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RAY-LIKE GRAPHS

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We prove that a graph Γ is coarsely equivalent to ray if and only if Γ is uniformly spherically bounded. We introduce and study some other classes of graphs asymptotically close to ray.

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Доказывается, что граф Γ грубо эквивалентен лучу тогда и только тогда, когда Γ равномерно сферически ограничен. Мы вводим и исследуем некоторые другие классы графов, асимптотически близких лучу.

1. INTRODUCTION. A ray is a (non-directed) graph $\mathbb{I}$ with the set of vertices $\omega = \{0, 1, ...\}$ and the set of edges $\{(i, i+1) : i \in \omega\}$. We are going to characterize some types of graphs that asymptotically look like ray. To make the adverb "asymptotically"precise we start with some general approach to asymptology.

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$ where X, P are non-empty sets, and for all $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a ball of radius α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the set of *radiuses*.

Given any $x \in X, A \subseteq X, \alpha \in P$, we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},\ B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called

- *lower symmetric* if, for any $\alpha, \beta \in P$ there exists $\alpha', \beta' \in P$ such that, for any $x \in X$,

$$B^*(x, \alpha') \subseteq B(x, \alpha),\ B(x, \beta') \subseteq B^*(x, \beta);$$

- *upper symmetric* if for any $\alpha, \beta \in P$ there exists $\alpha', \beta' \in P$ such that, for any $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$

- *lower multiplicative* if for any $\alpha, \beta \in P$ there exists $\gamma \in P$ such that, for any $x \in X$,

$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);$$

- *upper multiplicative* if for any $\alpha, \beta \in P$ there exists $\gamma \in P$ such that,for any $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

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Let $\mathcal{B} = (X, P, B)$ be a lower symmetric and lower multiplicative ball structure. Then the family

$$\bigcup_{x\in X} B(x,\alpha) \times B(x,\alpha) : \alpha \in P$$

is a base of entourages for some (uniquely determined) uniformity on X.On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, U) = \{y \in X : (x, y) \in \mathcal{U}\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with uniform topological spaces.

A ball structure is said to be *ballean* if it is upper symmetric and upper multiplicative. The balleans arouse in asymptotic geometry [6], asymptotic topology [1] under name of coarse structures, and later (but independently) in combinatorics [5].

Now we describe some types of morphisms of balleans. A mapping $f : X \to X$ is called a *$\prec$-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x,\alpha))) \subseteq B_2(f(x),\beta).$$

By this definition, $\prec$-mappings can be considered as asymptotic counterparts of uniformly continuous mappings between uniform topological spaces.

A mapping $f : X_1 \to X_2$ is called a *$\succ$-mapping* if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that, for every $x \in X_1$,

$$B_2(f(x),\beta) \subseteq f(B_1(x,\alpha)).$$

By this definition, $\succ$-mappings can be considered as asymptotic counterparts of uniformly open mappings between uniform topological spaces.

If $f : X_1 \to X_2$ is a bijection such that f is a $\prec$-mapping and f is a $\succ$-mapping, we say that f is an *asymorphism* between $\mathcal{B}_1$ and $\mathcal{B}_2$. Given an arbitrary ballean $\mathcal{B} = (X, P, B)$, we can replace every ball $B(x, \alpha)$ to $B(x, \alpha) \cap B^*(x, \alpha)$ for all $x \in X$, $\alpha \in P$.

More generally, a pair (f_1, f_2) of $\prec$-mappings $f_1 : X_1 \to X_2$, $f_2 : X_2 \to X_1$ is called a quasi-asymorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$ if there exists $\alpha \in P_1$, $\beta \in P_2$ such that, for all $x \in X_1$, $y \in X_2$,

$$f_2 f_1(x) \in B_1(x,\alpha), f_1 f_2(y) \in B_2(y,\beta).$$

In terminology of [6], quasi-asymorphic balleans are called *coarsely equivalent*.

Every metric space (X, d) determines the ballean $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$, where $\mathbb{R}^+$ is the set of non-negative integers, $B_d(x, r) = \{y \in X : d(x, y) \le r\}$.

A ballean is called *metrizable* if $\mathcal{B}$ is asymorphic to $\mathcal{B}(X, d)$ for some metric space (X, d). A criterion of metrizability of balleans can be found in [5, Theorem 9.1]. This criterion shows that every ballean quasi-asymorphic to metrizable ballean is metrizable. We note also that every quasi-isometry between metric spaces [3,Chapter 4] is a quasi-asymorphism between corresponding balleans.

From some point of view, the ballean $\mathcal{B}(\mathbb{R}^+, d)$, where $d(x, y) = |x-y|$, can be considered as asymptotic counterpart of the interval [0,1] with natural topology. If this so, we get the problem of characterization of balleans quasi-asymorphic to $\mathcal{B}(\mathbb{R}^+, d)$. Which language is appropriate for this goal?

Let Γ be connected graph with the set of vertices $V(\Gamma)$ and the set of edges $E(\Gamma)$. Given any $u, v \in V(\Gamma)$, we denote $d(u, v)$ the length of a shortest path between u and v. We denote by $\mathcal{B}(\Gamma)$ the metric ballean $\mathcal{B}(V(\Gamma), d)$. A ballean $\mathcal{B}$ is called a *graph ballean* if $\mathcal{B}$

is asymorphic to $\mathcal{B}(\Gamma)$ for some graph Γ. A criterion of graph balleans can be found in [5, Theorem 9.2]. It follows from this criterion that every ballean quasi-asymorphic to graph ballean is a graph ballean.

Let $f_1 : \omega \to \mathbb{R}^+$ be the canonical embedding, $f_2(x) = [x]$. Clearly, (f_1, f_2) is a quasi-asymorphism between $\mathcal{B}(\mathbb{I})$ and $\mathcal{B}(\mathbb{R}^+, d)$, so in the original problem we can replace $\mathcal{B}(\mathbb{R}^+, d)$ to $\mathcal{B}(\mathbb{I})$.

The main result of this paper characterizes all graphs Γ such that $\mathcal{B}(\Gamma)$ is quasi-asymorphic to $\mathcal{B}(\mathbb{I})$. Besides, we introduce and study some wider classes of graphs asymptotically close to $\mathbb{I}$.

2. EMBEDDINGS. Let $\mathcal{B} = (X, P, B)$ be a ballean, Y be a nonempty subset of X. The ballean $\mathcal{B}_Y = (Y, P, B_Y)$, where $B_Y(y, \alpha) = B(y, \alpha) \cap Y$, is called a *subballean* of X.

A subset $Y \subseteq X$ is called *bounded* if there exists $x \in X$, $\alpha \in P$ such that $Y \subseteq B(x, \alpha)$. A ballean is called bounded if its support is bounded.

A family Im of subsets of X is called *uniformly bounded* in $\mathcal{B}$ if there exists $\alpha \in P$ such that, for every $F \in \mathrm{Im}$, $F \subseteq B(x, \alpha)$ for some $x \in X$. Equivalently, Im is uniformly bounded if there exists $\beta \in P$ such that, for every $F \in \mathrm{Im}$, $F \subseteq B(x, \beta)$ for every $x \in F$.

Let Im be a uniformly bounded partition of X. Given any $F \in \mathrm{Im}$ and $\alpha \in P$, we put $B_{\mathrm{Im}}(F, \alpha) = \{F' \in \mathrm{Im} : F' \subseteq B(F, \alpha)\}$. It is easy to check that the ball structure $\mathcal{B}/\,\mathrm{Im}$ is a ballean which is called a *factor-ballean* of $\mathcal{B}$. We note also that $\mathcal{B}/\,\mathrm{Im}$ is the smallest (by $\prec$) ballean on Im such that the projection $pr : X \to \mathrm{Im}$ is a $\prec$-mapping, where $pr(x) = F$ if and only if $x \in F$.

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans, $f : X_1 \to X_2$. Clearly, f is a $\prec$-mapping if and only if, for every uniformly bounded family Im of subsets of X_1, the family $f(\mathrm{Im}) = \{f(F) : F \in \mathrm{Im}\}$ is uniformly bounded in $\mathcal{B}_2$. We assume that f is a $\prec$-mapping and consider the partition $ker f$ of X_1 determined by the equivalence: $x \sim y$ if and only if $f(x) = f(y)$. If the partition $ker f$ is uniformly bounded, we get the *canonical decomposition* $f = i_f \circ pr_f$, $pr_f : X_1 \to ker f$, $i_f : ker f \to X_2$. In this case pr_f is a surjective $\prec$-mapping of $\mathcal{B}_1$ onto $\mathcal{B}_1/ker f$, i_f is an injective $\prec$-mapping of $\mathcal{B}_1/ker f$ into $\mathcal{B}_2$.

A mapping $f : X_1 \to X_2$ is called an *asymorphic embedding* of $\mathcal{B}_1$ into $\mathcal{B}_2$ if f is an asymorphism between $\mathcal{B}_1$ and the subballean of $\mathcal{B}_2$ determined by the subset $f(X_1)$ of X_2.

A $\prec$-mapping $f : X_1 \to X_2$ is called *quasi-asymorphic embedding* of $\mathcal{B}_1$ into $\mathcal{B}_2$ if, for every $\beta \in P_2$ there exists $\alpha \in P_1$ such that, for all $x_1, x_2 \in X_1$, $f(x_1) \in B_2(f(x_2), \beta)$ implies $x_1 \in B_1(x_2, \alpha)$. Equivalently, a mapping $f : X_1 \to X_2$ is a quasi-asymorphic embedding if, for every uniformly bounded family Im_1 of subsets of X_1, the family $f(\mathrm{Im}_1)$ is uniformly bounded in $\mathcal{B}_2$, and, for every uniformly bounded family Im_2 of subsets of X_2, the family $f^{-1}(\mathrm{Im}_2) = \{f^{-1}(F) : F \in \mathrm{Im}_2\}$ is uniformly bounded in $\mathcal{B}_1$. We note also that a quasi-isomorphic embedding f is an asymorphic embedding if and only if f is injective. For the case of metric ballean the notion of quasi-asymorphic embedding was introduced by Gromov [2] under name uniform embedding.

Let $f : X_1 \to X_2$ be a quasi-asymorphic embedding of $\mathcal{B}_1$ into $\mathcal{B}_2$. Then the partition $ker f$ is uniformly bounded and the mapping $i_f : ker f \to X_2$ from the canonical decomposition $f = i_f \circ pr_f$ is an asymorphic embedding of $\mathcal{B}_1/ker f$ into $\mathcal{B}_2$. On the other hand, if some factor-ballean of $\mathcal{B}_1$ admits an asymorphic embedding into $\mathcal{B}_2$, then $\mathcal{B}_1$ admits a quasi-asymorphic embedding into $\mathcal{B}_2$.

Let $\mathcal{B} = (X, P, B)$ be a ballean. A subset $Y \subseteq X$ is called *large* if there exists $\alpha \in P$ such that $X = B(Y, \alpha)$.

Now we describe interrelation between quasi-asymorphisms and quasi-asymorphic embeddings. Let $f_1 : X_1 \to X_2$ be a quasi-asymorphic embedding of $\mathcal{B}_1$ into $\mathcal{B}_2$ such that the subset $f(X_1)$ is large in $\mathcal{B}_2$. We construct a mapping $f_2 : X_2 \to X_1$ such that the pair (f_1, f_2) is a quasi-asymorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$. For every $y \in f(X_1)$, we choose some element $g(y) \in f^{-1}(y)$, so we have the mapping $g : f(X_1) \to X_1$. Since $f(X_1)$ is large in $\mathcal{B}_2$, there exists $\beta \in P_2$ such that $B_2(f(X_1), \beta) = X_2$. To define the mapping $f_2 : X_2 \to X_1$ we take an arbitrary $z \in X_2$, choose $y \in f(x_1)$ such that $z \in B(y, \alpha)$ and put $f_2(z) = g(y)$.

On the other hand, if (f_1, f_2) is a quasi-asymorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$, then f_1 is quasi-asymorphic embedding and $f_1(X_1)$ is large in $\mathcal{B}_2$. Hence, $\mathcal{B}_1$ and $\mathcal{B}_2$ are quasi-asymorphic if and only if there exists a quasi-asymorphic embedding $f : X_1 \to X_2$ such that $f(X_1)$ is large in $\mathcal{B}_2$.

The rest of this section is to interprete all the above notions for the case of graph balleans. *In what follows all graphs under consideration are supposed to be connected.*

Proposition 1. *Let Γ_1, Γ_2 be graphs, $f : V(\Gamma_1) \to V(\Gamma_2)$. Then the following statements are equivalent:*

(i) f is a $\prec$-mapping of $\mathcal{B}(\Gamma_1)$ to $\mathcal{B}(\Gamma_2)$;

(ii) there exists a natural number k such that $f(B_1(v, 1)) \subseteq B_2(f(v), k)$ for every $v \in V(\Gamma_1)$, where B_1 and B_2 are balls in Γ_1 and Γ_2;

(iii) there exists a natural number k such that $d_2(f(v), f(u)) \le kd_1(v, u)$ for all $u, v \in V(\Gamma_1)$, where d_1, d_2 are the path metrics in Γ_1 and Γ_2.

Proof. $(i) => (ii)$ follows directly from definition of $\prec$-mapping.

$(ii) => (iii)$ If $d_1(v, u) = 1$, then $u \in B(v, 1)$ so $d_2(f(u), f(v)) \le k$. Given any $v, u \in V(\Gamma_1)$, we fix the shortest path $v = v_0, v_1, ..., v_n = u$ between u and v. Since $d_2(f(v_i), f(v_{i+1})) \le k$ for every $i \in \{0, 1, ..., n-1\}$, then $d_2(f(v), f(v)) \le kn = d_1(v, u)$.

$(iii) => (i)$. It is suffices to note that (iii) is equivalent to $f(B_1(v, n)) \subseteq B_2(f(v), kn)$ for every $v \in V(\Gamma_1)$. □

In other words Proposition 1 states that, in the case of graph balleans, $\prec$-mappings are exactly lipschitz mappings of corresponding metric spaces. We say that $f(V(\Gamma_1)) \to V(\Gamma_2)$ is a *$\prec$-mapping of scale k* if $f(B_1(v, 1)) \subseteq B_2(f(v), k)$ for every $v \in V(\Gamma_1)$.

Let Γ_1, Γ_2 be graphs, m be a natural number, $f : V(\Gamma_1) \to V(\Gamma_2)$. Clearly, if

$$d_1(v_1, v_2)/m \le d_2(f(v_1), f(v_2)) \le md_1(v_1, v_2)$$

for any $v_1, v_2 \in V(\Gamma_1)$, then f is a quasi-asymorphic embedding of $\mathcal{B}_1$ into $\mathcal{B}_2$. Following example shows that the above inequality is not necessary for f to be a quasi-asymorphic embedding, but if the subset $f(V(\Gamma_1))$ is large in $\mathcal{B}(\Gamma_2)$, this is so (Proposition 2).

Example 1. We consider the ray $\mathbb{I}$ and, for every natural number n, identify the vertices 2^n,2^{n+1} of $\mathbb{I}$ with the end-vertices of $\mathbb{I}_n$. Here $\mathbb{I}_n$ is a graph with the set of vertices $\{0, 1, ..., n\}$ and the set of edges $\{(i, i+1) : i \in \{0, ..., n-1\}\}$ .Denote by Γ the resulting graph. Fix a natural number k. If a natural number n is sufficiently large, the distance between the vertices $n, n+k \in \omega$ in Γ is k. It follows that the identity mapping $i : \omega \to V(\Gamma)$ is a quasi-asymorphic embedding of $\mathcal{B}(\mathbb{I})$ into $\mathcal{B}(\Gamma)$. On the other side, the distance between the vertices $2^n, 2^{n+1}$ in Γ is n, but the distance between $2^n, 2^{n+1}$ in $\mathbb{I}$ is 2^n. Hence, the left part of above inequality fails.

Proposition 2. *Let Γ_1, Γ_2 be graphs, k, l be natural numbers, $f : V(\Gamma_1) \to V(\Gamma_2)$ be a $\prec$-mapping of scale k, $B_2(f(V(\Gamma_1)), l) = V(\Gamma_2)$. Then the following statements are equivalent:*

(i) the family $\{f^{-1}(B_2(u, 2l+1)) : u \in V(\Gamma_1)\}$ is uniformly bounded;

(ii) there exists a natural number m such that, for all $v_1, v_2 \in V(\Gamma_1)$,

$$d(v_1, v_2)/m \le d_2(f(v_1), f(v_2)) \le m d_1(v_1, v_2).$$

Proof. $(ii) \Rightarrow (i)$ is evident.
$(i) \Rightarrow (ii)$. By (i), there exists a natural number m' such that $d_2(f(v_1), f(v_2)) \le 2l+1$ implies $d_1(v_1, v_2) \le m'$. We put $m = max\{m', k\}$. Since f is a $\prec$-mapping of scale k, we have

$$d_2(f(v_1), f(v_2)) \le k d_1(v_1, v_2) \le m d_1(v_1, v_2).$$

Let $d_2(f(v_1), f(v_2)) = t$. We choose the shortest path $w_0, w_1, ..., w_t, w_0 = f(v_1), w_t = f(v_2)$ between $f(v_1)$ and $f(v_2)$. Since $B_2(f(V(\Gamma_1)), l) = V(\Gamma_2)$, there exists $u_1, u_2, ..., u_{t-1} \in V(\Gamma_1)$ such that $f(u_i) \in B_2(w_i, l)$, $i \in \{1, ..., t-1\}$. Then
$d_2(f(v_1), f(u_1)) \le l+1$, $d_2(f(u_{t-1}), f(v_2)) \le l+1$, $d_2(f(u_i), f(u_{i+1})) \le 2l+1$, $i \in \{1, ..., t-1\}$.
By the choice of m', we have $d_1(v_1, u_1) \le m'$, $d_1(u_{t-1}, v_2) \le m'$, $d_1(u_i, u_{i+1}) \le m'$, $i \in \{1, ..., t-1\}$. Hence, $d_1(v_1, v_2) \le m't \le mt = d_2(f(v_1), v_2)$. □

Proposition 3. *Let Γ_1, Γ_2 be graphs, $f : V(\Gamma_1) \to V(\Gamma_2)$. Then the following statements are equivalent:*

(i) f is a $\prec$-mapping of $\mathcal{B}(\Gamma_1)$ to $\mathcal{B}(\Gamma_2)$;

(ii) there exists a natural number k such that $B_2(f(v), 1) \subseteq f(B_1(v, k))$ for every $v \in V(\Gamma_1)$;

(iii) there exists a natural number k such that $B_2(f(v), n) \subseteq f(B_1(v, kn))$ for all $v \in V(\Gamma_1)$, $n \in \omega$.

Proof. The implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are evident.
$(ii) \Rightarrow (iii)$. Fix any $v \in V(\Gamma_1), n \in \omega$. For every element $u \in B_2(f(v), n)$ we can choose the shortest path $u_0, ..., u_t, u_0 = f(v), u_t = u, t \le n$ between $f(v)$ and u. By (ii), there exist $v_0, ..., v_t \in V(\Gamma_1)$, $v_0 = v$ such that $v_{i+1} \in B_2(v_i, k)$, $i \in \{0, ..., t-1\}$ and $f(v_1) = u_1, f(v_2) = u_2, ..., f(v_t) = u_t = u$. It follows that $v_t \in B_2(v, kt) \subseteq B_1(v, kn)$ and $u \in f(B_1(v, kn))$. □

3. QUASI-RAYS. We say that a graph Γ is a *quasi-ray* if $\mathcal{B}(\Gamma)$ is quasi-asymorphic to $\mathcal{B}(\mathbb{I})$. Let Γ be an arbitrary graph, $v \in V(\Gamma)$, $n \in \omega$. We put

$$S(v, n) = \{u \in V(\Gamma) : d(u, v) = n\}.$$

Let r be a natural number. We say that a sequence $(a_n)_{n\in\omega}$ of vertices of Γ is an r-*arrow* if $d(a_i, a_{i+1}) = 1$ and $a_i \in S(a_0, i)$ for every $i \in \omega$. In the case $r = 1$, $(a_n)_{n\in\omega}$ is called an *arrow*. A graph Γ is called *locally finite* if $\rho(v)$ is finite for every $v \in V(\Gamma)$, where $\rho(v) = |B(v, 1)| - 1$. By Køning lemma, for every locally finite graph and every $v \in V(\Gamma)$, there exists an arrow starting at v.

A graph Γ is called *unbounded* if the ballean $\mathcal{B}(\Gamma)$ is unbounded (i.e. $S(v, n) \ne \varnothing$ for all $v \in V(\Gamma), n \in \omega$).

Theorem 1. *Let Γ be an unbounded graph, $v_0 \in V(\Gamma)$. Then the following statements are equivalent:*

(i) Γ *is a quasi-ray;*

(ii) the family $\{S(v_0, n) : n \in \omega\}$ *is uniformly bounded in* $\mathcal{B}(\Gamma)$*;*

(iii) there exist natural numbers r, s *and an* r*-arrow* $a(_n)_{n\in\omega}$ *such that* $a_0 = v_0$ *and* $V(\Gamma) = B(A, s)$*, where* $A = \{a_n : n \in \omega\}$.

Proof. (i) => (ii). Let $f : V(\Gamma) \to \omega$ be a quasi-asymorphic embedding of $\mathcal{B}(\Gamma)$ into $\mathcal{B}(\mathbb{I})$ such that the subset $f(V(\Gamma))$ is large in $\mathcal{B}(\mathbb{I})$. By Proposition 2, there exists a natural number m such that

$$d(u, v)/m \leq |f(u) - f(v)| \leq md(u, v), \qquad (*)$$

where d is the path metric on $V(\Gamma)$.

If the family $\{S(v, n) : n \in \omega\}$ is uniformly bounded in $\mathcal{B}(\Gamma)$ for some $v \in V(\Gamma)$, the family $\{S(u, n) : n \in \omega\}$ is uniformly bounded for every $u \in V(\Gamma)$. Hence, we may suppose that $f(v_0) = \min\{f(v) : v \in V(\Gamma)\}$ and, moreover, $f(v_0) = 0$.

We fix an arbitrary natural number n and show that $f(S(v_0, n)) \subseteq [i(n) - m, i(n) + m]$ for some $i(n) \in \omega$, where $[a, b]$ is a segment in ω with $[a, b] = [0, b]$ if $a < 0$. Since the family $\{[i-m, i+m] : i \in \omega\}$ is uniformly bounded in $\mathcal{B}(\mathbb{I})$ and f is a quasi-asymorphic embedding, the family $\{f^{-1}([i - m, i + m]) : i \in \omega\}$ is uniformly bounded in $\mathcal{B}(\Gamma)$, so $\{S(v_0, n) : n \in \omega\}$ is uniformly bounded in $\mathcal{B}(\Gamma)$.

We put $t = m^2(n + 1)$ and choose $v_0, v_1, ..., v_t \in V(\Gamma)$ such that $d(v_{i-1}, v_i) = 1$, $v_i \in S(v_0, i)$ for every $i \in \{1, 2, ..., t\}$. It is possible because Γ is unbounded. By (*), $f(v_t) \geq m(n + 1)$ and $|f(v_{i-1}) - f(v_i)| \leq m$ for every $i \in \{1, ..., t\}$. It follows that every segment $[k, k + m]$, $k \in \{0, ..., mn\}$ contains at least one element $f(v_0), f(v_1), ..., f(v_t)$.

We show that $f(S(v_0, n)) \subseteq [f(v_n) - m, f(v_n) + m]$, so we can take $i(n) = f(v_n)$. Suppose the contrary and choose $v \in S(v_0, n)$ such that $f(v) \notin [f(v_n) - m, f(v_n) + m]$. Since $d(v, v_0) = n$, we have $f(v) \in [0, mn]$. We pick $k \in [0, m(n + 1)]$ such that $f(v) \in [k - m, k]$ and $[k - m, k] \cap [f(v_n) - m, f(v_n) + m] = \varnothing$.

Then we take $j \in \{0, ..., t\}$ such that $f(v_j) \in [k - m, k]$. Since $|f(v_j) - f(v_n)| > m$, by $(*)$, we have $d(v_n, v_j) > 1$. Since $v \in S(v_0, n)$, then $d(v, v_j) > 1$. On the other hand, $|f(v) - f(v_j)| \leq m$, so $d(v, v_j) \leq 1$ and we got a contradiction.

(ii) => (iii). Since $\{S(v_0, n) : n \in \omega\}$ is uniformly bounded in $\mathcal{B}(\Gamma)$, there exists a natural number s such that $S(v_0, n) \subseteq B(v, s)$ for every $v \in S(v_0, n)$. We put $a_0 = v_0$ and, for every $n \in \omega$, choose $a_n \in S(v_0, n)$. Then $V(\Gamma) = B(A, s)$ and $d(a_n, a_{n+1}) \leq 2s + 1$, so we can put $r = 2s + 1$.

(iii) => (i). We define a mapping $f : \omega \to V(\Gamma)$ by the rule $f(i) = a_i$, $i \in \omega$. It is easy to check that f is a quasi-asymporphic embedding of $\mathcal{B}(\mathbb{I})$ to $\mathcal{B}(\Gamma)$ and $f(\omega)$ is large in $\mathcal{B}(\Gamma)$. Hence, Γ is a quasi-ray. □

Let T be a tree and let $(v_n)_{n\in\omega}$ be an arrow in T. After deletion of the edges $\{v_n, v_{n+1}\}$ the tree T disintegrates into the family $\{T(v_n) : n \in \omega\}$ of trees with the roots $\{v_n : n \in \omega\}$. If a tree $T(v_n)$ is bounded, we put $h(T(v_n)) = \max\{d(v_n, v) : v \in V(T_n)\}$.

Theorem 2. *An unbounded tree* T *is a quasi-ray if and only if there exists an arrow* $(v_n)_{n\in\omega}$ *in* T *and a natural number* k *such that* $h(T(v_n)) \leq k$ *for every* $n \in \omega$.

Proof. Let T be a quasi-ray. By Theorem1, there exists an r-arrow $(a_n)_{n\in\omega}$ in T such that the subset $\{a_n : n \in \omega\}$ is large in $\mathcal{B}(T)$. For every pair a_n, a_{n+1}, we find a shortest path between a_n, a_{n+1} and put the vertices of this path in $(a_n)_{n\in\omega}$ between a_n, a_{n+1}. After that we get a new sequence $(b_n)_{n\in\omega}$ containing $(a_n)_{n\in\omega}$ as a subsequence. We put $v_0 = a_0$. Since $(a_n)_{n\in\omega}$ is an r-arrow, for every $i \in \omega$, $S(v_0, i)$ contains only finite number of elements of $(b_n)_{n\in\omega}$. For every $i \in \omega$, we denote by v_i the last member of $(b_n)_{n\in\omega}$ such that $v_i \in S(v_0, i)$. Since T is a tree $(v_n)_{n\in\omega}$ is an arrow. By Theorem1, the family $\{S(v_0, n) : n \in \omega\}$ is uniformly bounded. It follows that there exists a natural number t such that $d(v_i, a_i) \leq t$, $i \in \omega$. Since $\{a_n : n \in \omega\}$ is large in $\mathcal{B}(T)$, then $\{v_n : n \in \omega\}$ is large in $\mathcal{B}(T)$.It follows that the heights of all trees $T(v_n)$ are bounded by some constant k.

If $(v_n)_{n\in\omega}$ is an arrow in T and $h(T(v_n)) \leq k$ for every $n \in \omega$, then $\{v_n : n \in \omega\}$ is large in $\mathcal{B}(T)$. By Theorem 1, T is a quasi-ray. □

Comparing Theorem 1 and Theorem 2 it is naturally to ask if there exists an arrow in every quasi-ray. The following example gives negative answer.

Example 2. For every $n \in \omega$, we consider the graph $\mathbb{I}_n$ and denote its vertices $a(0, n)$, $a(1, n), \ldots, a(n, n)$ with $(a(i, n), a(i + 1, n)) \in V(\mathbb{I}_n)$, $i \in \{0, .., n - 1\}$. We stick together all the vertices $\{a(0, n) : n \in \omega\}$ and call this new vertex by a(0,0). Then, for every natural number k, we consider the set of vertices $\{a(k, i) : i \in \{k, k+1, ...\}\}$ and connect all pairs of these vertices by the edges. Let Γ be the resulting graph. By the construction, $S(a(0, 0), n)$ is a complete graph for every $n \in \omega$. Hence, the family $\{S(a(0, 0), n) : n \in \omega\}$ is uniformly bounded and, by Theorem 1, Γ is a quasi-ray.

Assume that there is an arrow in Γ starting at $a(0, 0)$. Then this arrow must follow via one of the subgraphs $\mathbb{I}_n$ of Γ. But from the vertex $a(n, n)$ on the level $S(a(0, 0), n)$ there are no possibilities to get $S(a(0, 0), n + 1)$ in one step. The same arguments show that there are no arrows in Γ at all, i.e. starting at any vertex of Γ.

We say that a graph Γ is an *asyray* if the balleans $\mathcal{B}(\Gamma)$ and $\mathcal{B}(\mathbb{I})$ are asymorphic. Clearly, every asyray is a quasi-ray.

Theorem 3. *A quasi-ray Γ is an asyray if and only if there exists a natural number m such that $\rho(v) \leq m$ for every $v \in V(\Gamma)$.*

Proof. Assume that Γ is an asyray and fix an asymorphism $f : V(\Gamma) \to \omega$ between $\mathcal{B}(\Gamma)$ and $\mathcal{B}(\mathbb{I})$. By Lemma 1, there exists a natural number k such that $f(B(v, 1)) \subseteq [f(v)-k, f(v)+k]$ for every $v \in V(\Gamma)$. It follows that $\rho(v) \leq 2k$ for every $v \in V(\Gamma)$. Put $m = 2k$.

Suppose that $\rho(v) \leq m$ for every $v \in V(\Gamma)$. Fix $v_0 \in V(\Gamma)$. By Theorem 1, $\{S(v_0, n) : n \in \omega\}$ is uniformly bounded. It follows that there exists a natural number t such that $|S(v_0, n)| \leq t$ for every $n \in \omega$. We construct a bijection $f : \omega \to V(\Gamma)$ in the following way. Put $f(0) = v_0$, then we enumerate the elements of $S(v_0, 1), S(v_0, 2), ...$ It is easy to see that f is an asymorphism between $\mathcal{B}(\mathbb{I})$ and $\mathcal{B}(\Gamma)$. □

A direct characterization of asyrays is given in [4] in the following form.

Theorem 4. *Let Γ be an infinite graph, s be a natural number such that $\rho(v) \leq s$ for every $v \in V(\Gamma)$, $(a_n)_{n\in\omega}$ be an arrow in Γ. Then the following statements are equivalent:*

(i) Γ is an asyray;

(ii) the family $\{S(v_0, n) : n \in \omega\}$ is uniformly bounded in $\mathcal{B}(\Gamma)$;

(iii) there exists a natural number r such that $V(\Gamma) = B(\{a_n : n \in \omega\}, r)$.

4. RELATIONS $\prec$ AND $\succ$. Given any graphs Γ_1, Γ_2, we write $\mathcal{B}(\Gamma_1) \prec \mathcal{B}(\Gamma_2)$ (resp. $\mathcal{B}(\Gamma_1) \succ \mathcal{B}(\Gamma_2)$) if there exists an injective $\prec$-mapping (resp. surjective $\succ$-mapping) $f : V(\Gamma_1) \to V(\Gamma_2)$. The next two examples show that $\mathcal{B}(\Gamma_1) \prec \mathcal{B}(\Gamma_2)$ does not imply $\mathcal{B}(\Gamma_2) \succ \mathcal{B}(\Gamma_1)$, and $\mathcal{B}(\Gamma_1) \succ \mathcal{B}(\Gamma_2)$ does not imply $\mathcal{B}(\Gamma_2) \prec \mathcal{B}(\Gamma_1)$.

Example 3. Let K be a complete graph with the set of vertices $\{v_n : n \in \omega\}$. For every $n \in \omega$, we identify v_n with one of the end-points of $\mathbb{I}_n$. After this attachments we get some graph Γ. The identity mapping $f : V(K) \to V(\Gamma)$ is a $\prec$-mapping of scale 1, so $\mathcal{B}(K) \prec \mathcal{B}(\Gamma)$. Suppose that there exists a surjective $\succ$-mapping $f : V(\Gamma) \to V(K)$. Choose a natural number m such that $B_K(f(v), 1) \subseteq f(B_\Gamma(v, m))$ for every $v \in V(\Gamma)$. If $n > m$ and v_n is the end-point of the subgraph $\mathbb{I}_n$ of Γ, then $B_\Gamma(v, m)$ is finite, but every ball of unit radius in K coincides with K, a contradiction.

Example 4. Let G be a group with the identity e and finite set S of generators,$S = S^{-1}$, $e \notin S$. The Cayley graph $Cay(G, S)$ is a graph with the set of vertices G, and the set of (non-directed) edges $\{(a, b) : a, b \in G, a^{-1}b \in S\}$.

Let F_2 be a free group of rank 2 with generators a, b. Put $T = Cay(F_2, \{a, b, a^{-1}, b^{-1}\})$ and note that T is a tree of local degree 4.

Let A_2 be a free Abelian group of rank 2 with the set of generators c, d. Put $\Gamma = Cay(A_2, \{c, d, c^{-1}, d^{-1}\})$. Geometrically, Γ is a graph with the set of vertices $\mathbb{Z}^2$ and the set of edges connecting the pair of points on euclidian distance 1.

We consider homomorphism $f : F_2 \to A_2$ defined by $f(a) = c$, $f(b) = d$ and note that f is a $\succ$-mapping of $\mathcal{B}(T)$ onto $\mathcal{B}(\Gamma)$. Hence, $\mathcal{B}(T) \succ \mathcal{B}(\Gamma)$.

We show that $\mathcal{B}(\Gamma)$ can not be injectively $\prec$-embedded into $\mathcal{B}(T)$. Assume the contrary and fix some injective $\prec$-mapping $f : A_2 \to F_2$.

We identify $V(T)$ with the set of shortest path from the identity e of F_2 to the vertices of T. Let $\sigma(T)$ be the set of all arrows in T starting at e. We endow $V(T) \cup \sigma(T)$ with topology of pointwise convergence and note that $V(T)$ is dense discrete subspace of compact space $V(T) \cup \sigma(T)$. Since $h(A_2)$ is an infinite subset of $V(T)$, then $cl(h(A_2)) \cap \sigma(T) \neq \varnothing$, where cl is a closure in $V(T) \cup \sigma(T)$. We consider two cases.

Case $|cl(h(A_2)) \cap \sigma(T)| > 1$. Then there exists $v, v_1, v_2 \in V(T)$ such that $(v_1, v), (v_2, v) \in E(T)$ and the sets $V(T(v_1)) \cap h(A_2)$, $V(T(v_2)) \cap h(A_2)$ are infinite, where $T(v_1), T(v_2)$ are trees with the roots v_1, v_2 obtained by deletion of the edges (v, v_1), (v, v_2). Then we can choose two sequences $(u_n)_{n\in\omega}$, $(w_n)_{n\in\omega}$ in $V(A_2)$ such that the family $\{p_n : n \in \omega\}$ of the shortest paths between u_n and w_n is disjoint (by vertices), and $h(u_n) \in T(v_1)$, $h(w_n) \in T(v_2)$, $n \in \omega$. If the scale of h is k, then $h(p_n) \cap B_T(v, k) \neq \varnothing$. Since $B_T(v, k)$ is finite, the family $\{h(p_n) : n \in \omega\}$ is not disjoint and we get the contradiction to injectivity of h.

Case $|cl(h(A_2)) \cap \sigma(T)| = 1$. Then there exists an arrow $(v_n)_{n\in\omega}$ in T such that $V(T(v_n)) \cap h(A_2)$ is finite for every $n \in \omega$, where $\{T(v_n) : n \in \omega\}$ are trees with the roots $\{v_n : n \in \omega\}$ obtained by deletion of the edges $\{(v_n, v_{n+1}) : n \in \omega\}$. We choose a countable family $\{R_n : n \in \omega\}$ of pairwise disjoint (by vertices) arrow in Γ. Since h is a $\prec$-mapping and all the sets $\{V(T(v_n)) \cap h(A_2)\}$ are finite, we can choose $m, n \in \omega, n \neq m$ such that $h(R_n) \cap h(R_m) \neq \varnothing$, contradicting injectivity of h.

We note also that the statement "$\mathcal{B}(\Gamma)$ does not admit injective $\prec$-mapping into $\mathcal{B}(T)$" can be extracted from Lemma 9.16 [6] concerning asymptotic dimension.

5. $\prec$-RAYS. By [5, Theorem 10.1], $\mathcal{B}(\mathbb{I}) \prec \mathcal{B}(\Gamma)$ and $\mathcal{B}(\Gamma) \succ \mathcal{B}(\mathbb{I})$ for every infinite graph Γ. We say that a graph Γ is a *$\prec$-ray* if $\mathcal{B}(\Gamma) \prec \mathcal{B}(\mathbb{I})$. Clearly, every finite graph is a $\prec$-ray.

Problem 1. *Characterize all $\prec$-rays.*

We give one sufficient condition for graph to be $\prec$-ray and show that this condition is not necessary.

Theorem 5. *Let Γ be a graph, $v_0 \in \Gamma$, k be a natural number. If $|S(v_0, n)| \leq k$ for every $n \in \omega$, then Γ is a $\prec$-ray.*

Proof. Starting with v_0, we enumerate $v_1, ..., v_t$ the elements of $S(v_0, 1)$, then we enumerate $v_{t+1}, ..., v_s$ the elements of $S(v_0, 2)$ and so on. After that we get enumeration $(v_n)_{n\in\omega}$ of $V(\Gamma)$. We define a mapping $f : V(\Gamma) \to \omega$ by the rule $f(v_n) = n, n \in \omega$. If $u, v \in V(\Gamma)$ and $d(u, v) = 1$, then $|f(u) - f(v)| \leq 3k$. By Proposition 1, f is a $\prec$-mapping of $\mathcal{B}(\Gamma)$ into $\mathcal{B}(\mathbb{I})$. □

Example 5. We fix a natural number n and consider the following set of points of $\mathbb{Z}^2$:
$(2, 1), (2, 2)$
$(4, 1), (4, 2), (4, 3), (4, 4)$
....
$(2^{n-1}, 1), (2^{n-1}, 2), ..., (2^{n-1}, 2^{n-1})$
$(2^n, 1), (2^n, 2), ..., (2^n, 2^n)$
$(0, 0), (1, 0), ..., (2^n, 0)$.

Given any two points u, v on this table, we define the edge (u, v) if and only if the euclidian distance between u and v is 1. Denote by T_n the resulting tree and say that $(0, 0)$ is a left vertex of T_n, $(2^n, 0)$ is a right vertex of T_n.

Now we define a mapping $f : V(T) \to \{0, 1, ..., 3 \cdot 2^n\}$ by the rule: $f(i, 0) = 3i, i \in \{0, 1, ..., 2^n\}$; $f(2^k, j) = 3(2^k + j) + 1, k = \{1, ..., n-1\}, j = \{1, ..., 2^k\}$; $f(2^n, j) = 3(2^n - j) + 2, j = \{1, 2, ..., 2^n\}$. It is not hard to check that f is an injective $\prec$-mapping of scale 3 of $\mathcal{B}(T_n)$ to $\mathcal{B}(\mathbb{I}_{3\cdot 2^n})$.

Let us forget the geometric definition of T_n and consider the sequence $T_1, T_2, ..., T_n, ...$ of abstract graphs with pairwise disjoint sets of vertices. For every pair T_n, T_{n+1} we connect by edge the left vertex of T_n with the right vertex of T_{n+1}. Denote by T the resulting graph. By construction, there exists an injective 3-scale $\prec$-mapping of $\mathcal{B}(T)$ to $\mathcal{B}(\mathbb{I})$, so T is a $\prec$-ray. Let v be the right vertex of T_1, v_n be the left vertex of T_n and t be a distance between v and v_n. Then $|S(v, t+2^n)| = n$, so the family $\{S(v, k) : k \in \omega\}$ is not bounded by the cardinality of its members.

Theorem 6. *For every graph Γ, the following statements are equivalent: (i) there exists a finite-to-one mapping $f : V(\Gamma) \to \omega$ which is a $\prec$-mapping of $\mathcal{B}(\Gamma)$ to $\mathcal{B}(\mathbb{I})$; (ii) Γ is locally finite.*

Proof. $(ii) \Rightarrow (i)$. We fix an arbitrary vertex $v_0 \in V(\Gamma)$ and, for every $v \in V(\Gamma)$, put $f(v) = d(v, v_0)$. If $(u, v) \in E(\Gamma)$, then $|f(u) - f(v)| \leq 1$. Since Γ is locally finite, $S(v_0, n)$ is finite for every $n \in \omega$. It follows that f is a finite to one $\prec$-mapping of scale 1 of $\mathcal{B}(\Gamma)$ to $\mathcal{B}(\mathbb{I})$.

$(i) \Rightarrow (ii)$. Assume the contrary: Γ is not locally finite, but there exists a finite-to-one $\prec$-mapping $f : V(\Gamma) \to \omega$. We choose a vertex $v \in V(\Gamma)$ such that $B(v, 1)$ is infinite. Since f is $\prec$-mapping, the subset $f(B(v, 1))$ of ω is finite. Hence, f is not finite-to-one. □

The same arguments show that, for every unbounded graph Γ, there exists a surjective $\prec$-mapping $f : V(\Gamma) \to V(\mathbb{I})$ of scale 1.

6. $\succ$-RAYS. Example 4 shows that $\mathcal{B}(\Gamma_1) \succ \mathcal{B}(\Gamma_2)$ does not imply $\mathcal{B}(\Gamma_2) \prec \mathcal{B}(\Gamma_1)$, but in the case $\Gamma_1 = \mathbb{I}$ such an example impossible.

Theorem 7. *Let Γ be a graph. Then $\mathcal{B}(\mathbb{I}) \succ \mathcal{B}(\Gamma)$ if and only if Γ is a $\prec$-ray.*

Proof. Let $f : \omega \to V(\Gamma)$ be a surjective $\succ$-mapping of $\mathcal{B}(\mathbb{I})$ onto $\mathcal{B}(\Gamma)$. Let m be a natural number such that $B_\Gamma(f(n), 1) \subseteq f([n-m, n+m])$ for every $n \in \omega$. We define a mapping $h : V(\Gamma) \to \omega$ by the rule $h(v) = \min\{i : f(i) = v\}$, and prove that h is an injective $\prec$-mapping of scale m of $\mathcal{B}(\Gamma)$ into $\mathcal{B}(\mathbb{I})$, so Γ is a $\prec$-ray.

Let $v \in V(\Gamma)$ and $h(v) = n$. Suppose that $h(B_\Gamma(v,1)) \nsubseteq [n-m, n+m]$ and choose $u \in B_\Gamma(v,1)$ such that $h(u) \notin [n-m, n+m]$. We put $h(u) = k$ and consider two cases.

Case $k < n-m$. Since $v \in f([k-m, k+m])$, there exists $n' < n$ such that $f(n') = v$, contradicting $h(v) = n$.

Case $k > n-m$. Since $u \in f([n-m, n+m])$, there exists $n' < k$ such that $f(n') = u$, contradicting $h(u) = k$.

If $f : V(\Gamma) \to \omega$ is an injective $\prec$-mapping of $\mathcal{B}(\Gamma)$ into $\mathcal{B}(\mathbb{I})$, we eliminate all elements of ω, which have no preimages. Then we enumerate the rest of ω in natural order. Thus, we have defined a bijective $\prec$-mapping $h : V(\Gamma) \to \omega$ of $\mathcal{B}(\Gamma)$ onto $\mathcal{B}(\mathbb{I})$. Clearly, $h^{-1} : \omega \to V(\Gamma)$ is bijective $\succ$-mapping of $\mathcal{B}(\mathbb{I})$ onto $\mathcal{B}(\Gamma)$, so $\mathcal{B}(\mathbb{I}) \succ \mathcal{B}(\Gamma)$. □

We say that a graph Γ is a $\succ$ ray if there exists a bijective $\succ$-mapping $f : V(\Gamma) \to \omega$ of $\mathcal{B}(\Gamma)$ onto $\mathcal{B}(\mathbb{I})$. Equivalently, Γ is a $\succ$-ray if there exists a bijective $\prec$-mapping $h : \omega \to V(\Gamma)$ of $\mathcal{B}(\mathbb{I})$ onto $\mathcal{B}(\Gamma)$.

Problem 2. *Characterize all $\succ$-rays.*

In fact, for locally finite graphs, this problem has been solved in another terminology in [5, Chapter 3]. The next two theorems are paraphrases of Theorem 3.13 and 3.16 from [5].

Theorem 8. *A locally finite tree T is a $\succ$-ray if and only if there exists an arrow $(v_n)_{n\in\omega}$ in T such that all the trees $\{T(v_n) : n \in \omega\}$ are finite.*

Let T be a tree with the root x. We say that T is x-rooted and define a partial ordering $\le$ on the set $V(T)$ by the rule: $y \le z$ if and only if the shortest path from x to z goes through y.

Call an x-rooted spanning tree of a graph Γ *normal* if any pair of adjacent vertices of Γ is comparable in the partial ordering on $V(\Gamma)$ induced by T. For every locally finite graph Γ and every vertex $x \in V(\Gamma)$, there exists a normal x-rooted spanning tree of Γ [5, Theorem 3.15].

Theorem 9. *For every infinite locally finite graph Γ, the following statements are equivalent: (i) Γ is a $\succ$-ray; (ii) Γ has a normal rooted spanning tree which is a $\succ$-ray; (iii) every normal rooted spanning tree of Γ is a $\succ$-ray; (iv) there exists a bijection $f : \omega \to V(\Gamma)$ such that $d(f(i), f(i+1)) \le 3$ for every $i \in \omega$.*

In view of Theorem 8, the equivalence $(i) <=> (ii)$ of Theorem 9 can be considered as a characterization of locally finite $\succ$-rays. The equivalence $(i) <=> (iv)$ of Theorem 9 can be reformulated in the following way: if there exists a $\prec$-bijection $h : \omega \to V(\Gamma)$ of $\mathcal{B}(\mathbb{I})$ onto $\mathcal{B}(\Gamma)$, then there exists a $\prec$-bijection $f : \omega \to V(\Gamma)$ of scale 3 of $\mathcal{B}(\mathbb{I})$ onto $\mathcal{B}(\Gamma)$.

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