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MEROMORPHIC FUNCTIONS IN m-PUNCTURED COMPLEX PLANES

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An extension of the Nevanlinna value distribution theory for meromorphic functions in m-punctured complex planes is proposed. The main characteristics are one-parameter and possess the same properties as in the classical case. Analogs of the Jensen and of the First Fundamental Theorem for m-punctured complex planes are obtained. We propose proofs of Decomposition Lemma, Main Lemma on logarithmic derivative and Second Fundamental Theorem for meromorphic functions in m-punctured planes.

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Теория распределения значений Неванлинны распространяется на мероморфные функции в комплексной плоскости с m выколотыми точками. Введенные характеристики обладают такими же свойствами, как и в классическом случае. Получены аналоги теорем Йенсена и первой основной теоремы теории распределения значений. Доказаны аналоги леммы о декомпозиции, основной леммы о логарифмической производной и второй основной теоремы теории распределения значений.

1. Introduction. Many problems lead us to the study of meromorphic functions in multiconnected regions. In particular, considering the composition $f \circ \mathcal{R}$ of a transcendental meromorphic in $\mathbb{C}$ function f and a rational function $\mathcal{R}$ with m distinct poles in $\mathbb{C}$ we obtain a meromorphic function in an $m+1$ − connected region. The mentioned poles are its isolated or nonisolated essential singularities.

There is a principal difference in the topological sense between simply connected and multiconnected regions which is reflected on the theory of meromorphic functions. The fundamental (Poincaré) group of a simply connected region is trivial and this one of a multiconnected region is a free group of certain rank.

Meromorphic functions in multiconnected regions were studied by many authors [1]-[5]. Our approach allows to introduce one-parameter characteristics of meromorphic functions in m-punctured planes and generalize the Nevanlinna theory on the classes of these functions.

In this paper we prove an analog of the Jensen theorem [6] for an m-punctured plane, introduce the Nevanlinna characteristics, study their properties and prove the First and Second Fundamental Theorems of the value distribution theory for an m-punctured plane.

2. Definitions and notations. Fundamental group of an m-punctured plane. A *curve* in $\mathbb{C}$ is a continuous mapping γ of a parameter interval $[\alpha, \beta] \subset \mathbb{R}$ into $\mathbb{C}$. The range of γ is denoted by γ^*. A *path* is a piecewise continuously differentiable curve.

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If $[0,1]$ is the parameter interval of a path γ then $\gamma_1(t)=\gamma(1-t)$, $0\le t\le 1$, is the path *opposite* to γ and $\gamma^*=\gamma_1^*$.

Let γ be a closed path, and let $\Omega=\mathbb{C}\setminus\gamma^*$. *Index of z with respect to γ* is defined as

$$\mathrm{Ind}_\gamma(z)=\frac{1}{2\pi i}\int\limits_\gamma\frac{d\varsigma}{\varsigma-z},\qquad z\in\Omega.$$

Theorem A. $\mathrm{Ind}_\gamma(z)$ *is an integer-valued function on Ω which is constant in each component of Ω and which is 0 in the unbounded component of Ω.*

Proof see, for example, ([7], P. 93, [8], P. 219-220).

Closed curves γ_0, γ_1 in a region Ω, both with parameter interval $I=[0,1]$, are said to be Ω-*homotopic* if there is a continuous mapping H of the unit square $I^2=I\times I$ into Ω such that

$$H(s,0)=\gamma_0(s),\quad H(s,1)=\gamma_1(s),\quad H(0,t)=H(1,t)$$

for all $s\in I$ and $t\in I$.

Intuitively, this means that γ_0 can be continuously deformed to γ_1 by a one-parameter family of closed curves $\gamma_t(s)=H(s,t)$.

Denote by $\mathcal{L}(z_0)$ the set of closed curves in Ω with the parameter interval I and initial points z_0. *The product of curves* γ_1 and γ_2 from $\mathcal{L}(z_0)$ is defined by the relation

$$(\gamma_1\gamma_2)(t)=\begin{cases}\gamma_1(2t), & 0\le t\le\frac{1}{2},\\ \gamma_2(2t-1), & \frac{1}{2}\le t\le 1.\end{cases}$$

The homotopic curves γ_1 and γ_2 from $\mathcal{L}(z_0)$ are said to be equivalent. The class of closed curves equivalent to a curve γ is denoted by $[\gamma]$. The curve γ, $\gamma(I)=z_0$, is denoted by e.

Let $\pi_1(\Omega,z_0)$ be the set of equivalent classes $[\gamma]$, $\gamma\in\mathcal{L}(z_0)$.

Put $[\gamma_1][\gamma_2]=[\gamma_1\gamma_2]$. The element $[\gamma^{-1}]$ is inverse to $[\gamma]$, if we define $\gamma^{-1}(t)=\gamma(1-t)$. It is easy to verify that $\pi_1(\Omega,z_0)$ forms a group with the unity $[e]$.

The group $\pi_1(\Omega,z_0)$ is called *fundamental group of Ω at z_0*.

Let us consider the set of distinct points $c_j\in\mathbb{C}$, $j\in\{1,\dots,m\}$. Then $\Omega=\mathbb{C}\setminus\bigcup_{j=1}^m\{c_j\}$ is an $m-$punctured plane. We will consider those $m-$punctured planes for which $m\ge 2$. The case $m=1$ leads us to the consideration of annuli and was studied in [1]-[5]. The same approach as in [4],[5] allows to generalize the Nevanlinna theory for meromorphic functions in $m-$punctured planes.

Denote $d=\frac{1}{2}\min\{|c_j-c_k|: j\ne k\}$ and $r_0=1/d+\max\{|c_j|:1\le j\le m\}$. It is easy to see that $1/r_0<d$, $\overline{D}_{1/r_0}(c_j)\cap\overline{D}_{1/r_0}(c_k)=\varnothing$, $j\ne k$, and $\overline{D}_{1/r_0}(c_j)\subset D_{r_0}(0)$, $j\in\{1,2,\dots,m\}$, where $D_r(c)$ denotes a disk of radius $r>0$ centered at c. For an arbitrary $t\ge r_0$ we define

$$\Omega_t=D_t(0)\setminus\bigcup_{j=1}^m\overline{D}_{1/t}(c_j).$$

Using this notation we conclude that $\Omega_{r_0}\subset\Omega_r$, $r_0<r\le+\infty$.

Let γ be a closed path in Ω_r with the initial point $z_0\in\Omega_r$. Denote

$$k_j=\mathrm{Ind}_\gamma(c_j)=\frac{1}{2\pi i}\int\limits_\gamma\frac{d\varsigma}{\varsigma-c_j}.$$

The regions Ω_r are $m+1$ $-$connected. The fundamental group $\pi_1(\Omega_r,z_0)$, $z_0\in\Omega_r$, is a free group of rank m. If $r_0<r\le+\infty$ it can be represented as follows. Consider closed

paths with the parameter interval $I = [0; 1]$ and initial point $z_0 \in \Omega$ consisted of the circles of radii $1/r_0$ centered at c_j, rounded $|k_j|$ times in the positive or negative direction, $k_j \in \mathbb{Z}$, and intervals joined these circles and passed in both directions a certain number of times. This class of paths, including the path γ, $\gamma(I) = z_0$, represents the group $\pi_1(\Omega_r, z_0)$. Thus, any closed path γ in Ω_r is Ω_r-homotopic to a path from mentioned class. For details see, for example, [9].

3. Preliminary Lemma on the logarithmic derivative for m-punctured planes.

Lemma 1. *Let f be a holomorphic function in Ω_r, $r_0 < r < +\infty$, $f(z) \neq 0$ in Ω_r. Then for any closed path γ in Ω_r*

$$\int\limits_{\gamma} \frac{F'(z)}{F(z)}\, dz = 0, \tag{1}$$

where

$$F(z) = f(z) \prod_{j=1}^{m} (z - c_j)^{-m_j}, \quad m_j = \frac{1}{2\pi i} \int\limits_{|z-c_j|=\frac{1}{r_0}} \frac{f'(z)}{f(z)}\, dz. \tag{2}$$

Proof. First of all, note that $m_j \in \mathbb{Z}$. This follows immediately from Theorem A.

According to the representation of the fundamental group $\pi_1(\Omega_r, z_0)$ and to Cauchy's theorem

$$\frac{1}{2\pi i} \int\limits_{\gamma} \frac{F'(z)}{F(z)}\, dz = \sum_{j=1}^{m} \Big(\frac{k_j}{2\pi i} \int\limits_{|z-c_j|=\frac{1}{r_0}} \frac{f'(z)}{f(z)}\, dz - \frac{m_j\, k_j}{2\pi i} \int\limits_{|z-c_j|=\frac{1}{r_0}} \frac{dz}{z - c_j} dz \Big).$$

Using (2) we obtain (1). □

In order to formulate a generalization of Lemma 1 for arbitrary meromorphic function in Ω assume that neither zeros nor poles lie in $\overline{\Omega}_{r_0}$. By Ω^* denote Ω without the intervals $\{z = \tau\, a : 1 \leq \tau < +\infty\}$, $|a| > r_0$, and $\{z = (1-\tau)a + \tau c_j : 0 \leq \tau < 1\}$, $|a - c_j| < 1/r_0$, where a is a zero or pole of $f, j \in \{1, 2, \ldots, m\}$. Set $\Omega_r^* = \Omega_r \cap \Omega^*$.

Lemma 2. *Under the above assumptions equality (1) holds for any closed path γ in Ω^*, where $F(z)$ is determined by (2).*

Proof. Note only that first we prove (1) for any closed path γ in Ω_r^* and after taking into account that r is arbitrary from the interval $(r_0, +\infty)$ we will obtain the same equality (1) for Ω^*.

Assume that f has unique zero a in Ω_r^* which is simple. The function $g(z) = f(z)/(z-a)$ is holomorphic in Ω_r^* without zeros. According to Lemma 1

$$\int_{\gamma} \frac{G'(z)}{G(z)}\, dz = 0, \tag{3}$$

for each closed path γ in Ω_r^*, where $G(z) = g(z) \prod\limits_{j=1}^{m} (z - c_j)^{-\nu_j}$, $\nu_j = \frac{1}{2\pi i} \int_{|z-c_j|=1/r_0} \frac{g'(z)}{g(z)} dz$, $\nu_j \in \mathbb{Z}$. If there exists $q \in \{1, 2, \ldots, m\}$ such that $|a - c_q| < 1/r_0$ then we define $m_q = \nu_q + 1$

and $m_j = \nu_j$ as $j \neq q, j \in \{1, \dots, m\}$. Put $F(z) = f(z)\prod_{j=1}^{m}(z - c_j)^{-m_j}$. Then $F(z) = G(z)(z-a)/(z-c_q)$ and

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{F'(z)}{F(z)}\,dz = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{G'(z)}{G(z)}\,dz + \frac{1}{2\pi i}\int\limits_{\gamma}\frac{dz}{z-a} - \frac{1}{2\pi i}\int\limits_{\gamma}\frac{dz}{z-c_q}. \tag{4}$$

Since a and c_q belong to the same component of complement of Ω_r^*, using Theorem A we obtain that two last integrals of (4) are equal. This and (3) give (1).

If $|a| > r_0$ we put $F(z) = f(z)\prod_{j=1}^{m}(z-c_j)^{-m_j}$, $m_j = \nu_j$. Then $F(z) = G(z)(z-a)$, a belongs to the unbounded component of $\mathbb{C}\backslash\Omega_r^*$. Using Theorem A and (3) we have

$$\int_{\gamma}\frac{F'(z)}{F(z)}\,dz = \int_{\gamma}\frac{G'(z)}{G(z)}\,dz + \int_{\gamma}\frac{dz}{z-a} = 0.$$

Note that in both cases m_j is determined by (2).
If a is a unique simple pole of f in Ω_r^* then the reflections are similar.
The needed conclusion for Ω_r^* in the general case is now obtained by recurrences. □

4. Logarithm of $f(z)\prod_{j=1}^{m}(z-c_j)^{-m_j}$. Let f be a meromorphic function in Ω, $f(z) \neq 0, \infty$ in Ω_{r_0} and let $F(z)$ be determined by relation (2). Define $\log F(z)$ in Ω^* as follows. Choose $z_0 \in \Omega_{r_0}$ and a value of $\log F(z_0)$. Put

$$\log F(z) = \log F(z_0) + \int\limits_{z_0}^{z}\frac{F'(\zeta)}{F(\zeta)}\,d\zeta, \qquad z \in \Omega^*, \tag{5}$$

where the integral is taken over a path γ_1 in Ω^*, $\gamma_1(0) = z_0$, $\gamma_1(1) = z$. If γ_2 is another path in Ω^*, $\gamma_2(0) = z_0, \gamma_2(1) = z$, then $\gamma = \gamma_2^{-1}\gamma_1$ is a closed path in Ω^*. By Lemma 2 relation (1) holds. Therefore, $\int_{\gamma_1}\frac{F'(z)}{F(z)}\,dz = \int_{\gamma_2}\frac{F'(z)}{F(z)}\,dz$. In other words, the integral in (5) does not depend on the path of integration joined z_0 and z. Thus, relation (5) determines a branch of $\log F$ in Ω^*.

5. Generalization of Jensen's Theorem. Let f be a meromorphic function in an m-punctured plane Ω. By $n_0(r, f)$ denote the counting function of its poles in $\overline{\Omega}_r$, $r_0 \leq r < +\infty$. Put

$$N_0(r, f) = \int\limits_{r_0}^{r}\frac{n_0(t, f)}{t}\,dt.$$

Theorem 1 (Jensen's Theorem for m-punctured planes). *Let f be a meromorphic function in an m-punctured plane Ω not identically equal to zero and let $r_0 < r < +\infty$. Then*

$$N_0(r, \frac{1}{f}) - N_0(r, f) = \frac{1}{2\pi}\int\limits_{0}^{2\pi}\log|f(re^{i\theta})|\,d\theta + \frac{1}{2\pi}\sum_{j=1}^{m}\int\limits_{0}^{2\pi}\log\left|f\left(c_j + \frac{1}{r}\,e^{i\theta}\right)\right|\,d\theta -$$
$$- \frac{1}{2\pi}\int\limits_{0}^{2\pi}\log|f(r_0e^{i\theta})|\,d\theta - \frac{1}{2\pi}\sum_{j=1}^{m}\int\limits_{0}^{2\pi}\log\left|f\left(c_j + \frac{1}{r_0}\,e^{i\theta}\right)\right|\,d\theta. \tag{6}$$

Proof. First assume that neither zeros nor poles of f lie in $\overline{\Omega}_{r_0}$. Let F be determined by (2). Apply the argument principle to F in Ω_t, supposing that $\partial\Omega_t$ is free of zeros and poles of f, $r_0 < t < +\infty$. Since $n_0(t, F) = n_0(t, f)$ and $n_0(t, \frac{1}{F}) = n_0(t, \frac{1}{f})$, we have

$$n_0(t, \frac{1}{f}) - n_0(t, f) = \frac{1}{2\pi i} \int\limits_{\partial\Omega_t} \frac{F'(z)}{F(z)}\, dz. \tag{7}$$

Dividing both side of (7) by t and integrating over t from r_0 to r we obtain

$$N_0(r, \frac{1}{f}) - N_0(r, f) = \frac{1}{2\pi} \int\limits_{r_0}^{r} \Big(\int\limits_0^{2\pi} \frac{F'(te^{i\theta})}{F(te^{i\theta})}\, e^{i\theta}\, d\theta \Big) dt -$$

$$- \frac{1}{2\pi} \int\limits_{r_0}^{r} \Big(\sum_{j=1}^{m} \int\limits_0^{2\pi} \frac{F'\Big(c_j + \frac{1}{t} e^{i\theta}\Big)}{F\Big(c_j + \frac{1}{t} e^{i\theta}\Big)}\, e^{i\theta}\, \frac{1}{t^2}\, d\theta \Big) dt. \tag{8}$$

Using the Fubini Theorem rewrite the right-hand side of (8) in the form

$$\frac{1}{2\pi} \int\limits_0^{2\pi} \Big(\int\limits_{r_0 e^{i\theta}}^{re^{i\theta}} \frac{F'(\zeta)}{F(\zeta)}\, d\zeta \Big) d\theta + \frac{1}{2\pi} \sum_{j=1}^{m} \int\limits_0^{2\pi} \Big(\int\limits_{c_j + \frac{1}{r_0} e^{i\theta}}^{c_j + \frac{1}{r} e^{i\theta}} \frac{F'(\zeta)}{F(\zeta)}\, d\zeta \Big) d\theta, \tag{9}$$

where the inner integrals are taken over the intervals joining the respective points.

According to (5) $\int_{z_1}^{z_2} \frac{F'(\zeta)}{F(\zeta)}\, d\zeta = \log F(z_2) - \log F(z_1)$, z_1, $z_2 \in \Omega^*$. Hence, relations (8) and (9) yield

$$N_0(r, \frac{1}{f}) - N_0(r, f) = \frac{1}{2\pi} \int\limits_0^{2\pi} \log F(re^{i\theta})\, d\theta + \frac{1}{2\pi} \sum_{j=1}^{m} \int\limits_0^{2\pi} \log F\Big(c_j + \frac{1}{r}\, e^{i\theta}\Big) d\theta -$$

$$- \frac{1}{2\pi} \int\limits_0^{2\pi} \log F(r_0 e^{i\theta}) d\theta - \frac{1}{2\pi} \sum_{j=1}^{m} \int\limits_0^{2\pi} \log F\Big(c_j + \frac{1}{r_0}\, e^{i\theta}\Big) d\theta. \tag{10}$$

Observing that $\log |F(z)| = \log |f(z)| - \sum\limits_{k=1}^{m} m_k \log |z - c_k|$, and taking the real parts of both sides of (10) we obtain

$$N_0(r, \frac{1}{f}) - N_0(r, f) = \frac{1}{2\pi} \int\limits_0^{2\pi} \log |f(re^{i\theta})|\, d\theta + \frac{1}{2\pi} \sum_{j=1}^{m} \int\limits_0^{2\pi} \log \Big| f\Big(c_j + \frac{1}{r}\, e^{i\theta}\Big) \Big| d\theta -$$

$$- \frac{1}{2\pi} \int\limits_0^{2\pi} \log \Big| f(r_0 e^{i\theta}) \Big| d\theta - \frac{1}{2\pi} \sum_{j=1}^{m} \int\limits_0^{2\pi} \log \Big| f\Big(c_j + \frac{1}{r_0}\, e^{i\theta}\Big) \Big| d\theta -$$

$$- \frac{1}{2\pi} \sum_{j=1}^{m} m_j \int\limits_0^{2\pi} \Big(\log |re^{i\theta} - c_j| - \log |r_0 e^{i\theta} - c_j| \Big) d\theta - \sum_{j=1}^{m} m_j \Big(\log \frac{1}{r} - \log \frac{1}{r_0} \Big) -$$

$$-\frac{1}{2\pi}\sum_{j=1}^{m}\sum_{k\neq j} m_k\int\limits_0^{2\pi}\Big(\log\Big|c_j-c_k+\frac{1}{r}\,e^{i\theta}\Big|-\log\Big|c_j-c_k+\frac{1}{r_0}\,e^{i\theta}\Big|\Big)d\theta. \qquad (11)$$

In virtue of classical Jensen's Theorem for a disk $\frac{1}{2\pi}\int_0^{2\pi}\log|te^{i\theta}-c_j|d\theta=\log t$, $r_0\le t\le r$, $j\in\{1,\dots,m\}$, because $|c_j|<r_0$, and $\frac{1}{2\pi}\int_0^{2\pi}\log|c_j-c_k+\frac{1}{r}e^{i\theta}|d\theta=\log|c_j-c_k|$, $j\neq k$, because $|c_j-c_k|>1/r$, $j\neq k$. Combining these equalities and (11), we get (6).

Let now f admit zeros or poles in $\overline{\Omega}_{r_0}$. Consider the function $g(z)=f(z)\prod(z-b)/\prod(z-a)$, where the products are taking over poles b and zeros a of f lying in $\overline{\Omega}_{r_0}$. Apply (6) to g. Four integrals at the right-hand side of (6) written for $\prod(z-b)/\prod(z-a)$ give according to the classical Jensen formula $(n_0(r_0,f)-n_0(r_0,\frac{1}{f}))\log r/r_0$. Taking into account that $n_0(r,f)=n(r,g)+n_0(r_0,f)$ and $n_0(r,\frac{1}{f})=n_0(r,\frac{1}{g})+n_0(r_0,\frac{1}{f})$ we obtain (6) for f. This finishes the proof. □

6. Nevanlinna's characteristic and its properties. Let f be a meromorphic function in an m-punctured plane Ω. Denote

$$\begin{aligned} m_0(r,f) \;=\; & \frac{1}{2\pi}\int\limits_0^{2\pi}\log^+|f(re^{i\theta})|+\frac{1}{2\pi}\sum_{j=1}^{m}\int\limits_0^{2\pi}\log^+\Big|f\Big(c_j+\frac{1}{r}e^{i\theta}\Big)\Big|d\theta-\\ -\; & \frac{1}{2\pi}\int\limits_0^{2\pi}\log^+|f(r_0e^{i\theta})|-\frac{1}{2\pi}\sum_{j=1}^{m}\log^+\Big|f\Big(c_j+\frac{1}{r_0}e^{i\theta}\Big)\Big|d\theta, \qquad r_0\le r<+\infty. \end{aligned}$$

Let f be a meromorphic function in Ω. The function

$$T_0(r,f)=N_0(r,f)+m_0(r,f), \qquad r_0\le r<+\infty, \qquad (12)$$

is called *the Nevanlinna characteristic of f*.

Theorem 2 (Cartan's Theorem for m-punctured plane). *Let f be a meromorphic function in Ω. Then*

$$T_0(r,f)=\frac{1}{2\pi}\int\limits_0^{2\pi}N_0(r,\frac{1}{f-e^{i\varphi}})\,d\varphi, \qquad r_0\le r<+\infty. \qquad (13)$$

Proof. It is similar to the proof of the classical Cartan's theorem for a disc [10].

Let f be not identically equal to a constant. Rewrite (6) for the function $f(z)-e^{i\varphi}$. We have

$$\begin{aligned} N_0(r,\frac{1}{f-e^{i\varphi}}) \;-\; N_0(r,f) = \\ =\frac{1}{2\pi}\int\limits_0^{2\pi}\log|f(re^{i\theta})-e^{i\varphi}|\,d\theta \;+\; \frac{1}{2\pi}\sum_{j=1}^{m}\int\limits_0^{2\pi}\log\Big|f\Big(c_j+\frac{e^{i\theta}}{r}\Big)-e^{i\varphi}\Big|\,d\theta-\\ -\frac{1}{2\pi}\int\limits_0^{2\pi}\log|f(r_0e^{i\theta})-e^{i\varphi}|\,d\theta \;-\; \frac{1}{2\pi}\sum_{j=1}^{m}\int\limits_0^{2\pi}\log\Big|f\Big(c_j+\frac{e^{i\theta}}{r_0}\Big)-e^{i\varphi}\Big|\,d\theta. \qquad (14) \end{aligned}$$

Since Jensen's formula for a disk gives $\frac{1}{2\pi}\int_0^{2\pi}\log|w-e^{i\varphi}|\,d\varphi=\log^+|w|$, integrating (14) over φ, noting that the double integrals exist and using the Fubini theorem we obtain (13).

If f identically equals a constant, (13) is evident. This completes the proof. □

The Cartan's theorem allows to establish the following properties of the Nevanlinna characteristic $T_0(r,f)$ like its classical counterpart $T(r,f)$ for meromorphic functions in a disk and to Nevanlinna's characteristic $T_0(r,f)$ for meromorphic functions on annuli.

Theorem 3. *Let f, f_1, f_2 be meromorphic functions in an m-punctured plane Ω. Then*

(i) the function $T_0(r,f)$ is non-negative, continuous, non-decreasing and convex with respect to $\log r$ on $[r_0;+\infty)$, $T_0(r_0,f)=0$;

(ii) if f identically equals a constant, then $T_0(r,f)$ vanishes identically;

(iii) if f is not identically equal to zero, then $T_0(r,f)=T_0(r,\frac{1}{f})$, $r_0\le r<+\infty$;

(iv) $T_0(r,f_1f_2)\le T_0(r,f_1)+T_0(r,f_2)+O(1)$, $r_0\le r<+\infty$, and
$T_0(r,f_1+f_2)\le T_0(r,f_1)+T_0(r,f_2)+O(1)$, $r_0\le r<+\infty$,

Proof. The function $N_0(r,f)$ is non-negative, continuous, non-decreasing and convex with respect to $\log r$ on $[1,+\infty)$. By virtue of Cartan's Theorem, $T_0(r,f)$ is the integral of an one-parametric family of such functions. Hence, $T_0(r,f)$ satisfies (i).

Using (12) and denoting $x^-=(-x)^+$, rewrite (6) as follows

$$T_0(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^-|f(re^{i\theta})|\,d\theta+\frac{1}{2\pi}\sum_{j=1}^m\int_0^{2\pi}\log^-\left|f\left(c_j+\frac{1}{r}e^{i\theta}\right)\right|d\theta-$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\log^-|f(r_0e^{i\theta})|\,d\theta-\frac{1}{2\pi}\sum_{j=1}^m\int_0^{2\pi}\log^-\left|f\left(c_j+\frac{1}{r_0}e^{i\theta}\right)\right|d\theta+N_0(r,\frac{1}{f}). \tag{15}$$

The right-hand side of this equality is $T_0(r,\frac{1}{f})$, because $\log^-|f|=(-\log|f|)^+=\log^+|\frac{1}{f}|$. We have (iii).

Using the inequalities $\log^+ xy\le\log^+x+\log^+y$, $\log(x+y)\le\log^+x+\log^+y+\log 2$, for positive x,y and $n_0(r,f_1f_2)\le n_0(r,f_1)+n_0(r,f_2)$, $n_0(r,f_1+f_2)\le n_0(r,f_1)+n_0(r,f_2)$, we obtain (iv). The proof is complete. □

7. First Fundamental Theorem. The case $T_0(r,f)=O(\log r)$.

Theorem 4. *Let f be a non-constant meromorphic function in an m-punctured plane Ω. Then*

$$T_0\left(r,\frac{1}{f-a}\right)=T_0(r,f)+O(1), \tag{16}$$

for any fixed $a\in\mathbb{C}$ and all r, $r_0\le r<+\infty$.

Proof. Assertions (ii) and (iv) of Theorem 3 yield

$$T_0(r,f-a)\le T_0(r,f)+O(1)\le T_0(r,f-a)+O(1),\qquad r_0\le r<+\infty. \tag{17}$$

Assertion (iii) of the mentioned theorem together with (17) impliy (16). □

Theorem 5. *Let f be a meromorphic function in an $m-$punctured plane Ω. Then*

$$\liminf_{r\to+\infty}\frac{T_0(r,f)}{\log r}<+\infty \tag{18}$$

iff f is rational.

Proof. Let f be a rational function. Then $n_0(t,f)$ is bounded. Hence, $N_0(r,f)=O(\log r)$, $r\to+\infty$. Besides $\log^+|f(re^{i\theta})|=O(\log r)$ and $\log^+|f(c_j+1/re^{i\theta})|=O(\log r)$, $j\in\{1,\dots,m\}$, $r\to+\infty$, uniformly over θ, and we obtain (18).

Now let (18) hold. Then for each $r>1$ there is a $n_0(r,f)\int_r^{r_j}\frac{dt}{t}\le\int_r^{r_j}\frac{n(t,f)}{t}dt\le N_0(r_j,f)\le C\log r_j$, $1<r^2<r_j$, $C=const$. Hence

$$n_0(r,f)(\log r_j-\log r)\le C\log r_j. \tag{19}$$

As $1<r^2<r_j$ implies $\log r_j-\log r>\frac{1}{2}\log r_j$, (19) gives $n_0(r,f)\le 2C$. We have that $n_0(t,f)$ is bounded. Thus, the function f has a finite number of poles. According to statement (iii) of Theorem 3 it has a finite number of zeros as well.

Consider the function $g(z)=f(z)/\mathcal{R}(z)$ where $\mathcal{R}(z)$ is a rational function, zeros and poles of which coincide with the ones of f taking into account their multiplicities. We have, as above, $T_0(r,\mathcal{R})=O(\log r)$, $r\to+\infty$. Therefore,

$$\liminf_{r\to+\infty}\frac{T_0(r,g)}{\log r}<+\infty \tag{20}$$

due to (18) and assertion (iv) of Theorem 3. The function $g(z)$ has neither zeros nor poles in Ω. In virtue of Lemma 2 there are integers $m_1,m_2,\dots,m_m$ such that a branch of $\log G(z)$,

$$G(z)=g(z)\prod_{j=1}^{m}(z-c_j)^{-m_j}, \tag{21}$$

is determined in Ω. Consider its Laurent expansions $\log G(z)=\sum_{k\in\mathbb{Z}}c_{k\infty}z^k$ $(r_0<|z|)$ and $G(z)=\sum_{k\in\mathbb{Z}}c_{kj}(z-c_j)^k$ $(0<|z-c_j|<1/r_0)$, $j\in\{1,2,\dots,m\}$. Then
$\log|G(z+c_j)|=\sum_{k\in\mathbb{Z}}Re(c_{kj}z^k)=\frac{1}{2}\sum_{k\in\mathbb{Z}}(c_{kj}r^{-k}+c_{-kj}r^k)e^{ik\theta}$, $\quad r_0<r$. We have

$$\frac{1}{2}\left(c_{k\infty}r^k+c_{-k\infty}r^{-k}\right)=\frac{1}{2\pi}\int_0^{2\pi}\log|G(re^{i\theta})|e^{-ik\theta}\,d\theta,\qquad r_0<r,\qquad k\in\mathbb{Z}, \tag{22}$$

and for $j\in\{1,2,\dots,m\}$, $k\in\mathbb{Z}$

$$\frac{1}{2}\left(c_{kj}r^{-k}+c_{-kj}r^k\right)=\frac{1}{2\pi}\int_0^{2\pi}\log|G(c_j+\frac{1}{r}e^{i\theta})|e^{-ik\theta}\,d\theta,\qquad r_0<r.$$

It follows from assertions (iv) and (iii) of Theorem 3 that $T_0(r,G)=T_0(r,g)+O(\log r)$,

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|\log|G(re^{i\theta})|\Big|\,d\theta\le 2\,T_0(r,G)+O(1),\qquad r\to+\infty, \tag{23}$$

and for $j \in \{1, 2, \dots, m\}$

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\Big|\log|G(c_j+\frac{1}{r}e^{i\theta})|\Big|\,d\theta \le 2\,T_0(r,G)+O(1), \qquad r\to+\infty.$$

Relations (20), (23) and (24) imply $c_{-kj} = 0$ as $k \in \mathbb{N}$, $j \in \{1, 2, \dots, m\}$. Hence, the function $\log G(z)$ admits an analytic continuation to $\mathbb{C}$. Relations (20), (22) and (23) give $c_{k\infty} = 0$, $k \in \mathbb{N}$. Thus, the function $\log G(z)$ is bounded in $\mathbb{C}$. In virtue of Liouville's Theorem it is constant. Taking into account (21) we obtain that $g(z)$ is a rational function. The equality $f(z) = g(z)\mathcal{R}(z)$ yields that $f(z)$ is also rational. This completes the proof. □

8. Decomposition Lemma for meromorphic functions in m-punctured planes.

Lemma 3. (Compare with Theorem 1.2 from [1]) *Let f be a non-constant meromorphic function in an m-punctured plane Ω. Then there exist functions $f_0, f_1, \dots, f_m$ such that*

(i) f_0 is meromorphic in $\mathbb{C}$, f_j is meromorphic in $\mathbb{C}\backslash\{c_j\}$, $j \in \{1, 2, \dots, m\}$, and

$$f(z) = f_0(z)f_1(z)\dots f_m(z), \qquad z \in \Omega; \tag{24}$$

(ii) zero and pole sets of $f_0(z)$, taking into account the multiplicities, coincide with those of f(z) in $\Omega\backslash\bigcup\limits_{j=1}^{m} D_{1/r_0}(c_j)$, zero and pole sets of $f_j(z)$ coincide with those of f in $D_{1/r_0}(c_j)\backslash\{c_j\}, j \in \{1, 2, \dots, m\}$, except a possible pole or zero of $f_j(z)$ at ∞ of multiplicity $|k_j(f)|$.

Proof. First suppose that neither zeros nor poles of f lie on $\overline{\Omega}_{r_0}$. Let F be determined by

$$F(z) = f(z)\prod_{j=1}^{m}(z_j - c_j)^{-m_j}, \tag{25}$$

where

$$m_j = \frac{1}{2\pi i}\int\limits_{|z-c_j|=\frac{1}{r_0}}\frac{f'(z)}{f(z)}\,dz, \qquad j \in \{1, 2, \dots, m\}. \tag{26}$$

In view of Preliminary Lemma on logarithmic derivative for $m-$ punctured planes a branch of $\log F(z)$ can be determined in Ω_{r_0}. Secondly, the region Ω is the intersection of $\mathbb{C}$ with the set of points $\{c_j\}, j \in \{1, 2, \dots, m\}$. We will decompose $\log F(z)$ in the form of a sum

$$\log F(z) = \sum_{j=0}^{m}\Psi_j(z), \qquad z \in \Omega_{r_0}, \tag{27}$$

where the functions $\{\Psi_j(z)\}$ are holomorphic in some subsets of the intersection.

This decomposition generates a decomposition of $F(z)$ and, consequently, of $f(z)$ in the form of the products $F(z) = \prod\limits_{j=0}^{m}\exp\{\Psi_j(z)\}$, $f(z) = \exp\{\Psi_0(z)\}\prod\limits_{j=1}^{m}(z-c_j)^{m_j}\exp\{\Psi_j(z)\} :=$ $= f_0 f_1(z)\dots f_m(z)$, $z \in \Omega_{r_0}$. Continuing meromorfically each factor in the respective domain of the intersection mentioned above we can obtain (24).

Now, we begin the proof decomposing the logarithmic derivative $F'(z)/F(z)$. By the Cauchy formula

$$\frac{F'(z)}{F(z)} = \frac{1}{2\pi i} \int\limits_{\partial\Omega_{r_0}} \frac{F'(\zeta)}{F(\zeta)} \frac{d\zeta}{\zeta - z}, \qquad z \in \Omega_{r_0}. \tag{28}$$

Put

$$\psi_0(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=r_0} \frac{F'(\zeta)}{F(\zeta)} \frac{d\zeta}{\zeta - z}, \qquad |z| < r_0, \tag{29}$$

and

$$\psi_j(z) = \frac{1}{2\pi i} \int\limits_{|\zeta - c_j| = \frac{1}{r_0}} \frac{F'(\zeta)}{F(\zeta)} \frac{d\zeta}{z - \zeta}, \qquad |z - c_j| > \frac{1}{r_0}, \qquad j \in \{1, 2, \ldots, m\}. \tag{30}$$

Relations (28) - (30) imply

$$\frac{F'(z)}{F(z)} = \sum_{j=0}^{m} \psi_j(z), \qquad z \in \Omega_{r_0}. \tag{31}$$

The function $\psi_0(z)$ is holomorphic in the disk $D_{r_0}(0)$ and the function $\psi_j(z)$ is holomorphic outside $\overline{D}_{1/r_0}(c_j)$, $j \in \{1, 2, \ldots, m\}$. Besides,

$$\psi_j(z) = \frac{1}{2\pi i} \int\limits_{|\zeta - c_j| = \frac{1}{r_0}} \frac{f'(\zeta)}{f(\zeta)} \left(\frac{1}{z - \zeta} - \frac{1}{z - c_j} \right) d\zeta = \frac{1}{2\pi i} \int\limits_{|\zeta - c_j| = \frac{1}{r_0}} \frac{f'(\zeta)}{f(\zeta)} \frac{\zeta - c_j}{(z - \zeta)(z - c_j)} d\zeta,$$

by the definition of $F(z)$, $j \in \{1, 2, \ldots, m\}$, and Cauchy's Theorem. Hence, the function $\psi_j(z)$ has zero at ∞ of the multiplicity at least 2, $j \in \{1, 2, \ldots, m\}$.

Let $z_0 \in \Omega_{r_0}$. Denote $\Psi_0(z) = \int_{z_0}^{z} \psi_0(\zeta)\, d\zeta + \log F(z_0)$, $z \in D_{r_0}(0)$. Since the function $\psi_0(z)$ is holomorphic in $D_{r_0}(0)$, the integral does not depend on the path of integration lying in the disk $D_{r_0}(0)$ and determines there a holomorphic function.

If γ is a closed path in $\mathbb{C} \backslash \overline{D}_{1/r_0}(c_j)$, $j \in \{1, 2, \ldots, m\}$, then $\mathrm{Ind}_\gamma(c_j) = \mathrm{Ind}_\gamma(\zeta)$ for $\zeta \in \partial D_{1/r_0}(c_j)$, because the points c_j and ζ belong to the same component of the complement of γ^*. Thus, for $j \in \{1, 2, \ldots, m\}$

$$\frac{1}{2\pi i} \int\limits_{\gamma} \psi_j(z)\, dz = \frac{1}{2\pi i} \int\limits_{|z - c_j| = \frac{1}{r_0}} \frac{f'(\zeta)}{f(\zeta)} \left(\frac{1}{2\pi i} \int\limits_{\gamma} \frac{dz}{z - \zeta} \right) d\zeta - m_j\, \mathrm{Ind}_\gamma(c_j) = (m - m_j) \mathrm{Ind}_\gamma(c_j) = 0.$$

Therefore, the integral $\int_{z_0}^{z} \psi_j(\zeta)\, d\zeta = \Psi_j(z)$, $z \in \mathbb{C} \backslash \overline{D}_{1/r_0}(c_j)$, $j \in \{1, 2, \ldots, m\}$, does not depend on the path of integration and determines a holomorphic in $\mathbb{C} \backslash \overline{D}_{\frac{1}{r_0}}(c_j)$ function $\Psi_j(z)$, $j \in \{1, 2, \ldots, m\}$. The value $\Psi_j(\infty)$ is finite because $\psi_j(z)$ has zero at ∞ of the multiplicity at least 2, what was mentioned above.

Relation (31) implies (27). Denote $F_j(z) = \exp \Psi_j(z)$, $j \in \{0, 1, \dots, m\}$, $f_0(z) = F_0(z)$, and $f_j(z) = (z - c_j)^{m_j} F_j(z)$, $j \in \{1, 2, \dots, m\}$. It follows from (27) that $F(z) = \prod_{j=0}^{m} F_j(z)$, $z \in \Omega_{r_0}$, and

$$f(z) = \prod_{j=0}^{m} f_j(z), \qquad z \in \Omega_{r_0}. \tag{32}$$

Besides, $F_j(\infty) \neq 0, \infty$, $j \in \{1, 2, \dots, m\}$.

The equality

$$f_0(z) = f(z) / \big(f_1(z) f_2(z) \dots f_m(z)\big), \tag{33}$$

which follows from (32) and is true for $z \in \Omega_{r_0}$, provides also the meromorphic continuation of $f_0(z)$ in $\mathbb{C}$, because each function at the right-side of (33) is meromorphic in $\mathbb{C} \setminus \bigcup_{j=1}^{m} \overline{D}_{1/r_0}(c_j)$, and $f_0(z)$ is holomorphic in $D_{r_0}(0)$. The equality

$$f_j(z) = f(z) / \prod_{k \neq j} f_k(z), \qquad j \in \{1, 2, \dots, m\}, \tag{34}$$

which also follows from (32) and is true in Ω_{r_0}, gives the meromorphic continuation of $f_j(z)$ in $\mathbb{C} \setminus \{c_j\}$, $j \in \{1, 2, \dots, m\}$, because each function at the right-hand side of (34) is meromorphic in $\Omega \setminus \bigcup_{k=1}^{m} \overline{D}_{1/r_0}(c_k)$, and $f_j(z)$ is holomorphic in $\mathbb{C} \setminus \overline{D}_{1/r_0}(c_j)$. Therefore, we have assertion (i).

It follows from the definition of $f_j(z)$, $j \in \{0, 1, \dots, m\}$, that $f_0(z) \neq 0, \infty$ in $D_{r_0}(0)$ and $f_j(z) \neq 0, \infty$ in $\mathbb{C} \setminus \overline{D}_{1/r_0}(c_j)$, $j \in \{1, 2, \dots, m\}$. Together with (33) and (34) this shows that zero and pole sets of $f(z)$, taking into account the multiplicities, coincide with those of $f(z)$ in $\mathbb{C} \setminus D_{r_0}(0)$ and that zero and pole sets of $f_j(z)$ coincide with those of $f(z)$ in $D_{1/r_0}(0) \setminus \{c_j\}$, $j \in \{1, 2, \dots, m\}$, except a pole of $f_j(z)$ at ∞ of the multiplicity $m_j = m_j(f)$ if $m_j > 0$, or a zero of $f_j(z)$ at ∞ of the multiplicity $|m_j| = |m_j(f)|$ if $m_j < 0$, because $F_j(\infty) \neq 0, \infty$, $j \in \{1, 2, \dots, m\}$. Thus, assertion (ii) is true under the assumption $f(z) \neq 0, \infty$ in Ω_{r_0}.

In the general case apply the proved to the function

$$g(z) = f(z) \prod_{b_k \in \overline{\Omega}_{r_0}} (z - b_k) / \prod_{a_k \in \overline{\Omega}_{r_0}} (z - a_k) := f(z) \mathcal{R}(z),$$

where a_k is a zero and b_k is a pole of f respectively. Then $g(z) = g_0(z) g_1(z) \dots g_m(z)$, $z \in \Omega$. Setting $f_0(z) = g_0(z)/\mathcal{R}(z)$, $f_j(z) = g_j(z)$, $j \in \{1, 2, \dots, m\}$, and $k_j(f) = m_j(g)$ we obtain (ii) in the general case.

□

Let f be a meromorphic function in an m-punctured plane Ω. Representation (24), where $f_j(z)$, $j \in \{0, 1, \dots, m\}$, satisfies conditions (i) and (ii) of Lemma 3 is called *a decomposition of f*.

Lemma 3 provides the existence of a decomposition of arbitrary non-constant meromorphic function in Ω. The following lemma shows that the decomposition is unique up to constants.

Lemma 4. *Let f be a meromorphic function in an m-punctured plane Ω, let (24) and*

$$f(z) = \varphi_0(z)\varphi_1(z)\dots\varphi_m(z), \qquad z \in \Omega, \tag{35}$$

be decompositions of f. Then there are constants $\alpha_0, \alpha_1, \dots, \alpha_m$, such that $f_j(z)/\varphi_j(z) = \alpha_j$, $j \in \{0, 1, \dots, m\}$.

Proof. According to conditions (i) and (ii) of Lemma 3 the quotient $f_j(z)/\varphi_j(z)$ is holomorphic in $\mathbb{C}\backslash\{c_j\}$, $j \in \{1, 2, \dots, m\}$. It is bounded near ∞ because a pole or a zero of f_j and φ_j at ∞ of the multiplicity $|k_j(f)|$ is uniquely determined by f. Relations (24) and (35) imply

$$\frac{f_j(z)}{\varphi_j(z)} = \prod_{k \neq j} \varphi_k(z) / \prod_{k \neq j} f_k(z), \qquad j \in \{1, 2, \dots, m\}. \tag{36}$$

But the right-hand side of (24) is a holomorphic function in $D_{\frac{1}{r_0}}(0)$. Hence, the quotient $f_j(z)/\varphi_j(z)$ admits a holomorphic continuation for $\{c_j\}$, $j \in \{1, 2, \dots, m\}$. By the Liouville Theorem it is constant, say α_j, $\alpha_j \neq 0$, $j \in \{1, 2, \dots, m\}$. This and relations (24) and (35) imply also $f_0(z)/\varphi_0(z) = 1/(\alpha_1\alpha_2 \dots \alpha_m) := \alpha_0$. The proof is complete.

□

Lemma 5. *Let f be a meromorphic function in an m-punctured plane Ω, let (24) be its decomposition, and let $\widehat{f}_j(\zeta) = f_j\left(c_j + \frac{1}{\zeta}\right)$, $j \in \{1, 2, \dots, m\}$. Then the functions $\widehat{f}_j(\zeta)$, $j \in \{1, 2, \dots, m\}$, are meromorphic in $\mathbb{C}$, and there is a constant $C_0 = C_0(f, \Omega)$ such that*

$$\left| T_0(r, f) - T(r, f_0) - \sum_{j=1}^{m} T(r, \widehat{f}_j) \right| \leq C_0(1 + \log^+ r), \quad r_0 \leq r < +\infty, \tag{37}$$

where T denotes the classical Nevanlinna characteristic for a disk.

Proof. Using Lemma 3 we have $n_0(r, f) = n(r, f_0) + \sum\limits_{j=1}^{m} n(r, \widehat{f}_j) - \sum\limits_{j=1}^{m} k_j^+(f)$, $r_0 \leq r < +\infty$, where

$$k_j(f) = \frac{1}{2\pi} \int\limits_{|z-c_j|=\frac{1}{r_0}} \frac{g'(z)}{g(z)}\, dz, \quad g(z) = f(z) \prod (z - b_j) / \prod (z - a_j), \tag{38}$$

$\{a_j\}$ are zeros and $\{b_j\}$ are poles of f in $\overline{\Omega}_{r_0}$. Therefore,

$$N_0(r, f) = N(r, f_0) + \sum_{j=1}^{m} N(r, \widehat{f}_j) - \sum_{j=1}^{m} k_j^+(f) \log \frac{r}{r_0}, \qquad r_0 \leq r < +\infty. \tag{39}$$

Since the function $\widehat{f}_j(\zeta)$ has zero or pole at ∞ of the multiplicity $|k_j(f)|$, (24) implies

$$\frac{1}{2\pi} \int\limits_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = m(r, f_0) + O(\log^+ r), \qquad r_0 \leq r < +\infty. \tag{40}$$

The functions $f_k(z), k \neq j$, are without zeros, holomorphic and bounded in $\overline{D}_{1/r_0}(c_j)$ except the function $f_0(z)$ which can possess zeros or poles of $f(z)$ lying on $\partial D_{1/r_0}(c_j)$ and is holomorphic without zeros in $D_{1/r_0}(c_j)$. Thus, we deduce from (24)

$\frac{1}{2\pi}\int_0^{2\pi}\left|\log|f_0(c_j+\frac{1}{r}e^{i\theta})|\right|d\theta = O(1)$, $r_0 \le r < +\infty$, $j \in \{1,2,\dots,m\}$, and

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(c_j+\frac{1}{r}e^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log^+|f(c_j+\frac{1}{r}e^{i\theta})|d\theta +$$

$$+O(\log^+ r) = \frac{1}{2\pi}\int_0^{2\pi}\log^+|\widehat{f}(re^{i\theta})|d\theta + O(\log^+ r), \quad r_0 \le r < +\infty, \quad j \in \{1,2,\dots,m\}. \quad (41)$$

Relations (38) - (41) yield (37). This completes the proof.

□

9. Main Lemma on logarithmic derivative for m-punctured planes.

Lemma 6. *Let f be a non-constant meromorphic function in an m-punctured plane Ω. Then there is a constant $C = C(f,\Omega)$ such that*

$$m_0(r,\frac{f'}{f}) \le (m+1)\log^+\frac{RT_0(r,f)}{R-r} + \log^+\frac{CR(1+\log^+ r)}{r(R-r)} + C(1+\log^+ r), \quad (42)$$

for $r_0 \le r < R < +\infty$.

Proof. Let (24) be a decomposition of f. Then $f'/f = \sum_{j=0}^{m} f_j'/f_j$. Using the inequality $\log^+\sum_{j=0}^{m} x_j \le \sum_{j=0}^{m}\log^+ x_j + \log(m+1)$, $x_j \ge 0$, we find that

$$m_0(r,\frac{f'}{f}) \;\le\; \sum_{j=0}^{m} m_0(r,\frac{f_j'}{f_j}) + (m+1)\log(m+1), \qquad r_0 \le r < +\infty. \quad (43)$$

Since $f_j'(\zeta) = -f_j\left(c_j+\frac{1}{\zeta}\right)\frac{1}{\zeta^2}$, $j \in \{1,2,\dots,m\}$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{f_j'\left(c_j+\frac{1}{r}e^{i\theta}\right)}{f_j\left(c_j+\frac{1}{r}e^{i\theta}\right)}\right|d\theta \le m(r,\frac{\widehat{f_j}'}{\widehat{f_j}}) + \log^+ r^2, \quad r_0 \le r < +\infty, \quad j \in \{1,2,\dots,m\}. \quad (44)$$

The function $(z-c_j)^{-k_j(f)}f_j(z)$, where $k_j(f)$ is determined by (38) has no zeros outside $D_{1/r_0}(c_j)$, it is holomorphic and bounded there, $j \in \{1,2,\dots,m\}$. Hence,

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{f_j'(r_je^{i\theta})}{f_j(r_je^{i\theta})}\right|d\theta \le C_1(1+\log^+ r), \quad j \in \{1,2,\dots,m\}, \quad r_0 \le r < +\infty, \quad (45)$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{f_k'\left(c_j+\frac{1}{r}e^{i\theta}\right)}{f_k\left(c_j+\frac{1}{r}e^{i\theta}\right)}\right|d\theta \le C_1(1+\log^+ r), \quad j \in \{1,2,\dots,m\}, \quad r_0 \le r < +\infty, \quad (46)$$

where C_1 is a constant non-depending on a decomposition of f in virtue of Lemma 4.

Relations (43) - (46) imply

$$m_0(r, \frac{f'}{f}) \leq \sum_{j=1}^{m} m(r, \frac{\widehat{f}_j'}{\widehat{f}_j}) + m(r, \frac{f_0'}{f_0}) + C_2(1 + \log^+ r)\log(m+1), \quad r_0 \leq r < +\infty, \tag{47}$$

where C_2 is a constant depending on f and Ω only. Further we use the following improvement of the classical Logarithmic Derivative Lemma for a meromorphic in $\mathbb{C}$ function g, $g(0) = 1$, $m(t, g'/g) \leq \log^+ \left(\frac{T(R,g)}{t}\frac{R}{R-t}\right) + 4.86$, $0 < t < R < +\infty$. We deduce from it and (47)

$$m_0(r, \frac{f'}{f}) \leq \sum_{j=1}^{m} \log^+ \left(\frac{T(r, \widehat{f}_j)}{r}\frac{R}{R-r}\right) + \log^+ \left(\frac{T(r, f_0)}{r}\frac{R}{R-r}\right) + 5(m+1) + C_3(1 + \log^+ r) \tag{48}$$

for $r_0 \leq r < R < +\infty$, where C_3 is a constant.

Using (37) we find $T(r, f_0) \leq T_0(r, f) + C_0(1 + \log^+ r)$, $T(r, \widehat{f}_j) \leq T_0(r, f) + C_0(1 + \log^+ r)$ ($j \in \{1, 2, \dots, m\}$), and

$$m_0(r, \frac{f'}{f}) \leq (m+1)\log^+ \left(\frac{T_0(r, f) + C_0(1 + \log^+ r)}{r}\frac{R}{R-r}\right) + 5(m+1) + C_3(1 + \log^+ r)$$

for $r_0 \leq r < R < +\infty$, where C_0 is a constant. This implies $m_0(r, \frac{f'}{f}) \leq (m+1)\log^+ \frac{RT_0(r,f)}{R-r} + \log^+ \frac{C_0R(1+\log^+ r)}{r(R-r)} + (m+1)\log 2 + 5(m+1) + C_3(1 + \log^+ r)$, $r_0 \leq r < R < +\infty$. Taking $C = \max(C_0, C_3 + (m+1)(5 + \log 2))$ we obtain (42). □

10. Second Fundamental Theorem for m-punctured planes. Deficient values. If f is a meromorphic function in an m-punctured plane Ω, then $n_0(r, 1/f') = \sum_{a\in\mathbb{C}} \widetilde{n}_0(r, 1/(f-a))$, $r_0 \leq r < +\infty$, where $\widetilde{n}_0(r, 1/(f-a))$ is the counting function of a−points of f with the multiplicities reduced by 1, and the equalities

$$\widehat{n}_0(r, f) := \widetilde{n}_0(r, f) + \sum_{a\in\mathbb{C}} \widetilde{n}_0(r, \frac{1}{f-a}) = n_0(r, \frac{1}{f'}) + 2\,n_0(r, f) - n_0(r, f'),$$

and $\widehat{N}_0(r, f) = N_0(r, \frac{1}{f'}) + 2\,N_0(r, f) - N_0(r, f')$, where $\widehat{N}_0(r, f) = \int_1^r \widehat{n}_0(t, f)/t dt$ ($1 \leq r < R_0$), hold for $r \in [r_0, +\infty)$.

Theorem 6. *Let f be a non-constant meromorphic function in an m-punctured plane Ω, and let $a_1, a_2, \dots, a_q$ be distinct complex numbers. Then*

$$m_0(r, f) + \sum_{\nu=1}^{q} m_0(r, \frac{1}{f - a_\nu}) \leq 2T_0(r, f) - \widehat{N}_0(r, f) + S(r, f), \quad r_0 \leq r < +\infty, \tag{49}$$

where $\widehat{N}_0(r, f) = N_0(r, \frac{1}{f'}) + 2N_0(r, f) - N_0(r, f')$ and

$$S(r, f) = O(\log T_0(r, f)) + O(\log^+ r), \qquad r \to +\infty, \tag{50}$$

outside a set of finite measure.

Proof. We need the equality

$$S(r,f) = m_0(r,\frac{f'}{f}) + \sum_{\nu=1}^{q} m_0(r,\frac{f'}{f-a_\nu}) + O(1), \qquad r\to+\infty. \tag{51}$$

for meromorphic functions in Ω with $r\in[r_0,+\infty)$. Its proof can be obtained following ([11], p.128). Using Lemma 6, equality (51) and the well-known Borel-Nevanlinna Lemma we obtain (49), where $S(r,f)$ satisfies (50) outside a set of finite measure. □

Let f be a meromorphic function in an $m-$punctured plane. The *deficiency of f for a value $a\in\overline{\mathbb{C}}$* is

$$\delta_0(a,f) = \liminf_{r\to+\infty}\frac{m_0(r,\frac{1}{f-a})}{T_0(r,f)}, \qquad \delta_0(\infty,f) = \liminf_{r\to+\infty}\frac{m_0(r,f)}{T_0(r,f)}.$$

The value a is called *deficient value of f* if $\delta_0(a,f)>0$.

Note, first of all, that a meromorphic in an $m-$punctured plane function can omit $m+1$ values from $\mathbb{C}$. For example, the function $f(z)=z$ omits the values $\infty, c_1, c_2, \ldots, c_m$ in $\Omega=\mathbb{C}\setminus\bigcup_{j=1}^{m}\{c_j\}$. If a meromorphic in Ω function is non-rational, we are going to prove that it does not omit more then 2 values and the classical deficiency relation holds.

Theorem 7. *Let f be a meromorphic non-rational function in an $m-$punctured plane. Then there is at most countable set of deficient values of f and*

$$\sum_{a}\delta_0(a,f)\le 2. \tag{52}$$

Proof. According to (50) there is a sequence $\{r_k\}$ such that $r_k\to+\infty$ as $k\to+\infty$ and $S(r_k,f)=o(T_0(r_k,f))+O(\log r_k)$, $k\to+\infty$. Setting in (49) $r=r_k$ and dividing its both sides by $T_0(r_k,f)$ we obtain using Theorem 5

$$\underline{\lim}_{r\to+\infty}\ m_0(r,f)/T_0(r,f) + \textstyle\sum_{\nu=1}^{q}\ \underline{\lim}_{r\to+\infty}\ m_0(r,1(f-a_\nu))/T_0(r,f)\le 2.$$

Since $\{a_\nu\}$ is an arbitrary collection, this proves the first assertion of the theorem and implies (52). □

If f omits a value a from $\overline{\mathbb{C}}$, then $\delta_0(a,f) = 1-\overline{\lim}_{r\to+\infty}\ N_0(r,\frac{1}{f-a})/T_0(r,f)=1$ due to First Fundamental Theorem, and we deduce from (52) the following analog of the Picard Theorem.

Corollary 1. *A meromorphic non-rational function in an $m-$punctured plane does not omit more then 2 values from $\overline{\mathbb{C}}$.*

11. Estimates of characteristic $M_\infty(r,f)$.

Theorem 8. *Let f be a holomorphic function in an $m-$ punctured plane Ω not identically equal to zero. Then the function $\log M_\infty(r,f)$, where $M_\infty(r,f) = \max\left\{|f(z)| : z\in\overline{\Omega}_r\right\}$, $r_0\le r<+\infty$, is convex with respect to $\log r$ on $[r_0,+\infty)$, and*

$$T_0(r,f)\le \log^+ M_\infty(r,f) - C(r_0,f), \tag{53}$$

where

$$C(r_0,f)=\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+|f(r_0e^{i\theta})|\,d\theta+\frac{1}{2\pi}\sum_{j=1}^m\int\limits_0^{2\pi}\log^+\left|f\left(c_j+\frac{e^{i\theta}}{r_0}\right)\right|d\theta. \tag{54}$$

Proof. Set $M_0(t,f)=\max\{|f(z)|:|z|=t\}$, $r_0\le t<+\infty$, and $M_j(t,f)=\max\{|f(z)|: |z-c_j|=t\}$, $0<t<1/r_0$, $j\in\{1,2,\dots,m\}$. The functions $\log M_j(t,f)$, $j\in\{0,1,\dots,m\}$, are convex with respect to $\log t$ in virtue of the Hadamard Three Circles Theorem. Thus, the derivative from the right of $\log M_j(t,f)$, $j\in\{0,1,\dots,m\}$, with respect to $\log t$ does not decrease as t increases. If $t=1/r$, we have, meaning the derivative from the right,

$$r\frac{d}{dr}\log M_j\left(\frac{1}{r},f\right)=\frac{d}{dt}\log M_j(t,f)\frac{dt}{dr}=-t\frac{d}{dt}\log M_j(t,f),\qquad j\in\{1,2,\dots,m\}. \tag{55}$$

If r increases, then t decreases. Hence, the left-hand side of (55) is an non-decreasing function on $[r_0,+\infty)$. This implies that function $\log M_j(1/r,f)$, $j\in\{1,2,\dots,m\}$, is convex with respect to $\log r$. But, according to definition $M_\infty(r,f)=\max\{M_0(r,f),\, M_j(1/r,f): j\in\{1,2,\dots,m\}\}$. Thus, $\log M_\infty(r,f)$ is convex with respect to $\log r$ on $[r_0,+\infty)$.

The inequality (53) with $C(r_0,f)$ determined by (54) follows immediately from the definitions of Nevanlinna characteristic and $m_0(r,f)$. □

Theorem 9. *Let f be a holomorphic function in an m-punctured plane Ω. Then there is a constant $C=C(f,\Omega)$ such that*

$$\log^+ M_\infty(r,f)\le\frac{R+r}{R-r}\left(T_0(R,f)+C(1+\log^+ r)\right),\qquad r_0\le r<R<+\infty. \tag{56}$$

Proof. Let (24) be a decomposition of f, and let k_j be determined by (38). Then the functions $(z-c_j)^{-k_j}f_j(z)$, $j\in\{1,2,\dots,m\}$, are holomorphic and bounded outside $D_{r_0}(0)$. Hence, there is a constant C_0 such that $\max\{|f(z)|:|z|=r\}\le C_0\max\{|f_0(z)|:|z|=r\}r^{k_1+k_2+\cdots+k_m}$, $r_0\le r<R$. Similarly, $\max\{|f(z)|:|z-c_j|=1/r\}\le C_0\max\{|\widehat{f}_j(z)|:|z|=r\}(r)^{k_j}$, where $\widehat{f}_j(\zeta)=f_j(c_j+\frac{1}{\zeta})$. But ([12], p.23) $\log M_\infty(r,f_0)\le\frac{R+r}{R-r}T(R,f_0)$, $r_0\le r<R<+\infty$, and $\log M_\infty(r,\widehat{f}_j)\le\frac{R+r}{R-r}T(R,\widehat{f}_j)$, $r_0\le r<R<+\infty$, $j\in\{1,2,\dots,m\}$. Therefore, for $r_0\le r<R<+\infty$

$$\log M_\infty(r,f)\le\frac{R+r}{R-r}\Big(T(R,f_0)+\sum_{j=1}^m T(R,\widehat{f}_j)\Big)+C_1+2\sum_{j=1}^m k_j\log^+ r, \tag{57}$$

where C_1 is a constant. Using (57) and (37) we obtain (56). The estimates do not depend on decompositions of f because (see the proof of Lemma 4) $\alpha_0\alpha_1\dots\alpha_m=1$. □

Corollary 2. *Let f be a holomorphic function in an $m-$ punctured plane Ω. Let*

$$\rho_T(f)=\varlimsup_{r\to+\infty}\log T_0(r,f)/\log r,\quad \rho_\infty(f)=\varlimsup_{r\to+\infty}\log\log M_\infty(r,f)/\log r.$$

Then there is a constant K_0 such that

$$\log^+ M_\infty(r,f)\le 3T(2r,f)+K_0(1+\log^+ r),\qquad r_0\le r<+\infty, \tag{58}$$

and $\rho_T(f)=\rho_\infty(f)$.

Proof. Taking $R=2r$ we deduce (58) from (56) with $K_0=3C$. The last assertion of the corollary follows immediately from (58) and (53). □

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