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A REMARK ON ASYMPTOTIC DIMENSION AND DIGITAL DIMENSION OF FINITE METRIC SPACES

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Relationships between the asymptotic dimension (in the sense of Gromov) of a proper metric space and the so-called digital dimension of its finite subspaces are established.

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Утанавливаются соотношения между асимптотической размерностью (в смысле Громова) собственного метрического пространства и так называемой цифровой размерностью его конечных подпространств.

1. Introduction. The notion of asymptotic dimension of a proper metric space is introduced by M. Gromov [11]. This notion can be considered as a large scale counterpart of Ostrand's definition of the covering dimension [12]. The asymptotic dimension theory was developed in [4, 5, 6, 10] and other papers.

There are different conditions equivalent to Gromov's definition. In particular, the asymptotic dimension can be characterized in terms of mappings into polyhedra [11] (see also [7]), in terms of Lebesgue dimension of the Higson corona — the remainder of the compactification of a proper metric spaces generated by algebra of slowly oscillating functions [9]. Also, in [5] the notion of asymptotic inductive dimension is introduced and it is proved in [10] that in the class of asymptotically finite-dimensional spaces, the asymptotic dimension coincides with the inductive dimension. See also [8] for the discussion on another equivalent definitions of asymptotic dimension.

In the introduction to [7] A. Dranishnikov remarks that, to a finite metric space X, it is possible to assign a dimension on the scale $\lambda <$diameter of X by means of λ-approximations of X by finite polyhedra. In this note we use another (natural) definitions of "dimension on the scale λ" and, using the addition theorem for the asymptotic dimension proved in [2], show that the asymptotic dimension can be defined in terms of finite subspaces.

2. Preliminaries. Let us first recall some definitions. A metric space is *proper* if every its closed ball is compact. Given a set A of a metric space (X, d), one defines the *diameter* of A

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(written diam A) as $\sup\{d(a,b) \mid a,b \in A\}$. Given two sets A, B of a metric space (X,d), one defines the *distance* between A and B (written dist (A,B)) as $\inf\{d(a,b) \mid a \in A,\ b \in B\}$. By $O_r(x)$ we denote the open r-neighborhood of $x \in X$, $r > 0$. If $A \subset X$, we put $O_r(A) = \bigcup\{O_r(x) \mid x \in A\}$.

A family $\mathcal{A}$ of subsets of a metric space (X,d) is called

uniformly bounded if there exists a number $C > 0$ such that diam $A \le C$ for every $A \in \mathcal{A}$;

D-disjoint, where D is a number, if for any pair A, B of elements of $\mathcal{A}$ such that $A \neq B$ we have dist $(A,B) \ge D$.

For a family $\mathcal{A}$ of subsets of a metric space we let $\operatorname{mesh}\mathcal{A} = \sup\{\operatorname{diam} A : A \in \mathcal{A}\}$. It is evident that $\operatorname{mesh}\mathcal{A} < \infty$ iff $\mathcal{A}$ is uniformly bounded.

The following definition is due to M. Gromov [11].

Definition 2.1. The *asymptotic dimension* of a metric space X does not exceed $n \ge 0$ (written $\operatorname{asdim} X \le n$) if for every $D > 0$ there exists a cover $\mathcal{A}$ of the space X such that

(i) $\operatorname{mesh}\mathcal{A} < \infty$;

(ii) $\mathcal{A} = \mathcal{A}^0 \cup \dots \cup \mathcal{A}^n$, where all $\mathcal{A}^i$ are D-disjoint.

In the sequel, we will need the following known properties of the asymptotic dimension.

Theorem 2.1. *For a metric space X and its any bounded subset Y the following two conditions are equivalent:*

(i) $\operatorname{asdim} X \le n$*;*

(ii) $\operatorname{asdim}(X \setminus Y) \le n$.

Theorem 2.2. *Let X be a proper metric space and* $\operatorname{asdim} X < \infty$*. Then there exists a countable discrete subspace Y of X such that* $\operatorname{asdim} Y = \operatorname{asdim} X$.

The following addition theorem is proved in [2].

Theorem 2.3. *Let X be a proper metric space which the union of two subspaces X_1 and X_2. Then* $\operatorname{asdim} X = \max\{\operatorname{asdim} X_1, \operatorname{asdim} X_2\}$.

3. Results. We introduce the following notion.

Definition 3.1. *The digital dimension with parameters $D > 0, C \ge 0$ of a metric space X* does not exceed $n \ge 0$ (written $\dim_{D,C} X \le n$) if there exists a cover $\mathcal{A}$ of the space X such that

(i) $\operatorname{mesh}\mathcal{A} \le C$;

(ii) $\mathcal{A} = \mathcal{A}^0 \cup \dots \cup \mathcal{A}^n$, where all $\mathcal{A}^i$ are D-disjoint.

One can think of finite metric spaces X of $\dim_{D,C} X \le n$ as spaces D-close to $\le n$-dimensional at the scale C.

Remark 3.1. It is evident that $\operatorname{asdim} X \le n$ iff that for every $D > 0$ there exists $C \ge 0$ such that $\dim_{D,C} X \le n$.

Let us formulate simple properties of the digital dimensions.

Proposition 3.1. *Let X be a metric space. Then*

(i) For any set $A \subset X$ and any $D > 0$ and $C \geq 0$ we have $\dim_{D,C} A \leq \dim_{D,C} X$;

(ii) $\dim_{D,C} X \leq \dim_{D_1,C_1} X$ if $D \leq D_1$ and $C_1 \leq C$.

For any number x let us denote by int (x) the largest integer $\leq x$.

Example 3.1. Let $X_n = \{1, 2, \dots, n\} \subset R$. Then

(i) $\dim_{k,0} X_n = k - 1$ if $k + 1 \leq n$;

(ii) $\dim_{k,1} X_n = \begin{cases} \frac{k-1}{2} & \text{if } k = 2q \text{ for some integer } q \\ \text{int }(\frac{k-1}{2}) + 1 & \text{otherwise;} \end{cases}$

(iii) $\dim_{k,m} X_n = \begin{cases} \frac{k-1}{m} & \text{if } k = mq \text{ for some integer } q \\ \text{int }(\frac{k-1}{m}) + 1 & \text{otherwise.} \end{cases}$

The next statement connects the asymptotic dimension with the digital one.

Theorem 3.1. *For a proper metric space (X, d) the following conditions are equivalent:*

(i) $\operatorname{asdim} X \leq n$;

(ii) for every $D > 0$ there exists $C \geq 0$ such that for any set $A \subset X$ we have $\dim_{D,C} A \leq n$.

(iii) for every $D > 0$ there exists $C \geq 0$ such that for any finite set $A \subset X$ we have $\dim_{D,C} A \leq n$.

Proof. The implication $(i) \Rightarrow (ii)$ is valid due to Remark 3.1. and Proposition 3.1 (i). The implication $(ii) \Rightarrow (iii)$ is evident. Let us prove the implication $(iii) \Rightarrow (i)$. By Theorem 2.2 we may assume that the space X is a countable discrete metric space. Let p be a point of X. Put $A_i = \{x \in X : (i-1)^2 \leq d(x,p) \leq i^2\}$ and $X_1 = \bigcup_{i=1}^{\infty} A_{2i-1}$, $X_2 = \bigcup_{i=1}^{\infty} A_{2i}$. By Theorem 2.3 it is sufficient to prove that $\operatorname{asdim} X_i \leq n$ for every i. Observe that because of similarity we have only to prove that $\operatorname{asdim} X_1 \leq n$. Moreover, because of the properness of X_1, for every i the set A_{2i-1} is finite. For given $D > 0$, find $C \geq 0$ such that for every finite set A of X_1 we have $\dim_{D,C} A \leq n$. For every $i \in \{1, 2, \dots\}$ find a cover $\mathcal{A}_i$ of A_{2i-1} such that

(i) mesh $\mathcal{A}_i \leq C$;

(ii) $\mathcal{A}_i = \mathcal{A}_i^0 \cup \dots \cup \mathcal{A}_i^n$, where all $\mathcal{A}_i^k$ are D-disjoint.

Observe that there is a natural number N such that for any pair integers $i, j \geq N$ and $i \neq j$ we have $d(A_{2i-1}, A_{2j-1}) \geq D$. Put $Y = \bigcup_{i=1}^{N} A_{2i-1}$, $\mathcal{B}_0 = \{Y\} \cup \mathcal{A}^0_{2(N+1)-1} \cup \dots \cup \mathcal{A}^0_{2k-1} \cup \dots$ and $\mathcal{B}_i = \bigcup \mathcal{A}^i_{2(N+1)-1} \cup \dots \cup \mathcal{A}^i_{2k-1} \cup \dots$ for every $i \in \{1, \dots, n\}$. Note that the families $\mathcal{B}_i$, $i \in \{0, \dots, n\}$ are D-disjoint and their union $\mathcal{B} = \bigcup_{i=0}^{n} \mathcal{B}_i$ is a cover of X_1 with mesh $\mathcal{B} < \infty$ (in fact mesh $\mathcal{B} \leq \max\{C, 2(2N-1)^2\}$). It means that asdim $X_1 \leq n$. The theorem is proved. □

Recall that a number $\lambda > 0$ is a *Lebesgue number* of a cover $\mathcal{U}$ of a metric space X if for every $x \in X$ the ball $O_\lambda(x)$ is contained in an element of $\mathcal{U}$. A *multiplicity* $m(\mathcal{U})$ of an open cover $\mathcal{U}$ is the number $\sup\{|U \in \mathcal{U} \mid x \in U\}| \mid x \in X\}$.

In analogy to Definition 3.1, one can introduce the following notion.

Definition 3.2. *The digital dimension′ with parameters* $D > 0, C \geq 0$ of a metric space X does not exceed $n \geq 0$ (written $\dim'_{D,C} X \leq n$) if there exists a cover $\mathcal{A}$ of the space X such that

(i) $\operatorname{mesh} \mathcal{A} \leq C$;
(ii) D is a Lebesgue number of $\mathcal{A}$;
(iii) $m(\mathcal{A}) \leq n + 1$.

A counterpart of Theorem 3.1 can be proved also for the dimension $\dim'_{D,C}$.

Theorem 3.2. *For a proper metric space (X, d) the following conditions are equivalent:*

(i) $\operatorname{asdim} X \leq n$*;*
(ii) for every $D > 0$ there exists $C \geq 0$ such that for any set $A \subset X$ we have $\dim'_{D,C} A \leq n$.
(iii) for every $D > 0$ there exists $C \geq 0$ such that for any finite set $A \subset X$ we have $\dim'_{D,C} A \leq n$.

The proof is similar to that of Theorem 3.1. It uses the fact that the dimension $\operatorname{asdim} X$ of a proper metric space X can be defined as follows: $\operatorname{asdim} X \leq n$ if for every $D > 0$ there exists a uniformly bounded cover of X of multiplicity $\leq n+1$ and Lebesgue number D (see, e.g., [4]).

4. Remarks and open problems.

Question 4.1. Find relations between $\dim_{D,C}$ and $\dim'_{D,C}$.

One can easily prove that $\dim_{D,C} X \leq n$ implies $\dim'_{D,C+D} X \leq n$.

Let X be a proper metric space with base point x_0. Two closed subsets A, B of the space X are said to be *asymptotically disjoint* if $\lim_{r\to\infty} \operatorname{dist}(B \setminus O_r(x_0), A \setminus O_r(x_0)) = 0$.

A sequence $x_0, x_1, \ldots, x_k$ of a space X is said to be an *s-chain* in X if $d(x_{i-1}, x_i) < s$ for every $i \in \{1, \ldots, n\}$. An s-chain $x_0, x_1, \ldots, x_k$ *connects* subsets A and B of X if $x_0 \in A$ and $x_k \in B$. The following notion is introduced in [10]. A closed subset C of a proper metric space X is called a *homological separator* between A and B if for any $s > 0$ there exists $\lambda > 0$ such that every s-chain connecting A and B necessarily intersects $O_\lambda(C)$.

The *asymptotic homological inductive dimension* asInd_h is defined inductively as follows: (i) $\operatorname{asInd}_h X = -1$ if and only if X is a bounded metric space; (ii) if we have already defined the class of spaces Y with $\operatorname{asInd}_h Y \leq n-1$ then we say that $\operatorname{asInd}_h X \leq n$ whenever, for every pair A, B of asymptotically disjoint subsets of X, there exists a homological separator C between A and B with $\operatorname{asInd}_h C \leq n-1$.

Question 4.2. Find characterization of the asymptotic homological inductive dimension of a proper metric space in terms of dimension-like invariant of its finite subspaces (in the spirit of Theorems 3.1 and 3.2).

Note that the proofs of the mentioned theorems essentially use the addition theorem for the asymptotic dimension and that no addition theorem for the dimension asInd_h is known.

The asymptotic inductive dimension asInd of proper metric spaces is defined in [5]. The following question also remains open: characterize the asymptotic inductive dimension of proper metric spaces (see [5] for the definition) in terms of "digital inductive dimension" of its finite subspaces.

One may speculate whether the dimension invariants $\dim_{C,D}$ and $\dim'_{C,D}$ can be applied to the problems of pattern recognition, in particular, in the digital plane (space). This suggested the term "digital". Evaluation of the digital dimension of a finite metric space on a given scale can lead to computational and algorithmic problems.

The asymptotic counterpart of the notion of C-space is defined by Dranishnikov [4]. A proper metric space X is said to be a *C-space* if, for every sequence D_i of positive numbers there is a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of uniformly bounded covers of X such that every $\mathcal{U}_i$ is D_i-discrete and $\bigcup_{i=1}^{k}\mathcal{U}_i$ is a cover of X, for some k.

Question 4.3. Find a definition of C-space of a proper metric space by means of some dimension-like properties of its finite subspaces.

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