho_2$. Then

$$|w-\zeta^*| = \left|w-\zeta_2+\frac{\zeta_2-\zeta_1}{2}-\frac{\rho_2-\rho_1}{2}\omega\right| \le |w-\zeta_2| + \left|\frac{|\zeta_2-\zeta_1|}{2}\omega - \frac{\rho_2-\rho_1}{2}\omega\right| <$$
$$< \rho_2 + \frac{1}{2}||\zeta_2-\zeta_1|-(\rho_2-\rho_1)|.$$

Since $D_{j_1}\cap D_{j_2} \ne \varnothing$, we have $||\zeta_2-\zeta_1|-\rho_2| \le \rho_1$ by the triangle inequality. The latter two inequalities give $|w-\zeta^*| < \rho_2+\rho_1$, i.e. $w \in D(\zeta^*, \rho_1+\rho_2)$.

Repeating this procedure at most $p_k - 1$ times we obtain a disc $D(\zeta^k, \rho^k) \supset \Delta_k$, where $\rho^k = \sum_{m=1}^{p_k} r_{j_m}$ It means that we can enlarge the exceptional set changing Δ_k by the disc $D(\zeta^k, \rho^k)$ and preserving the properties (1) and (2) from Theorem C. By the construction $D(\zeta^k, \rho^k) \cap \mathcal{Z} \ne \varnothing$. □

Similar assertions to Theorems 1 and 3 can be proved for the logarithmic derivative of the meromorphic functions in the unit disk. Let $\sigma[f] = \varliminf_{r\uparrow 1} \dfrac{\log^+ T(r,f)}{-\log(1-r)}$ be the lower order of a meromorphic function f in the unit disk.

Theorem 4. *Let f be a meromorphic function in $\mathbb{D}$, $l \in \mathbb{N}$. Let $\beta \in (0,1)$ be a constant, ψ a positive non-decreasing function on $(0,+\infty)$ satisfying $\int_0^\infty \frac{dt}{\psi(t)} < +\infty$. Then there exists a sequence of disks $D_j = D(z_j, r_j)$ $(1 > |z_j| > 0)$ such that $\sum_j r_j/(1-|z_j|) < +\infty$ and*

$$\Big|\frac{f^{(l)}(z)}{f(z)}\Big| \le C(l,\beta)\Big(\frac{T(1-\beta(1-r),f)}{(1-r)^2}+$$
$$+\frac{n(1-\beta(1-r),0,\infty,f)}{1-r}\sqrt{\log^+ n(1-\beta(1-r),0,\infty,f)}\,\psi\Big(\log\frac{1}{\beta(1-r)}\Big)\Big)^l, \quad z \notin \bigcup_j D_j. \tag{1.7}$$

Corollary 3. *Suppose that the assumptions of Theorem 4 hold. There exists a set $E \subset [0,2\pi)$ which has linear measure zero, and a constant $C > 0$, such that if $\theta \in [0,2\pi) \setminus E$, then there is a constant $R = R(\theta) \in (0,1)$ such that for all z satisfying $\arg z = \theta$ and $R \le |z| < 1$, we have that (1.7) holds.*

The proof of the corollary is standard and similar to that in [8].

Definition 2. Let $A > 0$. We say that a function $\psi\colon (0,+\infty) \to (0,+\infty)$ belongs to the class $\Phi(A)$ if there exist a constant $t_0 \ge 1$ and a function $\phi(t)$ such that $\psi(t) \ge \phi(t)$ $(t \ge t_0)$, $\epsilon(x) = \phi(x)/x \ge 1$ is non-decreasing, $\int^\infty \frac{dt}{\phi(t)} < +\infty$ and $\mathcal{E}'(t) \le A$ $(t \ge t_0)$, where $\mathcal{E}(t) = \log \epsilon(e^{e^t})$.

Theorem 5. *Let f be a meromorphic function in $\mathbb{D}$, $\sigma[f] > 0$, $l \in \mathbb{N}$, $\varepsilon\colon (0,+\infty) \to (0,+\infty)$ such that $\psi(x) = x\varepsilon(x) \in \Phi(1)$. Then there exists a measurable set $F \subset [0,1)$ such that $\int_F \frac{dr}{1-r} < +\infty$ and*

$$\Big|\frac{f^{(l)}(z)}{f(z)}\Big| \le C(l)\Big(\frac{T(r,f)(\varepsilon(T(r,f)))^2}{(1-r)^2}+$$
$$+\frac{n(r,0,\infty,f)\varepsilon(n(r,0,\infty,f))\sqrt{\log^+ n(r,0,\infty,f)}}{1-r}\Big)^l, \quad r\uparrow 1,\ r \notin F. \tag{1.8}$$

Remark 5. It is plausible that Theorem 4 is best possible in the same sense that Theorem 2.

2. Preliminaries. Let f be a meromorphic function in $D(0,R_0)$, $0 < R_0 \le \infty$, $\{a_\mu\}$ and $\{b_\nu\}$ denote the sequences of all zeros and poles of f, respectively. We have the following representation [1, Theorem 2.4, p.17] $(|z| = r < R)$.

$$\Big(\frac{d}{dz}\Big)^{k-1}\frac{f'(z)}{f(z)} = \frac{k!}{\pi}\int_0^{2\pi}\frac{\log|f(Re^{i\theta})|Re^{i\theta}\,d\theta}{(Re^{i\theta}-z)^{k+1}}-$$
$$-(k-1)!\sum_{|a_\mu|<R}\Big(\frac{1}{(a_\mu-z)^k}-\frac{(\overline{a}_\mu)^k}{(R^2-\overline{a}_\mu z)^k}\Big)+(k-1)!\sum_{|b_\nu|<R}\Big(\frac{1}{(b_\nu-z)^k}-\frac{(\overline{b}_\nu)^k}{(R^2-\overline{b}_\nu z)^k}\Big). \tag{2.1}$$

Hence for $k = 1$ we have

$$\Big|\frac{f'(z)}{f(z)}\Big| \le \frac{R}{\pi}\int_0^{2\pi} \frac{|\log|f(Re^{i\theta})||\,d\theta}{|Re^{i\theta}-z|^2} + \Big|\sum_{|a_\mu|<R} \frac{1}{z-a_\mu}\Big| + \Big|\sum_{|b_\nu|<R} \frac{1}{z-b_\nu}\Big| +$$
$$+\Big|\sum_{|a_\mu|<R} \frac{1}{z-\frac{R^2}{\bar a_\mu}}\Big| + \Big|\sum_{|b_\nu|<R} \frac{1}{z-\frac{R^2}{\bar b_\nu}}\Big| \equiv I + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4. \tag{2.2}$$

The estimate of the integral I is standard:

$$I \le \frac{R}{\pi(R-r)^2}\int_0^{2\pi}\Big(\log^+|f(Re^{i\theta})| + \log^+\Big|\frac{1}{f(Re^{i\theta})}\Big|\Big)\,d\theta \le$$
$$\le \frac{2R}{(R-r)^2}(m(R,f)+m(R,1/f)) \le \frac{4R}{(R-r)^2}(T(R,f)+O(1)), \quad R\uparrow R_0. \tag{2.3}$$

In order to estimate Σ_j, $j\in\{1,2,3,4\}$ we use Theorem C, which plays a key role in our proof.

Lemma 1. *Let (α_ν) be a sequence of positive numbers increasing to $R_0 \in (0,+\infty]$, φ a non-decreasing positive function, (c_ν) a sequence of complex numbers in $D(0,R_0)$ without accumulation points in $D(0,R_0)$ listed according the multiplicities and ordered by increasing moduli and let $n(r)$ denote the counting function of the sequence (c_ν). Then for every $\nu\in\mathbb{N}$ and $\alpha_{\nu-1} < |z| \le \alpha_\nu$ we have*

$$\Big|\sum_{|c_\mu|\le\alpha_{\nu+1}} \frac{1}{z-c_\mu}\Big| + \Big|\sum_{|c_\mu|\le\alpha_{\nu+1}} \frac{1}{z-\frac{\alpha_{\nu+1}^2}{\bar c_\mu}}\Big| \le$$
$$\le \frac{2\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n(\alpha_{\nu+1})\sqrt{\log^+ n(\alpha_{\nu+1})} + \frac{2n(\alpha_{\nu-2})}{\alpha_{\nu-1}-\alpha_{\nu-2}}, \quad z\notin \bigcup_j D(z_{\nu j}, r_{\nu j}) \tag{2.4}$$

where

$$\alpha_{\nu-2} - \frac{c\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})} \le |z_{\nu j}| \le \frac{\alpha_{\nu+1}^2}{\alpha_{\nu-2}} + \frac{c\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}, \tag{2.5}$$

c is an absolute constant from Theorem C, $\sum_j r_{\nu j} \le \frac{2c\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}\frac{n(\alpha_{\nu+1})-n(\alpha_{\nu-2})}{n(\alpha_{\nu+1})}$.

Proof of the lemma. We write $A_\nu = \{\zeta : \alpha_{\nu-1} < |\zeta| \le \alpha_\nu\}$. Estimate the first sum in (2.4). First, we suppose that $n(\alpha_{\nu+1}) - n(\alpha_{\nu-2}) > 1$. We set

$$P_\nu = \frac{\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n(\alpha_{\nu+1})\sqrt{\log^+ n(\alpha_{\nu+1})}. \tag{2.6}$$

Applying Theorem C to the set of $\{c_\mu\}$ that are contained in $A_{\nu-1}\cup A_\nu\cup A_{\nu+1}$ we conclude that there exist disks $D_{\nu j} = D(z_{\nu j}, r_{\nu j})$, $1\le j\le j_\nu$ such that

$$\Big|\sum_{\alpha_{\nu-2}<|c_\mu|\le\alpha_{\nu+1}} \frac{1}{z-c_\mu}\Big| \le \frac{\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n(\alpha_{\nu+1})\sqrt{\log^+ n(\alpha_{\nu+1})}, \quad z\in A_\nu \setminus \bigcup_j D_{\nu j} \tag{2.7}$$

$$\sum_{j=1}^{j_\nu} r_{\nu j} \le c\frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}\frac{n(\alpha_{\nu+1})-n(\alpha_{\nu-2})}{n(\alpha_{\nu+1})}, \tag{2.8}$$

$$\alpha_{\nu-2} - c\frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})} \le |z_{\nu j}| \le \alpha_{\nu+1} + c\frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}. \tag{2.9}$$

On the other hand, for $z \in A_\nu$

$$\Big|\sum_{|c_\mu|\le \alpha_{\nu-2}} \frac{1}{z-c_\mu}\Big| \le \frac{n(\alpha_{\nu-2})}{\alpha_{\nu-1}-\alpha_{\nu-2}}. \tag{2.10}$$

Hence, using (2.7) we obtain

$$\Big|\sum_{|c_\mu|\le \alpha_{\nu+1}} \frac{1}{z-c_\mu}\Big| \le \frac{\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n(\alpha_{\nu+1})\sqrt{\log^+ n(\alpha_{\nu+1})} + \frac{n(\alpha_{\nu-2})}{\alpha_{\nu-1}-\alpha_{\nu-2}}. \tag{2.11}$$

Consider the second sum in (2.4). If $|c_\mu| \in [\alpha_{\nu-2}, \alpha_{\nu+1}]$, then $\alpha_{\nu+1} \le |c_\mu^*| \le \frac{\alpha_{\nu+1}^2}{\alpha_{\nu-2}}$, where $c_\mu^* = \alpha_{\nu+1}^2/\overline{c}_\mu$. Applying Theorem C to the set of all those $\{c_\mu^*\}$ such that $c_\mu \in A_{\nu-1}\cup A_\nu \cup A_{\nu+1}$ with $P = P_\nu$ we obtain a finite collection of disks $\widetilde{D}_{\nu j} = D(\widetilde{z}_{\nu j}, \widetilde{r}_{\nu j})$ $(1 \le j \le \widetilde{j}_\nu)$ such that

$$\Big|\sum_{\alpha_{\nu-2}<|c_\mu|\le \alpha_{\nu+1}} \frac{1}{z-\frac{\alpha_{\nu+1}^2}{\overline{c}_\mu}}\Big| \le \frac{2\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n(\alpha_{\nu+1})\sqrt{\log^+ n(\alpha_{\nu+1})}, \quad z \in A_\nu \setminus \bigcup_j \widetilde{D}_{\nu j}, \tag{2.12}$$

$$\sum_{j=1}^{j_\nu} \widetilde{r}_{\nu j} \le c\frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}\frac{n(\alpha_{\nu+1})-n(\alpha_{\nu-2})}{n(\alpha_{\nu+1})}, \tag{2.13}$$

$$\alpha_{\nu+1} - c\frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})} \le |\widetilde{z}_{\nu j}| \le \frac{\alpha_{\nu+1}^2}{\alpha_{\nu-2}} + c\frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}. \tag{2.14}$$

Similarly to (2.10) we have

$$\Big|\sum_{|c_\mu|\le \alpha_{\nu-2}} \frac{1}{z-\frac{\alpha_{\nu+1}^2}{\overline{c}_\mu}}\Big| \le \frac{n(\alpha_{\nu-2})}{\frac{\alpha_{\nu+1}^2}{\alpha_{\nu-2}}-\alpha_\nu}. \tag{2.15}$$

From the last inequality and (2.12) we obtain

$$\Big|\sum_{|c_\mu|\le \alpha_{\nu+1}} \frac{1}{z-\frac{\alpha_{\nu+1}^2}{\overline{c}_\mu}}\Big| \le \frac{\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n(\alpha_{\nu+1})\sqrt{\log^+ n(\alpha_{\nu+1})} + \frac{n(\alpha_{\nu-2})}{\frac{\alpha_{\nu+1}^2}{\alpha_{\nu-2}}-\alpha_\nu}. \tag{2.16}$$

Relationships (2.7), (2.11), (2.12), (2.16) yield

$$\Big|\sum_{|c_\mu|\le \alpha_{\nu+1}} \frac{1}{z-c_\mu}\Big| + \Big|\sum_{|c_\mu|\le \alpha_{\nu+1}} \frac{1}{z-\frac{\alpha_{\nu+1}^2}{\overline{c}_\mu}}\Big| \le$$

$$\le \frac{2\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n(\alpha_{\nu+1})\sqrt{\log^+ n(\alpha_{\nu+1})} + \frac{2n(\alpha_{\nu-2})}{\alpha_{\nu-1}-\alpha_{\nu-2}}, \quad z \in A_\nu \setminus \Big(\bigcup_{j=1}^{j_\nu} D_{\nu j} \cup \bigcup_{j=1}^{\widetilde{j}_\nu} \widetilde{D}_{\nu j}\Big), \tag{2.17}$$

because

$$\tfrac{\alpha_{\nu+1}^2}{\alpha_{\nu-2}} - \alpha_\nu \ge \alpha_\nu\Big(\tfrac{\alpha_\nu}{\alpha_{\nu-2}} - 1\Big) \ge \alpha_\nu - \alpha_{\nu-2} > \alpha_{\nu-1} - \alpha_{\nu-2}.$$

Moreover, according to (2.9) and (2.14), every $z_{\nu j}$, $1 \le j \le j_\nu$ and $\widetilde{z}_{\nu j}$, $1 \le j \le \widetilde{j}_\nu$ satisfies (2.5). By (2.8) and (2.13)

$$\sum_{j=1}^{j_\nu} r_{j\nu} + \sum_{j=1}^{\widetilde{j}_\nu} \widetilde{r}_{j\nu} \le \frac{2c\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}\frac{n(\alpha_{\nu+1})-n(\alpha_{\nu-2})}{n(\alpha_{\nu+1})}. \tag{2.18}$$

And the lemma is proved when $n(\alpha_{\nu+1}) - n(\alpha_{\nu-2}) > 1$.

If $n(\alpha_{\nu+1}) - n(\alpha_{\nu-2}) = 0$ inequalities (2.7) and (2.12) are trivial, and using (2.10) and (2.15) we obtain (2.4) without exceptional sets. If $n(\alpha_{\nu+1}) - n(\alpha_{\nu-2}) = 1$, we have $|z - c_\mu|^{-1} \le \varphi(\alpha_{\nu+1}) n(\alpha_{\nu+1})/\alpha_{\nu+1}$ and $|z - c_\mu^*|^{-1} \le \varphi(\alpha_{\nu+1}) n(\alpha_{\nu+1})/\alpha_{\nu+1}$ outside $D(c_\mu, \frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1}) n(\alpha_{\nu+1})}) \cup D(c_\mu^*, \frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1}) n(\alpha_{\nu+1})})$ for the unique $c_\mu \in A_{\nu-1} \cup A_\nu \cup A_{\nu+1}$. Therefore (2.7)–(2.9) and (2.12)–(2.14) hold. Consequently we obtain the statement of Lemma 1. □

The following two lemmas justify the definitions of the classes $\Psi(A)$ and $\Phi(A)$.

Lemma 2. *Let $A > 0$, $B > 0$, $\psi\colon (0, +\infty) \to (0, +\infty)$. Let $\int_0^\infty \frac{dt}{\psi(t)} < +\infty$ and there exist a constant $t_0 \ge 1$ such that $\epsilon(x) = \psi(x)/x$ is non-decreasing, and $E'(t)E(t) \le A$ $(t \ge t_0)$, where $E(t) = \log \epsilon(e^t)$. Then:*

(i) $\epsilon(BT\epsilon(T)) \le (e^A + o(1))\epsilon(T)$, as $T \to +\infty$; (ii) $\epsilon(BT) = (1 + o(1))\epsilon(T)$ as $T \to +\infty$.

Proof. Under our assumptions $E(t)$ is non-decreasing unbounded and $E'(t)E(t) \le A$ for all sufficiently large t. We put $R = \log T$. Let T be such that $\log B + E(R) > 0$. Then $\epsilon(BT\epsilon(T))/\epsilon(T) = \exp\Big\{\log \epsilon(BTe^{\log \epsilon(T)}) - \log \epsilon(T)\Big\} = \exp\Big\{E(R + \log B + E(R)) - E(R)\Big\} = \exp\{E'(\xi)(E(R) + \log B)\}$ where $\xi \in (R, R + E(R) + \log B)$ by the Largange theorem. Now, by the definition of $\Psi(A)$ we obtain $\epsilon(BT\epsilon(T))/\epsilon(T) \le \exp\Big\{\frac{A}{E(\xi)}(E(R) + \log B)\Big\} \le \exp\Big\{A + \frac{A \log B}{E(R)}\Big\} = e^A(1 + o(1))$, $r \to +\infty$, and (i) is proved. Part (ii) can be proved similarly. □

The next two lemmas concern the unit disk case.

Lemma 3. *Let $A_1, A_2 > 0$, $B > 0$. Let $\psi\colon (1, +\infty) \to (0, +\infty)$ satisfy $\int_0^\infty \frac{dt}{\psi(t)} < +\infty$ and there exist a constant $t_0 \ge 1$ such that $\varepsilon(x) = \psi(x)/x$ is non-decreasing, and $\mathcal{E}'(t) \le A_1$ $(t \ge t_0)$, where $\mathcal{E}(t) = \log \epsilon(e^{e^t})$. Let $b\colon (0, +\infty) \to (0, +\infty)$ satisfy $\log b(t) \le A_2 \log t$, $t \ge t_0$. Then*

$$\varepsilon(b(T)T) \le (1 + A_2)^{A_1} \varepsilon(T), \quad T \to +\infty.$$

The proof is similar to that of Lemma 2.

Lemma 4. *Let $v\colon [r_0, 1) \to [v_0, +\infty)$ be a non-decreasing unbounded function, let $\varepsilon\colon [v_0, +\infty) \to [1, +\infty)$ be a non-decreasing function such that $\int_{v_0}^\infty \frac{dv}{v\varepsilon(v)} < +\infty$. Then*

$$v\Big(r \exp\Big\{\frac{1-r}{\varepsilon(v(r))}\Big\}\Big) \le ev(r), \quad r \notin E \subset [0, 1), \quad \int_E \frac{dr}{1-r} < +\infty.$$

Proof. We use the following generalization of the classical Borel-Nevanlinna Theorem.

Theorem D. *Let $u\colon [r_0, +\infty) \to \mathbb{R}$ be a non-decreasing unbounded function, $(u_0 = u(r_0))$, $\varphi\colon [u_0, +\infty) \to \mathbb{R}$ be a non-increasing function such that $\varphi(u) \to 0$ as $u \to +\infty$ and $\int_{u_0}^\infty \varphi(u)\, du < +\infty$. Then for all $r \ge r_0$ except, possibly, a set of finite measure*

$$u\big(r + \varphi(u(r))\big) < u(r - 0) + 1.$$

In the case, when u and φ are continuous, this theorem is well-known ([1, 3]). The general case can be proved using similar arguments (see [14, Theorem 1.1, p.7], [3] for details.)

Following J.Miles [5] we write $1 + e^x = \frac{1}{1-r}$, i.e. $r = \frac{e^x}{e^x + 1}$,

$$u(x) \stackrel{\text{def}}{=} \log v\Big(\frac{e^x}{e^x + 1}\Big) = \log v(r) > 0$$

for $x \ge x_0 \log \frac{r_0}{1-r_0}$, provided that $v(r) > 1$ for $r \ge r_0$.

Since $\varepsilon(x)$ is non-decreasing we can choose a continuous function $\varepsilon_1\colon [v_0, +\infty) \to [1, +\infty)$ such that $\varepsilon_1(x) \le \varepsilon(x)/2$, $x \ge v_0$ and still $\int_{v_0}^{\infty} \frac{dv}{v\varepsilon_1(v)} < +\infty$.

We define $\beta(u) = \varepsilon_1(e^u)$. Then

$$\int_{u(x_0)}^{\infty} \frac{du}{\beta(u)} = \int_{e^{u(x_0)}}^{\infty} \frac{ds}{s\varepsilon(s)} < \infty.$$

By Theorem D

$$u\Big(x + \frac{1}{\beta(u(x))}\Big) \le u(x) + 1, \quad x \notin F \tag{2.19}$$

where $F \subset [0, +\infty)$ is of finite measure. Let $E = \{r = \frac{e^x}{e^x+1} : x \in F\}$. Then

$$\operatorname{mes} F = \int_{F\cap[x_0,\infty)} dx = \int_{E\cap[r_0,1)} \Big(\frac{1}{r} + \frac{1}{1-r}\Big) dr < +\infty,$$

i.e. E is of finite logarithmic measure on $[0, 1)$. Since $\beta(u(x)) = \varepsilon_1(v(r))$, (2.19) is equivalent to $\log v\Big(\frac{\exp\{x+1/\beta(u(x))\}}{\exp\{x+1/\beta(u(x))\}+1}\Big) \le \log v\Big(\frac{e^x}{e^x+1}\Big) + 1$, $\quad x \notin F$, or

$$v\Big(1 - \frac{1}{1 + \frac{r}{1-r} e^{\frac{1}{\varepsilon(v(r))}}}\Big) \le ev(r), \quad r \in [0, 1) \setminus E.$$

But as $r \uparrow 1$

$$1 - \frac{1}{1 + \frac{r}{1-r} e^{\frac{1}{\varepsilon(v(r))}}} = r\Big(1 + \frac{(1-r)(e^{\frac{1}{\varepsilon_1(v(r))}} - 1)}{1 + r(e^{\frac{1}{\varepsilon_1(v(r))}} - 1)}\Big) = r\Big(1 + \frac{(1 + o(1))(1-r)}{\varepsilon_1(v(r))}\Big).$$

Therefore, $v\Big(r \exp\Big\{\frac{1-r}{\varepsilon(v(r))}\Big\}\Big) \le ev(r)$ for $r \ge r_1$, $r \notin E$, $r_1 \in [0, 1)$ as required. □

3. Proofs of the main results.

3.1. The plane case.

Proof of Theorem 1. Let us estimate $\sum_j$, $1 \le j \le 4$ from (2.2). We may suppose that $n(r, 0, f)$ and $n(r, \infty, f)$ are unbounded. Otherwise, corresponding sums are bounded as $R \to +\infty$.

Given $\alpha > 1$ we put $\alpha_\nu = \sqrt{\alpha^\nu}$, $A_\nu = \{\zeta : \alpha_{\nu-1} < |\zeta| \le \alpha_\nu\}$, $\nu \in \mathbb{N}$, $\varphi(t) = \psi(\log^+ t)$. Fixing $\nu \in \mathbb{N}$ we apply Lemma 1 to the zero set $\{a_\mu\}$ of f, which lay in $A_{\nu-1} \cup A_\nu \cup A_{\nu+1}$, $R_0 = \infty$. There exists a finite collection of disks $D_{\nu j} = D(z_{\nu j}, r_{\nu j})$ $(1 \le j \le j_\nu)$ such that

$$\Big|\sum_{|a_\mu|\le\alpha_{\nu+1}} \frac{1}{z - a_\mu}\Big| + \Big|\sum_{|a_\mu|\le\alpha_{\nu+1}} \frac{1}{z - \frac{\alpha_{\nu+1}^2}{\bar{a}_\mu}}\Big| \le \frac{2n(\alpha_{\nu-2}, 0, f)}{\alpha_{\nu-1} - \alpha_{\nu-2}} +$$
$$+ \frac{2\psi(\log \alpha_{\nu+1})}{\alpha_{\nu+1}} n(\alpha_{\nu+1}, 0, f)\sqrt{\log^+ n(\alpha_{\nu+1}, 0, f)} \le$$
$$\le \frac{2\psi(\log(\alpha r))}{\sqrt{\alpha} r} n(\alpha r, 0, f)\sqrt{\log^+ n(\alpha r, 0, f)} + \frac{2n(\alpha r, 0, f)}{r(1 - \sqrt{\alpha})}, \quad z \in A_\nu \setminus \bigcup_{j=1}^{j_\nu} D_{\nu j}.$$

Similarly, there exists a finite collection of disks $D^*_{\nu j} = D(z^*_{\nu j}, r^*_{\nu j})$ $(1 \le j \le j^*_\nu)$ such that

$$\Big|\sum_{|b_\mu|\le\alpha_{\nu+1}} \frac{1}{z-b_\mu}\Big| + \Big|\sum_{|b_\mu|\le\alpha_{\nu+1}} \frac{1}{z-\frac{\alpha^2_{\nu+1}}{\bar b_\mu}}\Big| \le \frac{2n(\alpha_{\nu-2},\infty,f)}{\alpha_{\nu-1}-\alpha_{\nu-2}} + $$
$$+\frac{2\psi(\log\alpha_{\nu+1})}{\alpha_{\nu+1}} n(\alpha_{\nu+1},\infty,f)\sqrt{\log^+ n(\alpha_{\nu+1},\infty,f)} \le$$
$$\le \frac{2\psi(\log(\alpha r))}{\sqrt{\alpha} r} n(\alpha r,\infty,f)\sqrt{\log^+ n(\alpha r,\infty,f)} + \frac{2n(\alpha r,\infty,f)}{r(1-\sqrt{\alpha})}, \quad z\in A_\nu \setminus \bigcup_{j=1}^{j^*_\nu} D^*_{\nu j}.$$

Therefore

$$|\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4| \le \frac{2n(\alpha r,0,\infty,f)}{r}\sqrt{\log^+ n(\alpha r,0,\infty,f)}\psi(\log(\alpha r)). \tag{3.1}$$

By (2.3) $|I| \le \frac{4\alpha_{\nu+1}}{(\alpha_{\nu+1}-r)^2}(T(\alpha_{\nu+1},f)+O(1)) \le \frac{4\alpha}{(\sqrt{\alpha}-1)^2 r}(T(\alpha r,f)+O(1))$. Substituting the latter estimates in (2.2) we obatin ($|z|=r$)

$$\Big|\frac{f'(z)}{f(z)}\Big| \le \frac{4\alpha}{(\sqrt{\alpha}-1)^2}\Big(\frac{T(\alpha r,f)}{r} + \frac{n(\alpha r,0,\infty,f)}{r}\sqrt{\log^+ n(\alpha r,0,\infty,f)}\psi(\log(\alpha r))\Big), \tag{3.2}$$

$z\in\mathbb{C}\setminus\Omega$, where $\Omega = \bigcup_{\nu=1}^\infty\Big(\bigcup_{j=1}^{j_\nu} D_{\nu j} \cup \bigcup_{j=1}^{j^*_\nu} D^*_{\nu j}\Big)$. It remains to estimates the exceptional set Ω. By Lemma 1 we have

$$\alpha_{\nu+1}\Big(1/\sqrt{\alpha^3} - c/\psi(\log(\alpha_{\nu+1}))\Big) \le |z_{\nu j}| \le \alpha_{\nu+1}\Big((\sqrt{\alpha^3} + c/\psi(\log(\alpha_{\nu+1}))\Big).$$

Thus, $\alpha^{-2} \le |z_{\nu j}|/\alpha_{\nu+1} \le \alpha^2$ for all ν greater than some ν_1. Hence,

$$\sum_{\nu=\nu_1}^\infty\sum_{j=1}^{j_\nu}\frac{r_{\nu j}}{|z_{\nu j}|} \le \sum_{\nu=\nu_1}^\infty\sum_{j=1}^{j_\nu}\frac{\alpha^2 r_{\nu j}}{\alpha_{\nu+1}} \le \sum_{\nu=1}^\infty \frac{2c\alpha^2}{\psi(\log(\alpha_{\nu+1}))} = \sum_{\nu=1}^\infty\frac{2c\alpha^2}{\psi(\frac{\nu}{2}\log(\alpha))} < \infty.$$

Similarly, $\sum_{\nu=1}^\infty\sum_{j=1}^{j^*_\nu} r^*_{\nu j}/|z^*_{\nu j}| < \infty$. Theorem 1 is proved. □

Proof of Theorem 3. Let $A>0$, $\psi\in\Psi(A)$, $\phi(t)$ is a corresponding function from the definition of the class $\Psi(A)$, $\epsilon(t)=\phi(t)/t$. We shall write $T(r)$ and $n(r,a)$ instead of $T(r,f)$ and $n(r,a,f)$, respectively, and $n(r)$ instead of $n(r,0,\infty,f)$.

It is sufficient to prove (1.5) for ϵ instead of ε. We define a sequence (α_ν) by the induction. Let α_0 be such that $\min\{n(\alpha_0,0), n(\alpha_0,\infty)\}\ge 1$,

$$\alpha_{\nu+1} = \alpha_\nu \exp\Big\{\frac{1}{3e^A\epsilon(T(\frac{\alpha^2_{\nu+1}}{\alpha_\nu}))}\Big\}, \quad \nu\in\mathbb{Z}_+. \tag{3.3}$$

The definition is correct, because for $h_1(x)=x$, $h_2(x)=\alpha_\nu\exp\Big\{\frac{1}{3e^A\epsilon(T(\frac{x^2}{\alpha_\nu}))}\Big\}$, $\nu\in\mathbb{Z}_+$, $h_1(\alpha_\nu)<h_2(\alpha_\nu)$, $h_1(x)\uparrow+\infty$, $h_2(x)\downarrow 0$ as $x\to+\infty$. Therefore, there exists the unique $\alpha_{\nu+1}$ such that $h_1(\alpha_{\nu+1})=h_2(\alpha_{\nu+1})$. Moreover, from (3.3) it follows that $\alpha_\nu\uparrow+\infty$ $(\nu\to+\infty)$.

Without loss of generality we may assume that $f(0)=1$. Then, it is well-known that for $\Delta>0$, $\rho>0$, $\rho'=\rho e^\Delta$, $n(\rho,a)\le T(\rho')/\Delta$ for $a\in\{0,\infty\}$. We choose $\rho=\alpha_{\nu+1}$, $\rho'=\alpha^2_{\nu+1}/\alpha_\nu$, and consequently, $\Delta = 1/(3e^A\epsilon(T(\alpha^2_{\nu+1}/\alpha_\nu)))$. Then

$$n(\alpha_{\nu+1},a) \le 3e^A T\Big(\frac{\alpha^2_{\nu+1}}{\alpha_\nu}\Big)\epsilon\Big(T\Big(\frac{\alpha^2_{\nu+1}}{\alpha_\nu}\Big)\Big), \quad a\in\{0,\infty\}. \tag{3.4}$$

Applying Lemma 2 with $B=3e^A$, we obtain ($a\in\{0,\infty\}$)

$$\epsilon(n(\alpha_{\nu+1},a))\le\epsilon\Big(3e^AT\Big(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}\Big)\epsilon\Big(T\Big(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}\Big)\Big)\Big)\le(e^A+o(1))\epsilon\Big(T\Big(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}\Big)\Big),\quad \nu\to+\infty\,. \tag{3.5}$$

Let $\alpha_{\nu-1}<r\le\alpha_\nu$, and ν is sufficiently large. Using (3.5) and the definition of α_ν we deduce

$$n(\alpha_{\nu+1},a)=n\Big(\alpha_\nu\exp\Big\{\frac{1}{3e^A\epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}\Big\},a\Big)\le n\Big(\alpha_\nu\exp\Big\{\frac{1+o(1)}{3\epsilon(n(\alpha_{\nu+1},a))}\Big\},a\Big)\le$$
$$\le n\Big(\alpha_{\nu-1}\exp\Big\{\frac{1}{3e^A\epsilon(T(\frac{\alpha_\nu^2}{\alpha_{\nu-1}}))}+\frac{1+o(1)}{3\epsilon(n(\alpha_{\nu+1},a))}\Big\},a\Big)\le$$
$$\le n\Big(\alpha_{\nu-1}\exp\Big\{\frac{1+o(1)}{3\epsilon(n(\alpha_\nu,a))}+\frac{1+o(1)}{3\epsilon(n(\alpha_{\nu+1},a))}\Big\},a\Big)\le n\Big(r\exp\Big\{\frac{1}{\epsilon(n(r,a))}\Big\},a\Big),$$

as $\nu\to\infty$; $a\in\{0,\infty\}$. In particular, $\alpha_{\nu+1}\le r'=r\exp\Big\{\frac{1}{\epsilon(n(r,0)}\Big\}$. By (3.3) we have

$$\frac{\alpha_{\nu+1}}{\alpha_\nu}-1\ge\frac{1}{3e^A\epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))} \tag{3.6}$$

and $\alpha_{\nu+1}\sim\alpha_\nu$ as $\nu\to+\infty$. Now we apply Lemma 1 with $\varphi(t)=\epsilon(n(t,0))$. By (3) we have

$$\Big|\sum_{|a_\mu|\le\alpha_{\nu+1}}\frac{1}{z-a_\mu}\Big|+\Big|\sum_{|a_\mu|\le\alpha_{\nu+1}}\frac{1}{z-\frac{\alpha_{\nu+1}^2}{\overline{a}_\mu}}\Big|\le\frac{n(\alpha_{\nu-2},0)}{\alpha_{\nu-1}-\alpha_{\nu-2}}+$$
$$+\frac{n(\alpha_{\nu+1},0)\epsilon(n(\alpha_{\nu+1},0))}{\alpha_{\nu+1}}\sqrt{\log^+n(\alpha_{\nu+1},0)}\le$$
$$\le\frac{n(r',0)\epsilon(n(r',0))}{r}\sqrt{\log n(r',0)}(1+o(1))+\frac{n(r,0)}{r}3e^A\epsilon\Big(T\Big(\frac{\alpha_{\nu-1}^2}{\alpha_{\nu-2}}\Big)\Big), \tag{3.7}$$

where

$$z\in A_\nu\setminus\bigcup_{j=1}^{j_\nu}D_{\nu j},\quad \sum_j r_{\nu j}\le\frac{\alpha_{\nu+1}(n(\alpha_{\nu+1},0)-n(\alpha_{\nu-2},0))}{\epsilon(n(\alpha_{\nu+1},0))n(\alpha_{\nu+1},0)}.$$

If $\frac{\alpha_{\nu-1}^2}{\alpha_{\nu-2}}\le r$, then $T\Big(\frac{\alpha_{\nu-1}^2}{\alpha_{\nu-2}}\Big)\le T(r)$. Otherwise,

$$T\Big(\frac{\alpha_{\nu-1}^2}{\alpha_{\nu-2}}\Big)\le T\Big(\alpha_{\nu-2}\exp\Big\{\frac{2}{3e^A\epsilon\Big(T\Big(\frac{\alpha_{\nu-1}^2}{\alpha_{\nu-2}}\Big)\Big)}\Big\}\Big)\le T\Big(r\exp\Big\{\frac{2}{3e^A\epsilon(T(r))}\Big\}\Big)\equiv T(r'').$$

In any case $T\Big(\frac{\alpha_{\nu-1}^2}{\alpha_{\nu-2}}\Big)\le T(r'')$. Now we apply the following Borel-Nevanlinna type lemma ([5]).

Lemma 5. *Let* $\varepsilon\colon(0,+\infty)\to(0,+\infty)$, $\varepsilon(x)\nearrow\infty$, $\int_1^\infty\frac{dx}{x\varepsilon(x)}<\infty$, u *be a nondecreasing unbounded function on* $(0,\infty)$. *Then*

$$u\Big(r\exp\Big\{1/\varepsilon(u(r))\Big\}\Big)\le eu(r),\quad r\to+\infty$$

outside a set of finite logarithmic measure.

Remark 5. J.Miles deduces Lemma 5 from the classical Borel-Nevanlinna lemma [1, Theorem 1.2, p.120] when ε and u are supposed to be continuous. But the classical lemma (see Theorem D) holds even if ε and u are non-decreasing but not necessarily continuous. So, we have the assertion of Lemma 5 (see also the proof of Lemma 4).

Choosing $u(x) = n(x, 0)$, we find that $n(r', 0) \le en(r, 0)$ outside a set $E_1 \cup [0, 1]$, where $E_1 \subset [1, \infty)$ such that $\int_{E_1} \frac{dt}{t} < +\infty$. Let $F_\nu = \bigcup_j [|z_{\nu j}| - r_{\nu j}, |z_{\nu j}| + r_{\nu j}]$. By Lemma 1

$$\text{mes}\, F_\nu \le 2 \sum_j r_{\nu j} \le \frac{2\alpha_{\nu+1}(n(\alpha_{\nu+1}, 0) - n(\alpha_{\nu-2}, 0))}{\epsilon(n(\alpha_{\nu+1}, 0))n(\alpha_{\nu+1}, 0)}.$$

For $F = \bigcup_\nu F_\nu$ we have $(\nu \ge \nu_1 \ge 2)$

$$\sum_{\nu=\nu_1}^{\infty} \sum_j \frac{r_{\nu j}}{|z_{\nu j}|} \le 2 \sum_{\nu=\nu_1}^{\infty} \frac{\alpha_{\nu+1}(n(\alpha_{\nu+1}, 0) - n(\alpha_{\nu-2}, 0))}{(\alpha_{\nu-2} - \frac{c\alpha_{\nu+1}}{\epsilon(n(\alpha_{\nu+1}, 0))})\epsilon(n(\alpha_{\nu+1}, 0))n(\alpha_{\nu+1}, 0)} \le$$
$$\le 3 \sum_{\nu \ge \nu_1} \frac{n(\alpha_{\nu+1}, 0) - n(\alpha_{\nu-2}, 0)}{\epsilon(n(\alpha_{\nu+1}, 0))n(\alpha_{\nu+1}, 0)} \le 3 \sum_{\nu \ge \nu_1} \int_{(\alpha_{\nu-2}; \alpha_{\nu+1}]} \frac{dn(t, 0)}{\psi(n(t, 0))} \le$$
$$\le 9 \int_{\alpha_0}^{\infty} \frac{dn(t, 0)}{\psi(n(t, 0))} \le 9 \sum_{n=1}^{\infty} \frac{1}{\psi(n)} < \infty. \tag{3.8}$$

It means that the logarithmic mesure of F is finite. Therefore (3.7)–(3.8) yield

$$\Big| \sum_{|a_\mu| \le \alpha_{\nu+1}} \frac{1}{z - a_\mu} \Big| + \Big| \sum_{|a_\mu| \le \alpha_{\nu+1}} \frac{1}{z - \frac{\alpha_{\nu+1}^2}{\overline{a}_\mu}} \Big| \le \frac{n(r, 0)\epsilon(n(r, 0))}{r} \sqrt{\log n(r, 0)}(e + o(1)) +$$
$$+ \frac{n(r, 0)}{r}(3e^A + o(1))\epsilon\Big(T(r'')\Big), \quad r \to +\infty, r \notin E_1 \cup F. \tag{3.9}$$

Similarly,

$$\Big| \sum_{|b_\mu| \le \alpha_{\nu+1}} \frac{1}{z - b_\mu} \Big| + \Big| \sum_{|b_\mu| \le \alpha_{\nu+1}} \frac{1}{z - \frac{\alpha_{\nu+1}^2}{\overline{b}_\mu}} \Big| \le \frac{n(r, \infty)\epsilon(n(r, \infty))}{r} \sqrt{\log n(r, \infty)}(e + o(1)) +$$
$$+ \frac{n(r, \infty)}{r}(3e^A + o(1))\epsilon\big(T(r'')\big), \quad r \to +\infty, r \notin E_2, \tag{3.10}$$

where $E_2 \subset [1, +\infty)$ is of finite logarithmic measure.

For $\alpha_{\nu-1} < r \le \alpha_\nu$ we have

$$T\Big(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}\Big) = T\Big(\alpha_\nu \exp\Big\{\frac{2}{3e^A \epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}\Big\}\Big) \le$$
$$\le T\Big(\alpha_{\nu-1} \exp\Big\{\frac{2}{3e^A \epsilon(T(\frac{\alpha_\nu^2}{\alpha_{\nu-1}}))} + \frac{2}{3e^A \epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}\Big\}\Big) \le T\Big(r \exp\Big\{\frac{4}{3e^A \epsilon(T(r))}\Big\}\Big). \tag{3.11}$$

Applying Lemma 5 with $u(x) = T(x)$ and $\varepsilon(x) = \frac{4}{3e^A}\epsilon(x)$, we obtain that

$$T\Big(r \exp\Big\{\frac{4}{3e^A \epsilon(T(r))}\Big\}\Big) \le eT(r), \quad r \to +\infty, \ r \notin E_3, \tag{3.12}$$

where $E_3 \subset [1, \infty)$ is of finite logarithmic measure.

Since

$$\frac{\alpha_{\nu+1}}{r} = \frac{\alpha_\nu}{r} \exp\Big\{\frac{1}{3e^A \epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}\Big\} \ge 1 + \frac{1}{3e^A \epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}, \tag{3.13}$$

using (2.2), (3.11) and (3.12) we have

$$I \le \frac{4\alpha_{\nu+1}(T(\alpha_{\nu+1}) + O(1))}{(\alpha_{\nu+1} - r)^2} \le \frac{36(1+o(1))e^{2A} T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu})\epsilon^2(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}{r} \le$$
$$\le 37 e^{2A+1} \frac{T(r)\epsilon^2(T(r))}{r}, \quad r \to +\infty, r \notin E_3. \tag{3.14}$$

If $n(r) \ge T(r)$, we have $n(r)\epsilon(n(r)) \ge n(r)\epsilon(T(r))$, otherwise $T(r)\epsilon^2(T(r)) \ge n(r)\epsilon(T(r))$. Hence, (3.9)–(3.14) yield (1.5) for $l = 1$.

Now, let $l > 1$. As in [7, p.94] we deduce that for $R = \alpha_{\nu+1}$, $z \in A_\nu$

$$\Big|\Big(\frac{d}{dz}\Big)^{l-1} \frac{f'(z)}{f(z)}\Big| \le 5l! \frac{Rm(R, f)}{(R-r)^{l+1}} + \frac{(l-1)!n(R, 0, \infty, f)}{(R-r)^l} + (l-1)! \sum_{|c_\mu| \le R} \frac{1}{|z - c_\mu|^l}. \tag{3.15}$$

Using (3.3)–(3), and (3.12) we obtain

$$\frac{n(\alpha_{\nu+1}, 0)}{(\alpha_{\nu+1} - r)^l} \le C \frac{n(r', 0)\Big(3e^A \epsilon\big(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu})\big)\Big)^l}{r^l} \le C(A) \frac{n(r, 0)(\epsilon(T(r)))^l}{r^l}, \tag{3.16}$$

as $r \to +\infty, r \notin E_1$, where E_1 is of finite logarithmic measure.

In order to estimate $\sum_{|c_\mu| \le \alpha_{\nu+1}} |z - c_\mu|^{-l}$ we apply Cartan's lemma.

Lemma 6 ([13, pp.19–21]). *Let $a_1, a_2, \cdots, a_m$ be any finite collection of complex numbers, and let $d > 0$ be any given positive number. Then there exists a finite collection of closed disks $D_1, D_2, \cdots, D_q$ with corresponding radii $r_1, r_2, \cdots, r_q$ that satisfy $r_1 + r_2 + \cdots + r_q = 2d$, such that if $z \notin D_j$ for all $j \in \{1, 2, \cdots, q\}$, then there is a permutation of the points $a_1, a_2, \cdots, a_m$, say, $b_1, b_2, \cdots, b_m$, that satisfies $|z - b_k| > kd/m$ for $k = 1, 2, \cdots, m$, where the permutation may depend on z.*

First, using (3.11), (3.12) and Lemma 2 we obtain

$$\sum_{|c_\mu| \le \alpha_{\nu-2}} \frac{1}{|z - c_\mu|^l} \le \frac{n(\alpha_{\nu-2})}{(r - \alpha_{\nu-2})^l} \le Cn(r)\Big(\frac{\epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}{r}\Big)^l \le Cn(r)\Big(\frac{\epsilon(T(r))}{r}\Big)^l, \quad r \notin E_3.$$

Then, applying Lemma 6 to the points c_μ such that $\alpha_{\nu-2} < |c_\mu| \le \alpha_{\nu+1}$ with $d = 2\alpha_\nu(n(\alpha_{\nu+1}) - n(\alpha_{\nu-2}))/\phi(n(\alpha_{\nu+1}))$, we deduce that there exists a finite collection of closed disks $D_{\nu 1}, \ldots, D_{\nu j_\nu}$ whose radii have a total sum equal to $2d$ such that if $z \notin \bigcup_{j=1}^{j_\nu} D_{\nu j}$ then there is a permutation of the points c_μ, $\alpha_{\nu-2} < |c_\mu| \le \alpha_{\nu+1}$, say $\beta_1, \ldots, \beta_{n(\alpha_{\nu+1}) - n(\alpha_{\nu-2})}$ that satisfies $|z - \beta_k| > k\frac{\alpha_{\nu+1}}{\phi(n(\alpha_{\nu+1}))}$. Then

$$\sum_{\alpha_{\nu-2} < |c_\mu| \le \alpha_{\nu+1}} \frac{1}{|z - c_\mu|^l} = \sum_{k=1}^{n(\alpha_{\nu+1}) - n(\alpha_{\nu-2})} \frac{1}{|z - \beta_k|^l} \le$$
$$\le \Big(\frac{\phi(n(\alpha_{\nu+1}))}{\alpha_{\nu+1}}\Big)^l \sum_{k=1}^{n(\alpha_{\nu+1}) - n(\alpha_{\nu-2})} \frac{1}{k^l} \le C\Big(\frac{\phi(n(r))}{r}\Big)^l, \quad r \notin E_2, r \to +\infty,$$

where E_2 is a set of finite logarithmic measure. Thus,

$$\sum_{|c_\mu|\le\alpha_{\nu+1}} \frac{1}{|z-c_\mu|^l} \le C\Big(\frac{\phi(n(r))}{r}\Big)^l, \quad r\notin E_2, r\to+\infty, \tag{3.17}$$

Finally,

$$\frac{\alpha_{\nu+1}T(\alpha_{\nu+1})}{(\alpha_{\nu+1}-r)^{l+1}} \le C\frac{T(r)(\epsilon(T(r)))^{l+1}}{r^l}, \quad r\to+\infty,\ r\notin E_1. \tag{3.18}$$

Since $\frac{d}{dz}\Big(\frac{f'}{f}\Big) = \frac{f''}{f} - \Big(\frac{f'}{f}\Big)^2$, using (3.15) with $l=2$ and Theorem 3 for $l=1$, and (3.16)–(3.18) we can deduce that

$$\Big|\frac{f''(z)}{f(z)}\Big| \le \Big|\frac{d}{dz}\Big(\frac{f'}{f}\Big) + \Big(\frac{f'}{f}\Big)^2\Big| \le C\frac{T(r)(\epsilon(T(r)))^3}{r^2} + n(r)\Big(\frac{\epsilon(T(r))}{r}\Big)^2 +$$
$$+\Big(\frac{n(r)\epsilon(n(r))}{r}\Big)^2 + C\Big(\frac{T(r)(\epsilon(T(r)))^2}{r} + \frac{n(r)\epsilon(n(r))\sqrt{\log^+ n(r)}}{r}\Big)^2 \le$$
$$\le C\Big(\frac{T(r)(\epsilon(T(r)))^2}{r} + \frac{n(r)\epsilon(n(r))\sqrt{\log^+ n(r)}}{r}\Big)^2, \quad r\to+\infty, r\notin E.$$

In general, we have (cf. [3]).

Lemma 7. *Let there exist $f^{(m)}(x)$, $m\in\mathbb{N}$, $f(x)\ne 0$. Then*

$$\frac{f^{(m)}(x)}{f(x)} = \sum_{\substack{0\le i_s<m\\ \sum_s si_s=m}} a_{i_1\dots i_m} \prod_s \Big(\Big(\frac{f'(x)}{f(x)}\Big)^{(s-1)}\Big)^{i_s} + \Big(\frac{f'(x)}{f(x)}\Big)^m,$$

where the sum is taken over all nonnegative integers $i_1,\dots,i_m$ such that $\sum_{p=1}^m si_s = m$, $a_{i_1\dots i_m}$ are real.

Using this lemma, (3.15), and estimates (3.16)–(3.18), similar to [10, p.141–142] it can be deduced that for each positive integer l

$$\Big|\frac{f^{(l)}(z)}{f(z)}\Big| \le C\Big(\frac{T(r)(\epsilon(T(r)))^2}{r} + \frac{n(r)\epsilon(n(r))\sqrt{\log^+ n(r)}}{r}\Big)^l, \quad r\to+\infty, r\notin E,$$

where E is of finite logarithmic measure. Theorem 3 is proved. □

Proof of Corollary 2. From relationships (3.4), (3.5), (3.11) and (3.12), it follows that

$$n(r,0,\infty,f)\varepsilon(n(r,0,\infty,f)) \le C(A)T(r,f)\varepsilon^2(T(r,f)), \quad r\notin E,$$

where E is the set of finite logarithmic measure form the proof of Theorem 3. The inequality $\sqrt{\log n(r,0,\infty,f)} \le C\sqrt{\log T(r,f)}$ is easy to obtain with the aid of Lemma 2 similar to (3.5). Now, the statement of the corollary follows from Theorem 3 and the latter inequalities. □

3.2. The disk case.

Proof of Theorem 4. Proof of Theorem 4 is similar to that of Theorem 1 (cf. proof of Theorem 3.1 [8]). We should define $\alpha_\nu = 1-b^\nu$, $b=\sqrt{\beta}$, and $\varphi(t) = \frac{\psi(\log\frac{1}{1-t})}{1-t}$, and then apply Lemma 1. □

Proof of Theorem 5. We give the proof of Theorem 5 in details. Let $A > 0$, $\psi \in \Phi(A)$, $\varphi(t)$ is a corresponding function from the definition of the class $\Phi(A)$, $\epsilon(t) = \varphi(t)/t$. Let $A_1 = A\log(1+\frac{1}{\sigma})$. We define a sequence (α_ν) by the induction. Let $1 > \alpha_0 \ge \frac12$ be such that $\min\{n(\alpha_0, 0), n(\alpha_0, \infty)\} \ge 1$,

$$\alpha_{\nu+1} = \alpha_\nu \exp\Big\{\frac{1-\alpha_\nu}{3e^{A_1}\epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}\Big\}, \quad \nu \in \mathbb{Z}_+. \tag{3.19}$$

Since $e^x \le 1 + (e-1)x$ $(0 \le x \le 1)$ and $\epsilon(T) \ge 1$, it is easy to see that $\alpha_\nu < 1$ for all ν, moreover $\alpha_\nu \to 1$ $(\nu \to \infty)$.

We also note that and $1 - \alpha_{\nu+1} \sim 1 - \alpha_\nu$ as $\nu \to +\infty$, because

$$\log\frac{\alpha_{\nu+1}}{\alpha_\nu} = \frac{1-\alpha_\nu}{3e^{A_1}\epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))} = o(1-\alpha_\nu), \quad \nu \to \infty. \tag{3.20}$$

We choose $\rho = \alpha_{\nu+1}$, $\rho' = \alpha_{\nu+1}^2/\alpha_\nu$, and consequently, $\Delta = \frac{1-\alpha_\nu}{3e^{A_1}\epsilon(T(\alpha_{\nu+1}^2/\alpha_\nu))}$. Then

$$n(\alpha_{\nu+1}, a) \le \frac{3e^{A_1}}{1-\alpha_\nu} T\Big(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}\Big)\epsilon\Big(T\Big(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}\Big)\Big), \quad a \in \{0, \infty\}.$$

By the definition of the order $\sigma[T(r,f)] > 0$, for all $\eta > 0$ we have $\frac{3e^{A_1}}{1-\alpha_\nu} \le T(\alpha_\nu)^{\frac{1}{\sigma}+\eta}$ $(\nu \to +\infty)$. We can apply Lemma 3 with $b(T) = T^{\frac{1}{\sigma}+\eta}\epsilon(T)$. We obtain $(a \in \{0, \infty\})$

$$\epsilon(n(\alpha_{\nu+1}, a)) \le (e^{A_1} + o(1))\epsilon\Big(T\Big(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}\Big)\Big), \quad \nu \to +\infty\ . \tag{3.21}$$

Let $\alpha_{\nu-1} < r \le \alpha_\nu$, and ν is sufficiently large. Using (3.21), the definition of α_ν and (3.20) we deduce

$$\begin{aligned} n(\alpha_{\nu+1}, a) &= n\Big(\alpha_\nu \exp\Big\{\frac{1-\alpha_\nu}{3e^{A_1}\epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}\Big\}, a\Big) \le n\Big(\alpha_\nu \exp\Big\{\frac{(1+o(1))(1-\alpha_\nu)}{3\epsilon(n(\alpha_{\nu+1}, a))}\Big\}, a\Big) \le \\ &\le n\Big(\alpha_{\nu-1} \exp\Big\{\frac{1-\alpha_{\nu-1}}{3e^{A_1}\epsilon(T(\frac{\alpha_\nu^2}{\alpha_{\nu-1}}))} + \frac{(1+o(1))(1-\alpha_\nu)}{3\epsilon(n(\alpha_{\nu+1}, a))}\Big\}, a\Big) \le \\ &\le n\Big(\alpha_{\nu-1} \exp\Big\{\frac{(1+o(1))(1-\alpha_{\nu-1})}{3\epsilon(n(\alpha_\nu, a))} + \frac{(1+o(1))(1-\alpha_\nu)}{3\epsilon(n(\alpha_{\nu+1}, a))}\Big\}, a\Big) \le n\Big(r\exp\Big\{\frac{1-r}{\epsilon(n(r,a))}\Big\}, a\Big), \end{aligned}$$

as $\nu \to \infty$; $a \in \{0, \infty\}$. By (3.19) we have

$$\frac{\alpha_{\nu+1}}{\alpha_\nu} - 1 \ge \frac{1-\alpha_\nu}{3e^{A_1}\epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}. \tag{3.22}$$

Now we apply Lemma 1 with $\varphi(t) = \epsilon(n(t,0))/(1-t)$. By (3) we have $\big(r' = r\exp\{\frac{1-r}{\epsilon(n(r,0))}\}\big)$

$$\begin{aligned} &\Big|\sum_{|a_\mu| \le \alpha_{\nu+1}} \frac{1}{z - a_\mu}\Big| + \Big|\sum_{|a_\mu| \le \alpha_{\nu+1}} \frac{1}{z - \frac{\alpha_{\nu+1}^2}{\overline{a}_\mu}}\Big| \le \\ &\le \frac{n(r',0)\epsilon(n(r',0))}{1-r}\sqrt{\log n(r',0)}(1+o(1)) + \frac{n(r,0)}{1-r}3e^{A_1}\epsilon\Big(T\Big(\frac{\alpha_{\nu-1}^2}{\alpha_{\nu-2}}\Big)\Big), \end{aligned} \tag{3.23}$$

where $z \in A_\nu \setminus \bigcup_{j=1}^{j_\nu} D_{\nu j}$, $\sum_j r_{\nu j} \le \frac{(1-\alpha_{\nu+1})(n(\alpha_{\nu+1},0)-n(\alpha_{\nu-2},0))}{\epsilon(n(\alpha_{\nu+1},0))n(\alpha_{\nu+1},0)}$. And,

$$T\Big(\frac{\alpha_{\nu-1}^2}{\alpha_{\nu-2}}\Big) \le T\Big(r\exp\Big\{\frac{2(1-\alpha_\nu)}{3e^{A_1}\epsilon(T(r))}\Big\}\Big) \equiv T(r'').$$

Now we apply Lemma 4. Choosing $u(x) = n(x,0)$, we find that

$$n(r',0) \le en(r,0), \quad r \notin E_1 \subset [0,1) \tag{3.24}$$

such that $\int_{E_1} \frac{dt}{1-t} < +\infty$. Let $F_\nu = \bigcup_j [|z_{\nu j}| - r_{\nu j}, |z_{\nu j}| + r_{\nu j}]$. By Lemma 1

$$\operatorname{mes} F_\nu \le 2\sum_j r_{\nu j} \le \frac{2(1-\alpha_{\nu+1})(n(\alpha_{\nu+1},0)-n(\alpha_{\nu-2},0))}{\epsilon(n(\alpha_{\nu+1},0))n(\alpha_{\nu+1},0)}.$$

For $F = \bigcup_\nu F_\nu$ we have $(\nu \ge \nu_1 \ge 2)$

$$\sum_{\nu=\nu_1}^{\infty}\sum_j \frac{r_{\nu j}}{1-|z_{\nu j}|} \le 2\sum_{\nu=\nu_1}^{\infty} \frac{(1-\alpha_{\nu+1})(n(\alpha_{\nu+1},0)-n(\alpha_{\nu-2},0))}{(1-\alpha_{\nu+1}-\frac{c(1-\alpha_{\nu+1})}{\epsilon(n(\alpha_{\nu+1},0))})\epsilon(n(\alpha_{\nu+1},0))n(\alpha_{\nu+1},0)} \le$$
$$\le 3\sum_{\nu\ge\nu_1} \frac{n(\alpha_{\nu+1},0)-n(\alpha_{\nu-2},0)}{\psi(n(\alpha_{\nu+1},0))} \le 9\int_{\alpha_0}^{\infty} \frac{dn(t,0)}{\psi(n(t,0))} < \infty.$$

It means that the logarithmic mesure of F is finite. Therefore (3.23)–(3.24) yield

$$\Big|\sum_{|a_\mu|\le\alpha_{\nu+1}} \frac{1}{z-a_\mu}\Big| + \Big|\sum_{|a_\mu|\le\alpha_{\nu+1}} \frac{1}{z-\frac{\alpha_{\nu+1}^2}{\bar a_\mu}}\Big| \le \frac{n(r,0)\epsilon(n(r,0))}{1-r}\sqrt{\log n(r,0)}(e+o(1))+$$
$$+\frac{n(r,0)}{1-r}(3e^A+o(1))\epsilon\Big(T(r'')\Big), \quad r\to+\infty, r\notin E_1\cup F. \tag{3.25}$$

Similarly,

$$\Big|\sum_{|b_\mu|\le\alpha_{\nu+1}} \frac{1}{z-b_\mu}\Big| + \Big|\sum_{|b_\mu|\le\alpha_{\nu+1}} \frac{1}{z-\frac{\alpha_{\nu+1}^2}{\bar b_\mu}}\Big| \le \frac{n(r,\infty)\epsilon(n(r,\infty))}{r}\sqrt{\log n(r,\infty)}(e+o(1))+$$
$$+\frac{n(r,\infty)}{r}(3e^A+o(1))\epsilon\big(T(r'')\big), \quad r\to 1, r\notin E_2, \tag{3.26}$$

where $E_2 \subset [0,1)$ is of finite logarithmic measure.

For $\alpha_{\nu-1} < r \le \alpha_\nu$ we have

$$T\Big(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}\Big) = T\Big(\alpha_\nu \exp\Big\{\frac{2(1-\alpha_\nu)}{3e^{A_1}\epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}\Big\}\Big) \le$$
$$\le T\Big(\alpha_{\nu-1}\exp\Big\{\frac{2(1-\alpha_{\nu-1})}{3e^{A_1}\epsilon(T(\frac{\alpha_\nu^2}{\alpha_{\nu-1}}))} + \frac{2(1-\alpha_\nu)}{3e^{A_1}\epsilon(T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}\Big\}\Big) \le T\Big(r\exp\Big\{\frac{(4+o(1))(1-r)}{3e^{A_1}\epsilon(T(r))}\Big\}\Big).$$

Applying Lemma 4 with $v(x) = T(x)$ and $\varepsilon(x) = \frac{2}{e^{A_1}}\epsilon(x)$, we obtain that

$$T\Big(r\exp\Big\{\frac{2(1-r)}{e^A\epsilon(T(r))}\Big\}\Big) \le eT(r), \quad r\to 1,\ r\notin E_3, \tag{3.27}$$

where $E_3 \subset [0,1)$ is of finite logarithmic measure.

Then, using (2.2) and (3.22) we have

$$I \le \frac{4(T(\alpha_{\nu+1})+O(1))}{(\alpha_{\nu+1}-r)^2} \le \frac{36(1+o(1))e^{2A_1}T(\frac{\alpha_{\nu+1}^2}{\alpha_\nu})\epsilon^2(T((\frac{\alpha_{\nu+1}^2}{\alpha_\nu}))}{(1-r)^2} \le$$
$$\le 37e^{2A_1+1}\frac{T(r)\epsilon^2(T(r))}{(1-r)^2}, \quad r\to 1, r\notin E_3. \tag{3.28}$$

Hence, (3.25)–(3.28) yield (1.8). Theorem 5 is proved. □

4. An example.

Proof of Theorem 2. Given $\rho \in (0,+\infty)$ we set $q=[\rho]$, $r_n = 4^{\frac{n}{\rho}}$, $n\in\mathbb{Z}_+$ and define a Weierstrass product of the form

$$B(z)=\prod_{n=1}^{\infty}\prod_{k=1}^{4^n}E\Big(\frac{z}{a_{nk}},q\Big),$$

where $E(w,q)$ is a Weierstrass primary factor of genus q ([1], [2]), zeros a_{nk} satisfy $r_{n-1}<|a_{nk}|\le r_n$, $1\le k\le 4^n$, and will be specified later. Now we just note that since $n(r_n,0,B)=4(4^n-1)/3$, we have $n(t,0,B)\asymp t^\rho$ as $t\to+\infty$. It is well-known that the product is absolutely convergent in $\mathbb{C}$. Moreover,

$$\frac{B'(z)}{B(z)}=\sum_{n=1}^{\infty}\sum_{k=1}^{4^n}\frac{E'(\frac{z}{a_{nk}},q)}{E(\frac{z}{a_{nk}},q)}=\sum_{n=1}^{\infty}\sum_{k=1}^{4^n}\frac{z^q}{a_{nk}^q(z-a_{nk})}. \tag{4.1}$$

We use a construction due to V.Eiderman and J.Anderson ([11], [12]). Set $E^{(0)}=[-\frac12,\frac12]$ and at the ends of $E^{(0)}$ take subintervals $E_j^{(1)}$ of length $\frac14$, $j\in\{1,2\}$. Let

$$E^{(1)}=\bigcup_{j=1}^{2}E_j^{(1)}=\big[-\tfrac12,-\tfrac14\big]\cup\big[\tfrac14,\tfrac12\big].$$

We then construct, in a similar manner, two sub-intervals $E_{j,i}^{(2)}$ of length 4^{-2} in each $E_j^{(1)}$ and denote by $E^{(2)}$ the union of the four intervals $E_{j,i}^{(2)}$. Continuing this process we obtain a sequence of sets $E^{(n)}$ consisting of 2^n intervals of length 4^{-n}. We define $E_n=E^{(n)}\times E^{(n)}$ the Cartesian product, and note that E_n consists of 4^n squares $E_{n,k}$, $k\in\{1,\dots,4^n\}$ with sides parallel to the coordinate axes.

Theorem E. *Let $P>0$ be given and set $E=(100P)^{-1}\sqrt{n}4^nE_n$ is the set defined above. Let ν be the measure formed by 4^{n+1} Dirac masses located at the cornes z_k of the squares which form E_n. Then for any covernig $\{D(w_j,\rho_j)\}$ of the set $\mathcal{Z}=\Big\{z\in\mathbb{C}:\Big|\sum_{k=1}^{4^{n+1}}\frac{1}{z-z_k}\Big|>P\Big\}$ we have $\sum_j\rho_j\ge c_2\frac{4^{n+1}}{P}\sqrt{\log 4^{n+1}}$ where $c_2>0$ is an absolute constants.*

Moreover, the projection of $\bigcup_j D(w_j,\rho_j)$ onto the straight line $y=x/2$ has measure at least $c_3\frac{4^{n+1}}{P}\sqrt{\log 4^{n+1}}$ where $c_3>0$ is an absolute constant.

Remark 6. [11, 12] Let $z'_{n,k}$ be the centers of $E_{n,k}$ which form E in Theorem E. Then $\mathcal{Z}\supset\bigcup_{k\in\mathcal{K}_n}D(z'_{n,k},0.001\sqrt{n}/P)$, and $\operatorname{card}\mathcal{K}_n\ge c_4 4^n$ where $c_4>0$ is an absolute constant.

We choose

$$P_n=\frac{4^{n+1}}{r_{n+1}}\sqrt{\log 4^{n+1}}\psi(\log r_{n+1}),\ \widetilde{E}_n=\frac{\sqrt{n}4^n}{100P_n}E=\frac{r_{n+1}}{400\sqrt{(1+1/n)\log 4}\psi(\log r_{n+1})}E.$$

Hence the side length d_n of $\widetilde{E}_n$ satisfies

$$\frac{r_{n+1}}{800\psi(\log r_{n+1})}\le d_n\le\frac{r_{n+1}}{400\psi(\log r_{n+1})}. \tag{4.2}$$

We rotate $\widetilde{E}_n$ on the anlge $\pi/6$ clockwise and move along Ox such that the right vertex of the square coincides with r_{n+1}. We denote by $a_{n+1,k}$, $1 \le k \le 4^{n+1}$ the vertecies of the squares that form the obtained square E_n^*.

Then by Theorem E for any covering $\{D(w_{n,j}, \rho_{n,j})\}$ of the set

$$M_n \stackrel{\text{def}}{=} \Big\{z : \Big|\sum_{k=1}^{4^{n+1}} \frac{1}{z - a_{n+1,k}}\Big| \ge \frac{4^{n+1}}{r_{n+1}} \sqrt{\log 4^{n+1}}\psi(\log r_{n+1})\Big\}$$

we have $\sum_j \rho_{n,j} \ge c_2 r_{n+1}/\psi(\log r_{n+1})$. Let $z \in M_n$. According to Remark 6 we can assume that $z \in \bigcup_{k\in\mathcal{K}_n} D(z'_{n,k}, 10^{-3}\sqrt{n+1}/P_n) \subset M_n$. Then, by (4.2)

$$\begin{aligned} |z| &\le r_{n+1} + 0.001\frac{n+1}{P_n} \le r_{n+1} + \frac{r_{n+1}}{100\psi(\log r_{n+1})}, \\ |z| &\ge r_{n+1} - d_n\frac{2}{\sqrt{3}} - 0.001\frac{n+1}{P_n} \ge r_{n+1} - \frac{r_{n+1}}{100\psi(\log r_{n+1})}. \end{aligned} \tag{4.3}$$

We split the sum from (4.1)($A_{mk} = \frac{z^q}{a_{mk}^q(z-a_{mk})}$)

$\sum_{m=1}^{\infty}\sum_{k=1}^{4^m} A_{mk} = \sum_{m=1}^{n}\sum_{k=1}^{4^m} A_{mk} + \sum_{k=1}^{4^{n+1}} A_{n+1,k} + \sum_{m=n+2}^{\infty}\sum_{k=1}^{4^m} A_{mk} \equiv \Sigma_1 + \Sigma_2 + \Sigma_3$.

Using (4.2) and (4.3) we estimate Σ_3 for $z \in M_n$, $|z| = r$

$$\begin{aligned} |\Sigma_3| &\le \sum_{m=n+2}^{\infty}\sum_{k=1}^{4^m} \frac{|z|^q}{|a_{mk}|^q|z - a_{mk}|} \le 2r^q \sum_{m=n+2}^{\infty} \frac{4^m}{r_m^{q+1}(1 - \frac{r_{n+1}}{r_m})} \le \\ &\le C_1(\rho)|z|^q \sum_{m=n+2}^{\infty} r_m^{\rho-q-1} = C_2(\rho) r^q r_{n+2}^{\rho-q-1} \le C_3(\rho)\frac{n(r)}{r}, \quad r \to +\infty. \end{aligned} \tag{4.4}$$

Similarly,

$$\begin{aligned} |\Sigma_1| &\le \sum_{m=1}^{n}\sum_{k=1}^{4^m} \frac{|z|^q}{|a_{mk}|^q|z - a_{mk}|} \le 2C_4(\rho) r^{q-1} \sum_{m=1}^{n} \frac{4^m}{r_m^q} = C_5(\rho) r^{q-1} \cdot \begin{cases} r_n^{\rho-q}, & \rho > q \\ n, & \rho = q \end{cases} \le \\ &\le C_6(\rho) \cdot \begin{cases} r^{\rho-1}, & \rho \notin \mathbb{N} \\ r^{\rho-1}\log r, & \rho \in \mathbb{N} \end{cases} \le C_6 \frac{n(r)\log r}{r}, \quad r \to +\infty. \end{aligned} \tag{4.5}$$

Finally, let us estimate the difference

$$\Big|\sum_{k=1}^{4^{n+1}} \frac{z^q}{a_{n+1,k}^q(z - a_{n+1,k})} - \sum_{k=1}^{4^{n+1}} \frac{1}{z - a_{n+1,k}}\Big| \le \sum_{k=1}^{4^{n+1}} \frac{|z^{q-1} + z^{q-2}a_{n+1,k} + \cdots + a_{n+1,k}^{q-1}|}{|a_{n+1,k}|^q} \le$$

$\le C_6(\rho)\frac{q4^{n+1}}{r_{n+1}} \le C_7(\rho)\frac{n(r)}{r}$. Hence, it follows from (4.1), (4.4)–(4.5), the latter estimate and the definition of M_n that for $z \in M_n$, $|z| = r$

$$\begin{aligned} \Big|\frac{B'(z)}{B(z)}\Big| &\ge \Big|\sum_{k=1}^{4^{n+1}} \frac{1}{z - a_{n+1,k}}\Big| - \Big|\sum_{k=1}^{4^{n+1}} \frac{z^q}{a_{n+1,k}^q(z - a_{n+1,k})} - \sum_{k=1}^{4^{n+1}} \frac{1}{z - a_{n+1,k}}\Big| - \\ &- \Big|\sum_{m\ne n+1}\sum_{k=1}^{4^m} \frac{z^q}{a_{mk}^q(z - a_{mk})}\Big| \ge C_8(\rho)\frac{n(r)\sqrt{\log n(r)}\psi(\log r)}{r} - O\Big(\frac{n(r)\log r}{r}\Big) \ge \\ &\ge C_9(\rho)\frac{n(r)\sqrt{\log n(r)}\psi(\log r)}{r}, \quad r \to +\infty. \end{aligned}$$

On the other hand,

$$\sum_n \sum_j \frac{\rho_{n,j}}{r_n} \ge c_2 \sum_n \frac{r_{n+1}}{r_n \psi(\log r_{n+1})} \ge c_2 \sum_n \frac{1}{\psi(n \log 4)} = +\infty.$$

Theorem 2 is proved. □

I would like to thank Prof. V. Eiderman, Prof. G. Gundersen and Prof. O. Skaskiv for valuable remarks.

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Received 29.08.2006
Revised 5.01.2007