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A MONAD FOR THE INCLUSION HYPERSPACE FUNCTOR IS UNIQUE

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It is proved that there exists a unique monad in the category of compacta with the inclusion hyperspace functor as the functorial part, namely the classical inclusion hyperspace monad.

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Доказано, что классическая монада гиперпространств включения является единственной монадой в категории компактов с функтором гиперпространств включения в качестве функториальной части.

Introduction. Monads (or triples) [7] are both important objects of study and useful tools, e.g., for investigating of analytical and topological properties by algebraic means. Further we restrict our attention to monads in the category of compacta. We can recall, for example, characterization of convex compacta as algebras for the probability measure monad (see [8]). Because of importance of monad structure a natural question of its unicity for a particular functor arise. Problems of this kind were solved for classical monads in the category of compacta, e.g. for the hyperspace monad $\mathbb{H} = (\exp, s, u)$ [4], for the superextension monad $\mathbb{L} = (\lambda, \eta_L, \mu_L)$ [1] and for the probability measure monad $\mathbb{P} = (P, \eta_P, \mu_P)$ [5]. It is proved that for each of the functors exp, λ and P there are no other monads in the category of compacta. Nevertheless, for the inclusion hyperspace monad the problem of unicity remained open for a long time.

The aim of this paper is to provide an affirmative answer to this question. The common approach that allowed to solve the problems mentioned above, was to investigate images under the multiplication (in the sense of monad) of some "generic" elements in the corresponding spaces of closed subsets, maximal linked systems or probability measures. The case of the inclusion hyperspace functor is relatively difficult because of much more combinatorial complexity even for not very large finite spaces. Therefore it was necessary to develop special tools, namely "fine" equivalence relations on sets of nonempty subsets of cartesian products of finite spaces, such that it suffices to study only "main representatives" of each inclusion hyperspace. These "auxiliary statements" constitute the largest by size part of the paper. After that we use all equalities, included into the definition of monad, to gradually reduce the set of all alternatives possible for images of "generic elements", until the answer becomes quite unambiguous: there can be no more that one monad in the category of compacta with the inclusion hyperspace functor as the functorial part. Since such monad is well known, it is unique for this functor.

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1. Preliminaries. First we recall some necessary definitions and facts. A *compactum* is a compact Hausdorff topological space. We write $A \underset{\mathrm{cl}}{\subset} X$ (resp. $A \underset{\mathrm{op}}{\subset} X$) if A is a closed (resp. open) subset of X. See [7] for the definitions of category, functor, natural transformation. The category of compacta $\mathcal{C}omp$ consists of all compact Hausdorff spaces as objects and all continuous mappings of compacta as arrows.

The *hyperspace* [6] of a compactum X is the set $\exp X$ of all nonempty closed subsets of X with the *Vietoris topology.* The standard base of this topology consists of all sets of the form

$$\langle U_1, U_2, \ldots, U_n\rangle = \{G \in \exp X \mid G \subset U_1 \cup U_2 \cup \ldots U_n, \quad G \cap U_i \neq \varnothing, i \in \{1, 2, \ldots, n\}\},$$

where $n \in \mathbb{N}$ and sets U_i are open in X. The hyperspace of a compactum is also a compactum, thus for a compactum X the spaces $\exp^2 X = \exp(\exp X)$, $\exp^3 X = \exp(\exp^2 X)$, ... are compacta as well.

For a continuous mapping $f \colon X \to Y$ of compacta the mapping $\exp f \colon \exp X \to \exp Y$, that is defined by the formula $\exp f(A) = \{f(x) \mid x \in A\}$ for $A \in \exp X$, is continuous.

The *inclusion hyperspace* [6] on a compactum X is a subset $\mathcal{F} \subset \exp X$ such that

1) $\mathcal{F}$ is nonempty and closed in $\exp X$;

2) if $A, B \underset{\mathrm{cl}}{\subset} X$, then $A \in \mathcal{F}$, $A \subset B$ imply $B \in \mathcal{F}$.

By definition each inclusion hyperspace G is an element of $\exp^2 X$. We denote by GX the set of all inclusion hyperspaces on X with topology induced by the topology on $\exp^2 X$. The subspace GX, being closed, is a compactum.

If $f \colon X \to Y$ is a continuous mapping of compacta, let $Gf \colon GX \to GY$,

$$Gf(\mathcal{F}) = \{B \underset{\mathrm{cl}}{\subset} Y \mid \text{ exists } A \in \mathcal{F}, f(A) \subset B\}.$$

Then Gf is well-defined and continuous.

The assignments exp, G are *functors* in the category of compacta, i.e. they preserve sources and targets of arrows, compositions and identity maps [7]. Therefore the powers $\exp^n = \underbrace{\exp \circ \exp \circ \cdots \circ \exp}_{n}$ and $G^n = \underbrace{G \circ G \circ \cdots \circ G}_{n}$ are functors $\mathcal{C}omp \to \mathcal{C}omp$ as well. We also need an another (trivial) example of functor $\mathcal{C}omp \to \mathcal{C}omp$, namely, the *identity functor* $\mathbf{1}_{\mathcal{C}omp}$ that sends each compactum or mapping to itself.

The functor G preserves monomorphisms, that makes possible for an arbitrary embedding of compacta $i \colon X_0 \hookrightarrow X$ to identify each element $\mathsf{A} \in GX_0$ with its image $Gi(\mathsf{A}) = \{B \in \exp X \mid B \supset A \text{ for some } A \in \mathsf{A}\} \in GX$, and the entire space GX_0 with the image $Gi(GX_0) \subset GX$. The functor G also preserves intersections, i.e. $\mathsf{A} \in GX_\alpha$ for all elements of a family $\{X_\alpha \mid \alpha \in A\}$ of closed subsets of X implies $\mathsf{A} \in G(\bigcap_{\alpha \in A} X_\alpha)$. Thus we can define the *support* $\operatorname{supp} \mathsf{A}$ of an element $\mathsf{A} \in GX$ in usual way as the least closed subspace $X_0 \subset X$ such that $\mathsf{A} \in GX_0$.

It is straightforward to show that each element of an inclusion hyperspace on a compactum contains a minimal with respect to inclusion element of this inclusion hyperspace. Any inclusion hyperspace is uniquely determined by its minimal elements. It is obvious that the minimal elements of the above mentioned inclusion hyperspaces $\mathsf{A} \in GX_0$ and $Gi(\mathsf{A}) \in GX$ coincide, which again confirms legitimacy of the used identification. For all $X_0 \underset{\mathrm{cl}}{\subset} X$ and $\mathsf{A} \in GX$ we have $\mathsf{A} \in GX_0$ if and only if all minimal elements of A are contained in X_0.

We denote for an element $\mathsf{B} \in \exp^2 X$:

$$rX(\mathsf{B}) = \{A \in \exp X \mid A \supset B \text{ for some } B \in \mathsf{B}\}.$$

Then rX is a continuous retraction $\exp^2 X \to GX$.

For a compactum X the mappings $\eta X \colon X \to GX$, $\mu X \colon G^2 X \to GX$ are defined by the formulae: $\eta X(a) = \{F \underset{\mathrm{cl}}{\subset} X \mid F \ni a\}$, $\mu X(\mathsf{F}) = \bigcup_{\mathcal{A} \in \mathsf{F}} \bigcap \mathcal{A}$. They are well-defined and continuous, and the collections $\eta = (\eta X)_{X \in Ob\,\mathcal{C}omp}$, $\mu = (\mu X)_{X \in Ob\,\mathcal{C}omp}$ are *natural transformations* [7] $\mathbf{1}_{\mathcal{C}omp} \to G$ and $G^2 \to G$ resp. Moreover, they are the unit and the multiplication of the monad $\mathbb{G} = (G, \eta, \mu)$ in $\mathcal{C}omp$, i.e., the diagrams

$$\begin{tikzcd} GX \arrow[r, "\eta GX"] \arrow[d, "G\eta X"'] \arrow[dr, "\mathbf{1}_{GX}"] & G^2X \arrow[d, "\mu X"] \\ G^2X \arrow[r, "\mu X"'] & GX \end{tikzcd} \qquad \begin{tikzcd} G^3X \arrow[r, "G\mu X"] \arrow[d, "\mu GX"'] & G^2X \arrow[d, "\mu X"] \\ G^2X \arrow[r, "\mu X"'] & GX \end{tikzcd}$$

commute for each object X of the category $\mathcal{C}omp$.

See [2], [3] on the properties of this monad. It also has deep connections with the capacity monad in the category of compacta, which in turn has applications to the decision making theory.

It is easy to show that η is the unique natural transformation $\mathbf{1}_{\mathcal{C}omp} \to G$. The aim of this paper is to prove that the natural transformation μ is unique in the following sense: if (G, η, μ') is a monad in the category of compacta, then $\mu' = \mu$.

2. Auxiliary statements. We denote by n the set $\{0, 1, 2, \dots, n-1\}$ with the discrete topology. We regard the elements of a cartesian product of n sets as sequences indexed by numbers 0, 1, …, $n-1$. Nevertheless, it is convenient to number *factors* of this product by 1, 2, …, n.

For any $k \in \{0, 1, 2, \dots\}$, $n_1, n_2, \dots, n_k \in \{1, 2, 3 \dots\}$ we assume that subsets of the cartesian product $n_1 \times n_2 \times \dots \times n_k$ are ordered by inclusion, i.e. A precedes B if $A \subset B$. For $k = 0$ we consider this product as the set $\{()\}$ with the empty sequence $()$ being its unique element. For $0 \leqslant l \leqslant k$, $0 \leqslant i_1 \leqslant n_1 - 1$, $0 \leqslant i_2 \leqslant n_2 - 1$, …, $0 \leqslant i_l \leqslant n_l - 1$ call the set

$$[A]_{i_1, i_2, \dots, i_l} = \underset{(l+1)(l+2)\dots k}{\mathrm{pr}} (A \cap (\{i_1\} \times \{i_2\} \times \dots \times \{i_l\} \times n_{l+1} \times \dots \times n_k)) \subset n_{l+1} \times \dots \times n_k$$

the $(i_1, i_2, \dots, i_l)$*-th section* of the set A. As the restriction of the projection $\mathrm{pr}_{(l+1)(l+2)\dots k}$ to the intersection $A \cap (\{i_1\} \times \{i_2\} \times \dots \times \{i_l\} \times n_{l+1} \times \dots \times n_k)$ is a bijection onto $[A]_{i_1, i_2, \dots, i_l}$, we identify the section with the corresponding intersection. In this sense we talk about restriction of a mapping from A to $[A]_{i_1, i_2, \dots, i_l}$ etc. Specifically we identify the product $\{i\} \times n_2 \times \dots \times n_k$ with $n_2 \times \dots \times n_k$.

We extend lexicographically the strict partial order "$\subsetneq$" on the set of all subsets of the cartesian product $n_1 \times n_2 \times \dots \times n_k$ to a strict linear order $\prec$. For $k = 0$ let $\varnothing \prec \{()\}$. If $k \geqslant 1$, $A, B \subset n_1 \times n_2 \times \dots \times n_k$, we put $A \prec B$ iff there is $i \in \{0, 1, \dots, n_1 - 1\}$ such that $[A]_0 = [B]_0$, $[A]_1 = [B]_1$, …, $[A]_{i-1} = [B]_{i-1}$, $[A]_i \prec [B]_i$. The order reverse to "$\prec$" is denoted as "$\succ$". We write $A \preceq B$ if $A \prec B$ or $A = B$, and $A \succeq B$ if $A \succ B$ or $A = B$.

Consider all products of the form $n_1 \times n_2 \times \dots \times n_k$ and inductively define a class of their subsets called *regular*. For $k = 0$ we regard both $\varnothing$ and $\{()\}$ as regular subsets. If $k > 0$, we say that a subset $A \subset n_1 \times n_2 \times \dots \times n_k$ is regular if:

1) all sections $[A]_i$, $0 \leqslant i \leqslant n_1 - 1$, are regular subsets in $n_2 \times \cdots \times n_k$;

2) in the set of sections $[A]_i$, $0 \leqslant i \leqslant n_1 - 1$, each element is either minimal of maximal with respect to $\subset$;

3) $[A]_0 \succeq [A]_1 \succeq \cdots \succeq [A]_{n_1-1}$, and only a section that is minimal with respect to $\subset$ can appear more than once.

Note that 1) implies that *all* sections $[A]_{i_1,i_2,\dots,i_l}$ of a regular subset A are regular subsets in the corresponding products.

For $k = 1$ and $n_1 > 1$ we have three regular subsets in $n_1 = \{0, 1, \dots, n_1 - 1\}$, namely $\varnothing$, $\{0\}$ and n_1 itself. For $k = 2$ and $n_1, n_2 > 1$ there are six such subsets in $n_1 \times n_2 : \mathcal{A} = n_1 \times n_2$, $\mathcal{B} = n_1 \times \{0\} \cup \{0\} \times n_2$, $\mathcal{C}_1 = \{0\} \times n_2$, $\mathcal{C}_2 = n_1 \times \{0\}$, $\mathcal{D} = \{(0, 0)\}$, $\mathcal{E} = \varnothing$.

We call a mapping $f\colon n_1 \times n_2 \times \cdots \times n_k \to m_1 \times m_2 \times \cdots \times m_k$ a *regular* bijection (surjection), if every its restriction onto a section $[n_1 \times n_2 \times \cdots \times n_k]_{i_1,i_2,\dots,i_l}$, where $0 \leqslant l \leqslant k$, is a bijection (surjection) to some section $[m_1 \times m_2 \times \cdots \times m_k]_{j_1,j_2,\dots,j_l}$. The definition implies that the composition of regular bijections (surjections) is a regular bijection (surjection), and the mapping f should be a bijection (surjection) itself.

We call subsets $A, B \subset n_1 \times n_2 \times \cdots \times n_k$ *equivalent* and write $A \sim B$ if there is a regular bijection $b\colon n_1 \times n_2 \times \cdots \times n_k \to n_1 \times n_2 \times \cdots \times n_k$ such that $b(A) = B$.

Subsets $A \subset n_1 \times n_2 \times \cdots \times n_k$, $B \subset m_1 \times m_2 \times \cdots \times m_k$ are called *weakly equivalent* (denote $A \approx B$) if at least one of the following holds:

1) $k = 0$ and $A = B$, i.e. $A = B = \varnothing$ or $A = B = \{()\}$;

2) $k > 0$, for each $i \in \{0, 1, \dots, n_1 - 1\}$ there exists $i' \in \{0, 1, \dots, m_1 - 1\}$ such that $[B]_{i'} \approx [A]_i$, and for each $j \in \{0, 1, \dots, m_1 - 1\}$ there exists $j' \in \{0, 1, \dots, n_1 - 1\}$ such that $[A]_{j'} \approx [B]_j$.

It is obvious that $A \sim B \implies A \approx B$, and both "$\sim$" are "$\approx$" are equivalence relations.

A regular subset $A \subset n_1 \times n_2 \times \cdots \times n_k$ is called *quite regular*, if it is the least with respect to $\prec$ of all regular sets that are weakly equivalent to A. An equivalent definition by induction by k can be given:

1) all sections $[A]_i$, $0 \leqslant i \leqslant n_1 - 1$, are quite regular subsets in $n_2 \times \cdots \times n_k$;

2) in the set of sections $[A]_i$, $0 \leqslant i \leqslant n_1 - 1$, any element is either minimal or maximal with respect to $\subset$;

3) $[A]_0 \succeq [A]_1 \succeq \cdots \succeq [A]_{n_1-1}$, and only the last section $[A]_{n_1-1}$ can be repeated.

In the sequel assume that for some $k \in \{0, 1, 2, \dots\}$ and all $n_1, n_2, \dots, n_k \in \{1, 2, 3 \dots\}$ an inclusion hyperspace $\mathcal{K}(n_1, n_2, \dots, n_k) \in G(n_1 \times n_2 \times \cdots \times n_k)$ is fixed, and

(!) *if* $f\colon n_1 \times n_2 \times \cdots \times n_k \to m_1 \times m_2 \times \cdots \times m_k$ *is a regular surjection, then* $Gf(\mathcal{K}(n_1, n_2, \dots, n_k)) = \mathcal{K}(m_1, m_2, \dots, m_k)$.

Lemma 1. *Each minimal element* $A \in \mathcal{K}(n_1, n_2, \dots, n_k)$ *is equivalent to some regular set* $A_0 \in \mathcal{K}(n_1, n_2, \dots, n_k)$.

Proof. Assume that $1 \leqslant l \leqslant k$ and a regular bijection $b_l\colon n_1 \times n_2 \times \cdots \times n_k \to n_1 \times n_2 \times \cdots \times n_k$ has already been built such that all sections of the set $A_l = b_l(A)$ of the form $[A_l]_{i_1,i_2,\dots,i_l}$ are regular subsets in $n_{l+1} \times n_{l+2} \times \cdots \times n_k$.

Note that (!) implies that the set of all elements of the inclusion hyperspace $\mathcal{K}(n_1, n_2, \dots, n_k)$, as well as the set of all its minimal elements, are invariant with respect to any regular bijection $n_1 \times n_2 \times \cdots \times n_k \to n_1 \times n_2 \times \cdots \times n_k$. Therefore the set A_l is also a minimal element of $\mathcal{K}(n_1, n_2, \dots, n_k)$.

Assume that the section $[A_l]_{i_1,i_2,\dots,i_{l-1},i_l}$ is neither minimal nor unique and maximal with respect to $\subset$ among $[A_l]_{i_1,i_2,\dots,i_{l-1},0}, \dots, [A_l]_{i_1,i_2,\dots,i_{l-1},n_l-1}$. Then there are $i', i'' \neq i_l$, $i' \neq i''$

such that the section $[A_l]_{i_1,i_2,\dots,i_{l-1},i'}$ is maximal, the section $[A_l]_{i_1,i_2,\dots,i_{l-1},i''}$ is minimal, and $[A_l]_{i_1,i_2,\dots,i_{l-1},i''} \subset [A_l]_{i_1,i_2,\dots,i_{l-1},i_l} \subset [A_l]_{i_1,i_2,\dots,i_{l-1},i'}$. We define regular surjections

$$\psi, \theta\colon n_1 \times n_2 \times \dots \times n_{l-1} \times (n_l + 1) \times n_{l+1} \times \dots \times n_k \to n_1 \times n_2 \times \dots \times n_k$$

by the formulae

$$\psi(x_1, x_2, \dots, x_n) = \begin{cases} (x_1, x_2, \dots, x_n) & \text{if } x_l \leqslant n_i - 1; \\ (x_1, \dots, x_{l-1}, i'', x_{l+1}, \dots, x_n) & \text{if } x_l = n_l; \end{cases}$$

$$\theta(x_1, x_2, \dots, x_n) = \begin{cases} (x_1, x_2, \dots, x_n) \text{ if } x_1 = i_1, \dots, x_{l-1} = i_{l-1}, \\ \qquad\qquad x_l \neq i_l, x_l \neq n_l; \\ (x_1, \dots, x_{l-1}, i', x_{l+1}, \dots, x_n) \\ \qquad\qquad \text{if } x_1 = i_1, \dots, x_{l-1} = i_{l-1}, x_l = i_l; \\ (x_1, \dots, x_{l-1}, i_l, x_{l+1}, \dots, x_n) \\ \qquad\qquad \text{if } x_1 = i_1, \dots, x_{l-1} = i_{l-1}, x_l = n_l; \\ \psi(x_1, x_2, \dots, x_n) \text{ otherwise.} \end{cases}$$

Then $\psi^{-1}(A) \in \mathcal{K}(n_1, \dots, n_{l-1}, n_l + 1, n_{l+1}, \dots, n_k)$, thus $A'_l = \theta(\psi^{-1}(A_l)) \in \mathcal{K}(n_1, \dots, n_k)$. The sets A_l and A'_l coincide except for sections with the index $(i_1, i_2, \dots, i_{l-1}, i_l)$. We obtain $[A'_l]_{i_1,i_2,\dots,i_{l-1},i_l} = [A_l]_{i_1,i_2,\dots,i_{l-1},i''} \subsetneq [A_l]_{i_1,i_2,\dots,i_{l-1},i_l}$, and $A'_l \subsetneq A_l$, which contradicts to minimality of A_l. Thus the assumption is false, and each section $[A_l]_{i_1,i_2,\dots,i_{l-1},i_l}$ is either minimal or unique and maximal with respect $\subset$ among $[A_l]_{i_1,i_2,\dots,i_{l-1},0}, \dots, [A_l]_{i_1,i_2,\dots,i_{l-1},n_l-1}$.

For each $i_1 \in \{1, 2, \dots, n_1 - 1\}$, $i_2 \in \{1, 2, \dots, n_2 - 1\}$, …, $i_{l-1} \in \{1, 2, \dots, n_{l-1} - 1\}$ choose a permutation $\sigma = \sigma_{i_1,i_2,\dots,i_{l-1}}\colon n_l \to n_l$ such that

$$[A_l]_{i_1,i_2,\dots,i_{l-1},\sigma(n_l-1)} \preceq [A_l]_{i_1,i_2,\dots,i_{l-1},\sigma(n_l-2)} \preceq \dots \preceq [A_l]_{i_1,i_2,\dots,i_{l-1},\sigma(1)} \preceq [A_l]_{i_1,i_2,\dots,i_{l-1},\sigma(0)}.$$

We define a mapping $\phi_l\colon n_1 \times n_2 \times \dots \times n_k \to n_1 \times n_2 \times \dots \times n_k$ by the formula

$$\phi_l\colon (i_1, i_2, \dots, i_n) = (i_1, i_2, \dots, i_{l-1}, \sigma_{i_1,i_2,\dots,i_{l-1}}(i_l), i_{l+1}, i_k).$$

Obviously ϕ_l is a regular bijection, the set $A_{l-1} = \phi_l(A_l)$ is in $\mathcal{K}(n_1, n_2, \dots, n_k)$, and for all $i_1 \in \{1, 2, \dots, n_1 - 1\}$, …, $i_{l-1} \in \{1, 2, \dots, n_{l-1} - 1\}$ the sections satisfy a condition

$$[A_{l-1}]_{i_1,i_2,\dots,i_{l-1},n_l-1} \preceq [A_{l-1}]_{i_1,i_2,\dots,i_{l-1},n_l-2} \preceq \dots \preceq [A_{l-1}]_{i_1,i_2,\dots,i_{l-1},1} \preceq [A_{l-1}]_{i_1,i_2,\dots,i_{l-1},0}.$$

Thus the regular bijection $b_{l-1} = \phi_l \circ b_l$ maps the set A to a set $A_{l-1} \in \mathcal{K}(n_1, n_2, \dots, n_k)$ such that all sections of the latter of the form $[A_{l-1}]_{i_1,i_2,\dots,i_{l-1}}$ are regular sets.

To start the induction, note that for $l = k$ we can take the the identity map of $n_1 \times n_2 \times \dots \times n_k$ onto itself as a required bijection, and put $A_k = A$. Thus, decreasing l, we obtain for $l = 0$ a regular bijection $b_0\colon n_1 \times n_2 \times \dots \times n_k \to n_1 \times n_2 \times \dots \times n_k$ such that the section with the empty index of the set $A_0 = b_0(A)$, i.e. the set A_0 itself, is regular. □

Lemma 2. *Let $A \subset n_1 \times n_2 \times \dots \times n_k$, $A' \subset n'_1 \times n'_2 \times \dots \times n'_k$, and $A \approx B$. Then $A \in \mathcal{K}(n_1, n_2, \dots, n_k)$ if and only if $A' \in \mathcal{K}(n'_1, n'_2, \dots, n'_k)$.*

Proof. Put $m_1 = \max\{n_1, n_1'\}$, $m_2 = \max\{n_2, n_2'\}$, $\dots$, $m_k = \max\{n_k, n_k'\}$, and choose arbitrary regular surjections $f\colon m_1\times m_2\times\dots\times m_k \to n_1\times n_2\times\dots\times n_k$ and $f'\colon m_1\times m_2\times\dots\times m_k \to n_1'\times n_2'\times\dots\times n_k'$. If $B = f^{-1}(A)$, $B' = (f')^{-1}(A')$, then $A \approx B$, $A' \approx B'$, thus $B \approx B'$, and $A \in \mathcal{K}(n_1, n_2, \dots, n_k) \iff B \in \mathcal{K}(m_1, m_2, \dots, m_k)$, $A' \in \mathcal{K}(n_1', n_2', \dots, n_k') \iff B' \in \mathcal{K}(m_1, m_2, \dots, m_k)$. Therefore it suffices to prove that $B \in \mathcal{K}(m_1, m_2, \dots, m_k) \iff B' \in \mathcal{K}(m_1, m_2, \dots, m_k)$ for $B \approx B'$, $B, B' \subset m_1 \times m_2 \times \dots \times m_k$. Let $B'' \subset m_1 \times m_2 \times \dots \times m_k$ be a subset that is the least with respect "$\prec$" among all subsets that are weakly equivalent to B and B'. We will prove that $B \in \mathcal{K}(m_1, m_2, \dots, m_k) \iff B'' \in \mathcal{K}(m_1, m_2, \dots, m_k)$, $B' \in \mathcal{K}(m_1, m_2, \dots, m_k) \iff B'' \in \mathcal{K}(m_1, m_2, \dots, m_k)$.

Assume that for some $l \in \{0, 1, 2, \dots, k\}$ a subset $B_l \subset m_1 \times m_2 \times \dots \times m_k$ has been constructed such that:

1) for any $i_1 \in m_1$, $i_2 \in m_2$, $\dots$, $i_l \in m_l$ the section $[B_l]_{i_1,i_2,\dots,i_l}$ is the least with respect to "$\prec$" among all subsets $m_{l+1} \times m_{l+2} \times \dots \times m_k$ that are weakly equivalent to $[B]_{i_1,i_2,\dots,i_l}$;

2) $B \in \mathcal{K}(m_1, m_2, \dots, m_k) \iff B_l \in \mathcal{K}(m_1, m_2, \dots, m_k)$.

Then $B_l \approx B$. Define a set $B_{l-1} \subset m_1\times m_2\times\dots\times m_k$ by the following condition: for each $i_1 \in m_1$, $i_2 \in m_2$, $\dots$, $i_{l-1} \in m_{l-1}$ the section $[B_{l-1}]_{i_1,i_2,\dots,i_{l-1}}$ is the least with respect to "$\prec$" among all subsets $m_l\times m_{l+1}\times\dots\times m_k$ that are equivalent to $[B_l]_{i_1,i_2,\dots,i_{l-1}}$. It is obvious that $B_{l-1} \approx B_l$. We prove that $B_{l-1} \in \mathcal{K}(m_1, m_2, \dots, m_k) \iff B_l \in \mathcal{K}(m_1, m_2, \dots, m_k)$. For any $i_1 \in m_1$, $i_2 \in m_2$, $\dots$, $i_{l-1} \in m_{l-1}$ the set of sections $\{[B_{l-1}]_{i_1,i_2,\dots,i_l} \mid 0 \leqslant i_l \leqslant m_l - 1\}$ coincide with the set of sections $\{[B_l]_{i_1',i_2',\dots,i_l'} \mid 0 \leqslant i_l' \leqslant m_l - 1\}$. Thus each of the two sets B_{l-1} and B_l can be obtained from the other one by a sequence of transformations on subsets of $m_1 \times m_2 \times \dots \times m_k$ of the following two types:

1) fix $i_1 \in m_1$, $i_2 \in m_2$, $\dots$, $i_{l-1} \in m_{l-1}$, $i_l, i_l' \in m_l$, $i_l \neq i_l'$, and from a set $A \subset m_1\times m_2\times\dots\times m_k$ obtain a set A' by the permutation of sections $[A]_{i_1,i_2,\dots,i_{l-1},i_l}$ and $[A]_{i_1,i_2,\dots,i_{l-1},i_l'}$;

2) fix $i_1 \in m_1$, $i_2 \in m_2$, $\dots$, $i_{l-1} \in m_{l-1}$, $i_l, i_l', i_l'' \in m_l$, $i_l \neq i_l'$, such that $[A]_{i_1,i_2,\dots,i_{l-1},i_l} = [A]_{i_1,i_2,\dots,i_{l-1},i_l'} \neq [A]_{i_1,i_2,\dots,i_{l-1},i_l''}$ for a set $A \subset m_1 \times m_2 \times \dots \times m_k$, replace $[A]_{i_1,i_2,\dots,i_{l-1},i_l'}$ by $[A]_{i_1,i_2,\dots,i_{l-1},i_l''}$ and obtain a set A'.

It is sufficient to show that transformations 1), 2) preserve the class of elements of $\mathcal{K}(m_1, m_2, \dots, m_k)$. Consider the transformation 1) and define a regular bijection $\Phi\colon m_1 \times m_2 \times \dots \times m_k \to m_1 \times m_2 \times \dots \times m_k$ by the formula

$$\Phi(j_1, j_2, \dots, j_k) = \begin{cases} (i_1, i_2, \dots, i_l', j_{l+1}, \dots, j_k) \text{ if } j_1 = i_1, \dots, j_{l-1} = i_{l-1}, j_l = i_l; \\ (i_1, i_2, \dots, i_l, j_{l+1}, \dots, j_k) \text{ if } j_1 = i_1, \dots, j_{l-1} = i_{l-1}, j_l = i_l'; \\ (j_1, j_2, \dots, j_k) \text{ otherwise.} \end{cases}$$

Then $\Phi(A) = A'$, thus $A \in \mathcal{K}(m_1, m_2, \dots, m_k) \iff A' \in \mathcal{K}(m_1, m_2, \dots, m_k)$.

For transformation 2) define regular surjections

$$\Psi, \Psi'\colon m_1 \times m_2 \times \dots \times m_{l-1} \times (m_l + 1) \times m_{l+1} \times \dots \times m_k \to m_1 \times m_2 \times \dots \times m_k$$

by the formulae

$$\Psi(j_1, j_2, \dots, j_k) = \begin{cases} (j_1, j_2, \dots, j_{l-1}, i_l'', j_{l+1}, \dots, j_k) \text{ if} \\ \qquad j_l = m_l; \\ (j_1, j_2, \dots, j_k) \text{ otherwise;} \end{cases}$$

$$\Psi'(j_1,j_2,\dots,j_k)=\begin{cases}(j_1,j_2,\dots,j_{l-1},i''_l,j_{l+1},\dots,j_k)\text{ if}\\ \qquad j_l=m_l,(j_1,j_2,\dots,j_{l-1})\neq(i_1,i_2,\dots,i_{l-1});\\ (i_1,i_2,\dots,i_{l-1},i'_l,j_{l+1},\dots,j_k)\text{ if}\\ \qquad (j_1,j_2,\dots,j_{l-1},j_l)=(i_1,i_2,\dots,i_{l-1},m_l);\\ (i_1,i_2,\dots,i_{l-1},i_l,j_{l+1},\dots,j_k)\text{ if}\\ \qquad (j_1,j_2,\dots,j_{l-1},j_l)=(i_1,i_2,\dots,i_{l-1},i'_l);\\ (j_1,j_2,\dots,j_k)\text{ otherwise.}\end{cases}$$

Then $A\in\mathcal{K}(m_1,m_2,\dots,m_k)$, thus $\Psi^{-1}(A)\in\mathcal{K}(m_1,m_2,\dots,m_{l-1},m_l+1,m_{l+1},\dots,m_k)$, and $A'=\Psi'(\Psi^{-1}(A))\in\mathcal{K}(m_1,m_2,\dots,m_k)$.

Therefore $B_{l-1}\in\mathcal{K}(m_1,m_2,\dots,m_k)\iff B_l\in\mathcal{K}(m_1,m_2,\dots,m_k)$, and $B_{l-1}\approx B$, which implies $B_{l-1}\in\mathcal{K}(m_1,m_2,\dots,m_k)\iff B\in\mathcal{K}(m_1,m_2,\dots,m_k)$. For $l=k$ we can put $B_l=B$ and use induction to obtain that there exists a subset B_0 that is the least with respect to "$\prec$" among all subsets of $m_1\times m_2\times\dots\times m_k$ that are weakly equivalent to B, and $B_0\in\mathcal{K}(m_1,m_2,\dots,m_k)\iff B\in\mathcal{K}(m_1,m_2,\dots,m_k)$. But this implies $B_0=B''$, therefore $B\in\mathcal{K}(m_1,m_2,\dots,m_k)\iff B''\in\mathcal{K}(m_1,m_2,\dots,m_k)$, and $B'\in\mathcal{K}(m_1,m_2,\dots,m_k)\iff B''\in\mathcal{K}(m_1,m_2,\dots,m_k)$ as well. Thus we obtain $B\in\mathcal{K}(m_1,m_2,\dots,m_k)\iff B'\in\mathcal{K}(m_1,m_2,\dots,m_k)$. □

This lemma implies that the inclusion hyperspace $\mathcal{K}(n_1,n_2,\dots,n_k)$ is uniquely determined by the set of all sets of sections $\{[A]_0,[A]_1,\dots,[A]_{n_1-1}\}$ for *regular* minimal elements $A\in\mathcal{K}(n_1,n_2,\dots,n_k)$.

For $n_1,n_2\in\{1,2,\dots\}$ we denote by V_{n_1,n_2} the element $G(n_1\times n_2)$ with minimal elements $\{0\}\times n_2$, $\{1\}\times n_2$, …, $\{n_1-1\}\times n_2$. Now by induction define $V_{n_1,n_2,\dots,n_{2k-1},n_{2k}}$ for $k>1$ as the element of the space $G^k(n_1\times n_2\times\dots\times n_{2k-1}\times n_{2k})$ with minimal sets of the form

$$\{G^{k-1}j_{i_1,0}(V_{n_3,n_4,\dots,n_{2k-1},n_{2k}}),G^{k-1}j_{i_1,1}(V_{n_3,n_4,\dots,n_{2k-1},n_{2k}}),\dots,G^{k-1}j_{i_1,n_2-1}(V_{n_3,n_4,\dots,n_{2k-1},n_{2k}})\}$$

for $0\leqslant i\leqslant n_1-1$, where the embeddings $j_{i_1,i_2}\colon n_3\times n_4\times\dots\times n_{2k-1}\times n_{2k}\to n_1\times n_2\times\dots\times n_{2k-1}\times n_{2k}$ are defined by the formulae $j_{i_1,i_2}(i_3,\dots,i_{2k})=(i_1,i_2,i_3,\dots,i_{2k})$.

Let $n_1,n_2,\dots,n_{2k}\in\{1,2,\dots\}$, and for some $1\leqslant l_1<l_2<\dots<l_s\leqslant 2k$ we have $n_{l_2}=n_{l_2}=\dots=n_{l_s}=1$. Then the projection $p\colon n_1\times n_2\times\dots\times n_{2k}\to\prod\limits_{\substack{1\leqslant l\leqslant 2k,\\ l\notin\{l_1,l_2,\dots,l_s\}}}n_l$ is a bijection. We denote the image

$$G^kp(V_{n_1,n_2,\dots,n_{2k-1},n_{2k}})\in G^k(\prod_{\substack{1\leqslant l\leqslant 2k,\\ l\notin\{l_1,l_2,\dots,l_s\}}}n_l)$$

by $V_{\dot n_1,\dot n_2,\dots,\dot n_{2k-1},\dot n_{2k}}$, where the sequence $\dot n_1,\dot n_2,\dots,\dot n_{2k-1},\dot n_{2k}$ is obtained from $n_1,n_2,\dots,n_{2k-1},n_{2k}$ by replacement of all elements with indices $l_1,l_2,\dots,l_s$ (that are equal to 1) by dots.

E.g. $V_{\cdot,m,n,\cdot}\in G^2(m\times n)$ is the image of the inclusion hyperspace $V_{1,m,n,1}\in G^2(1\times m\times n\times 1)$ under $G^2\,\mathrm{pr}_{23}$, where $\mathrm{pr}_{23}\colon 1\times m\times n\times 1\to m\times n$ is the projection.

Lemma 3. *Let $n_1,n_2,\dots,n_{2k-1},n_{2k}\in\{\cdot,1,2,\dots\}$, and $n_{2p-1}=n_{2p}=\cdot$ for some $1\leqslant p\leqslant k$. Then*

$$V_{n_1,n_2,\dots,n_{2k-1},n_{2k}}=G^{p-1}\eta G^{k-p}\prod_{\substack{1\leqslant l\leqslant 2k,\\ n_l\neq\cdot}}n_l\ (V_{n_1,n_2,\dots,n_{2p-2},n_{2p+1},\dots,n_{2k}}).$$

Proof. For $p=1$ and arbitrary $k\geqslant p$ the inclusion hyperspace $V_{\cdot,\cdot,n_3,\dots,n_{2k-1},n_{2k}}$ has the unique minimal set $\{V_{n_3,\dots,n_{2k-1},n_{2k}}\}$, therefore is equal to $\eta G^{k-1}\prod_{\substack{1\leqslant l\leqslant 2k,\\ n_l\neq\cdot}} n_l\ (V_{n_3,\dots,n_{2k-1},n_{2k}})$, and the statement is true. Assume that is holds for all $p\leqslant p_0$ and all $k\geqslant p$. Consider the case $n_1,n_2\in\{1,2,\dots\}$, $p=p_0+1$. The inclusion hyperspace $V_{n_1,n_2,\dots,n_{2k-1},n_{2k}}$ has minimal sets

$$\{G^{k-1}j_{i_1,0}(V_{n_3,n_4,\dots,n_{2k-1},n_{2k}}),G^{k-1}j_{i_1,1}(V_{n_3,n_4,\dots,n_{2k-1},n_{2k}}),\\ \dots,G^{k-1}j_{i_1,n_2-1}(V_{n_3,n_4,\dots,n_{2k-1},n_{2k}})\}$$

for $0\leqslant i\leqslant n_1-1$, where the embedding j_{i_1,i_2} are defined above. But by the inductive assumption these sets can be written as

$$\{G^{k-1}j_{i_1,0}(G^{p_0-1}\eta G^{k-p_0-1}\prod_{\substack{3\leqslant l\leqslant 2k,\\ n_l\neq\cdot}} n_l\ (V_{n_3,n_4,\dots,n_{2p-2},n_{2p+1},\dots,n_{2k-1},n_{2k}})),$$
$$G^{k-1}j_{i_1,1}(G^{p_0-1}\eta G^{k-p_0-1}\prod_{\substack{3\leqslant l\leqslant 2k,\\ n_l\neq\cdot}} n_l\ (V_{n_3,n_4,\dots,n_{2p-2},n_{2p+1},\dots,n_{2k-1},n_{2k}})),\dots,$$
$$G^{k-1}j_{i_1,n_2-1}(G^{p_0-1}\eta G^{k-p_0-1}\prod_{\substack{3\leqslant l\leqslant 2k,\\ n_l\neq\cdot}} n_l\ (V_{n_3,n_4,\dots,n_{2p-2},n_{2p+1},\dots,n_{2k-1},n_{2k}}))\}.$$

Note that $\eta\colon \mathbf{1}_{\mathcal{C}omp}\to G$ is a natural transformation and $p_0-1<k$, therefore

$$G^{k-1}j_{i_1,i_2}\circ G^{p_0-1}\eta G^{k-p_0-1}\prod_{\substack{3\leqslant l\leqslant 2k,\\ n_l\neq\cdot}} n_l = G^{p_0-1}\eta G^{k-p_0-1}\prod_{\substack{1\leqslant l\leqslant 2k,\\ n_l\neq\cdot}} n_l\circ G^{k-2}j_{i_1,i_2}.$$

Thus the minimal sets of the inclusion hyperspace $V_{n_1,n_2,\dots,n_{2k-1},n_{2k}}$ are the images of the minimal sets of the inclusion hyperspace $V_{n_1,n_2,\dots,n_{2p-2},n_{2p+1},\dots,n_{2k-1},n_{2k}}$ under the mapping $G^{p_0-1}\eta G^{k-p_0-1}\prod_{\substack{1\leqslant l\leqslant 2k,\\ n_l\neq\cdot}} n_l$. Since $p_0+1=p$, we obtain the required equality

$$V_{n_1,n_2,\dots,n_{2k-1},n_{2k}}=G^{p-1}\eta G^{k-p}\prod_{\substack{1\leqslant l\leqslant 2k,\\ n_l\neq\cdot}} n_l\ (V_{n_1,n_2,\dots,n_{2p-2},n_{2p+1},\dots,n_{2k-1},n_{2k}}).$$

Passing to cases, when one or two of the indices n_1,n_2 are equal to "$\cdot$", is trivial.

Thus the statement of the lemma holds for all natural p.

□

3. Proof of the main result. In the sequel we assume that $\mathbb{G}=(G,\eta,\mu)$ is a monad in the category of compacta with the inclusion hyperspace functor as the functorial part, and the natural transformations η and μ are arbitrary. As we remarked above, ambiguity is possible only in the choice of the natural transformation $\mu\colon G^2\to G$.

We investigate images under this transformation of certain simple inclusion hyperspaces. By Lemma 3 we have $\mu(m\times n)(V_{\cdot,\cdot,m,n})=\mu(m\times n)\circ\eta G(m\times n)(V_{m,n})=V_{m,n}$. Analogously $\mu(m\times n)(V_{m,n,\cdot,\cdot})=\mu(m\times n)\circ G\eta(m\times n)(V_{m,n})=V_{m,n}$.

It is obvious that for any regular surjection $f\colon m\times n\to m'\times n'$ the equality $G^2f(V_{\cdot,m,n,\cdot})=V_{\cdot,m',n',\cdot}$, $G^2f(V_{m,\cdot,n,\cdot})=V_{m',\cdot,n',\cdot}$, $G^2f(V_{\cdot,m,\cdot,n})=V_{\cdot,m',\cdot,n'}$, $G^2f(V_{m,\cdot,\cdot,n})=V_{m',\cdot,\cdot,n'}$ holds. Thus each of the collections of inclusion hyperspaces $\mu(m\times n)(V_{\cdot,m,n,\cdot})$, $\mu(m\times n)(V_{m,\cdot,n,\cdot})$, $\mu(m\times n)(V_{\cdot,m,\cdot,n})$, $\mu(m\times n)(V_{m,\cdot,\cdot,n})$, where $m,n\in\mathbb{N}$, satisfies condition (!). Therefore all minimal sets of any of these inclusion hyperspaces can be obtained by regular bijections $m\times n\to m\times n$ from its regular minimal elements.

Let $\mathsf{A}_1=\mu(m\times n)(V_{m,\cdot,\cdot,n})$. Since

$$G\operatorname{pr}_1(\mathsf{A}_1)=G\operatorname{pr}_1\circ\mu(m\times n)(V_{m,\cdot,\cdot,n})=\mu m\circ G^2\operatorname{pr}_1(V_{m,\cdot,\cdot,n})=\mu_m(V_{m,\cdot,\cdot,\cdot})=V_{m,\cdot}\in G\,m$$

is the inclusion hyperspace with minimal sets $\{0\}$, $\{1\}$, …, $\{m-1\}$, the inclusion hyperspace A_1 contains all sets $\operatorname{pr}_1{}^{-1}(\{0\})=\{0\}\times n$, $\operatorname{pr}_1{}^{-1}(\{1\})=\{1\}\times n$, …, $\operatorname{pr}_1{}^{-1}(\{m-1\})=\{m-1\}\times n$. If the regular set $\mathcal{C}_1=\{0\}\times n$ is not minimal, it should contain a less set X that is in A_1 and is equivalent to a regular set. The unique possible form of this set X is $\{(0,i)\}$ for $0\leqslant i\leqslant n_1-1$. Thus all singletons in $m\times n$ are minimal in A_1. Then the inclusion hyperspace $G\operatorname{pr}_2(\mathsf{A})\in G\,n$ also should contain all singletons in n, which is impossible because

$$G\operatorname{pr}_2(\mathsf{A}_1)=G\operatorname{pr}_2\circ\mu(m\times n)(V_{m,\cdot,\cdot,n})=\mu m\circ G^2\operatorname{pr}_2(V_{m,\cdot,\cdot,n})=\mu m(V_{\cdot,\cdot,\cdot,n})=V_{\cdot,n}=\{n\}\in G\,n.$$

Thus the set $\{0\}\times n$, as well as the sets $\{1\}\times n$, …, $\{m-1\}\times n$ that are equivalent to it, is minimal in A_1. Assume that A_1 contains a minimal set Y that is not equivalent to C_1. Then Y is equivalent to a regular set that in incomparable with $\mathcal{C}_1$ with respect to "$\subset$", i.e. is equivalent to $\mathcal{C}_2$. But the assumption $\mathcal{C}_2\in\mathsf{A}_1$ is also incompatible with the form of $G\operatorname{pr}_2(\mathsf{A}_1)$. Thus the collection of minimal elements of A_1 has the form $\{\{0\}\times n,\{1\}\times n,\dots,\{m-1\}\times n\}$, i.e. $\mathsf{A}_1=V_{m,n}$.

Let $\mathsf{A}_2=\mu(m\times n)(V_{\cdot,m,n,\cdot})$. We have

$$G\operatorname{pr}_1(\mathsf{A}_2)=G\operatorname{pr}_1\circ\mu(m\times n)(V_{\cdot,m,n,\cdot})=$$
$$=\mu m\circ G^2\operatorname{pr}_1(V_{\cdot,m,n,\cdot})=\mu m(V_{\cdot,m,\cdot,\cdot})=V_{\cdot,m}=\{m\}\in G\,m,$$

therefore all elements of A_2 under the first projection should map onto m. We also have

$$G\operatorname{pr}_2(\mathsf{A}_2)=G\operatorname{pr}_2\circ\mu(m\times n)(V_{\cdot,m,n,\cdot})=$$
$$=\mu\,n\circ G^2\operatorname{pr}_2(V_{\cdot,\cdot,n,\cdot})=\mu\,n(V_{\cdot,\cdot,n,\cdot})=V_{n,\cdot}=\exp n\in G\,n,$$

therefore A_2 contains all sets $\operatorname{pr}_2{}^{-1}(\{0\})=m\times\{0\}$, $\operatorname{pr}_2{}^{-1}(\{1\})=m\times\{1\}$, …, $\operatorname{pr}_2{}^{-1}(\{n-1\})=m\times\{n-1\}$, i.e. sets that are equivalent to $\mathcal{C}_2$. Since all subsets of $m\times n$ with surjective projection to m contain subsets that are equivalent to $\mathcal{C}_2$, the set of minimal elements of A_2 consists only of these sets. In other words, A_2 consists of all subsets of $m\times n$ that contain subsets of the form $\{(0,i_0),(1,i_1),\dots,(m-1,i_{m-1})\}$, where $0\leqslant i_k\leqslant n$ for $0\leqslant k\leqslant m-1$.

Let $\mathsf{A}_3=\mu(m\times n)(V_{m,\cdot,n,\cdot})$. We have

$$G\operatorname{pr}_1(\mathsf{A}_3)=G\operatorname{pr}_1\circ\mu(m\times n)(V_{m,\cdot,n,\cdot})=$$
$$=\mu\,m\circ G^2\operatorname{pr}_1(V_{m,\cdot,n,\cdot})=\mu\,m(V_{m,\cdot,\cdot,\cdot})=V_{m,\cdot}=\exp m\in G\,m,$$

therefore A_3 contains all sets $\operatorname{pr}_1{}^{-1}(\{0\})=\{0\}\times n$, $\operatorname{pr}_1{}^{-1}(\{1\})=\{1\}\times n$, …, $\operatorname{pr}_1{}^{-1}(\{m-1\})=\{m-1\}\times n$. Analogously $G\operatorname{pr}_2(\mathsf{A}_3)=V_{n,\cdot}=\exp n\in G\,n$ implies that A_3 contains the set $\operatorname{pr}_2{}^{-1}(\{0\})=m\times\{0\}$, as well as all subsets $m\times n$ that are equivalent to it, i.e. sets of the form $\{(0,i_0),(1,i_1),\dots,(m-1,i_{m-1})\}$, where $0\leqslant i_0,i_1,\dots,i_{m-1}\leqslant n-1$.

Only two mutually exclusive assumptions are possible:

(*) *sets* $\{0\} \times n$, $\{1\} \times n$, $\dots$, $\{m-1\} \times n$ *and* $m \times \{0\}$, $m \times \{1\}$, $\dots$, $m \times \{n-1\}$ *exhaust the collection of all minimal sets of the inclusion hyperspace* A_3*;*

(**) *there exists a minimal set* $X \in \mathsf{A}_3$ *that is distinct from* $\{0\} \times n$, $\{1\} \times n$, $\dots$, $\{m-1\} \times n$ *and* $m \times \{0\}$, $m \times \{1\}$, $\dots$, $m \times \{n-1\}$.

In the case (**) only $X = \{(i,j)\}$, where $0 \leqslant i \leqslant m-1$, $0 \leqslant j \leqslant n-1$, is possible. Therefore minimal elements of A_3 are *all* singletons $\{(i,j)\}$, $0 \leqslant i \leqslant m-1$, $0 \leqslant j \leqslant n-1$, i.e. for (**) we have $\mathsf{A}_3 = \exp(m \times n)$.

Let $\mathsf{A}_4 = \mu(m \times n)(V_{\cdot,m,\cdot,n})$. We have

$$G\,\mathrm{pr}_1(\mathsf{A}_4) = G\,\mathrm{pr}_1 \circ \mu(m \times n)(V_{\cdot,m,\cdot,n}) = \\ = \mu\, m \circ G^2\,\mathrm{pr}_1(V_{\cdot,m,\cdot,n}) = \mu\, m(V_{\cdot,m,\cdot,\cdot}) = V_{\cdot,m} = \{m\} \in G\,m.$$

Analogously

$$G\,\mathrm{pr}_2(\mathsf{A}_4) = G\,\mathrm{pr}_2 \circ \mu(m \times n)(V_{\cdot,m,\cdot,n}) = \\ = \mu\, n \circ G^2\,\mathrm{pr}_2(V_{\cdot,m,\cdot,n}) = \mu\, n(V_{\cdot,m,\cdot,\cdot}) = V_{\cdot,m} = \{n\} \in G\,n.$$

Thus both the first and the second projections of any element of A_4 should be surjective. Therefore we can obtain two cases:

(+) *minimal elements of the inclusion hyperspace* A_4 *are equivalent to* $A = m \times n$, *i.e.* $\mathsf{A}_4 = \{m \times n\}$*;*

(++) *minimal elements of the inclusion hyperspace* A_4 *are equivalent to* $B = \{0\} \times n \cup m \times \{0\}$, *i.e.* A_4 *consists of all subsets of* $m \times n$ *that contain subsets of the form* $\{l\} \times n \cup \{(0,i_0),(1,i_1),\dots,(m-1,i_{m-1})\}$, *where* $0 \leqslant l \leqslant m-1$, $0 \leqslant i_0, i_1, \dots, i_{m-1} \leqslant n-1$.

We omit an easy proof of the following

Lemma 4. *If any of the statements (*), (**), (+), (++) holds for some* $m, n \geqslant 2$, *then it holds for all* $m, n \geqslant 2$.

Similar arguments are applicable also to the inclusion hyperspace $\mathsf{B}_1 = \mu(m \times n \times k)(V_{\cdot,m,n,k})$. We have

$$G\,\mathrm{pr}_{23}(\mathsf{B}_1) = G\,\mathrm{pr}_{23} \circ \mu(m \times n \times k)(V_{\cdot,m,n,k}) = \mu(n \times k) \circ G^2\,\mathrm{pr}_{23}(V_{\cdot,m,n,k}) = \\ = \mu(n \times k)(V_{\cdot,\cdot,n,k}) = V_{n,k}, G\,\mathrm{pr}_{12}(\mathsf{B}_1) = G\,\mathrm{pr}_{12} \circ \mu(m \times n \times k)(V_{\cdot,m,n,k}) = \\ = \mu(m \times n) \circ G^2\,\mathrm{pr}_{12}(V_{\cdot,m,n,k}) = \mu(m \times n)(V_{\cdot,m,n,\cdot}) = \mathsf{A}_2,$$

$$G\,\mathrm{pr}_{13}(\mathsf{B}_1) = G\,\mathrm{pr}_{13} \circ \mu(m \times n \times k)(V_{\cdot,m,n,k}) = \\ = \mu(m \times k) \circ G^2\,\mathrm{pr}_{13}(V_{\cdot,m,n,k}) = \mu(m \times k)(V_{\cdot,m,\cdot,k}) = \mathsf{A}_4$$

(if we replace $n \leftarrow k$).

For each set of the form $X = \{(0,i_0),(1,i_1),\dots,(m-1,i_{m-1})\} \subset m \times n$, where $0 \leqslant i_0, i_1, \dots, i_{m-1} \leqslant n-1$, the preimage ${\mathrm{pr}_{12}}^{-1}(X) = \{0\} \times \{i_0\} \times k \cup \{1\} \times \{i_1\} \times k \cup \dots \cup \{m-1\} \times \{i_{m-1}\} \times k$ is in B_1. This preimage is equivalent to a regular set $Y \subset m \times n \times k$, such that all its sections $[Y]_i$ are equal to $\{0\} \times k$, i.e. have the type $\mathcal{C}_1$. If we assume that there is a regular element $Z \in \mathsf{B}_1$ that is incomparable with Y with respect to "$\subset$", then one of its sections $[Z]_i$ should not contain $[Y]_i$, i.e. should have the type $\mathcal{C}_2$, $\mathcal{D}$ or $\mathcal{E}$. In all this cases $\mathrm{pr}_{12}(Z) \subsetneq m \times n$.

If we assume $(+)$, then each elements B_1 under pr_{13} should map to $m \times k$, that contradicts to the existence of Z. Thus under $(+)$ the set of minimal sets of the inclusion hyperspace B_1 consists of all sets of the form $\{0\} \times \{i_0\} \times k \cup \{1\} \times \{i_1\} \times k \cup \dots \cup \{m-1\} \times \{i_{m-1}\} \times$, i.e. of all sets equivalent to $m \times \{0\} \times k$. Consider the case $(++)$.

Lemma 5. *Let $Y \in \mathsf{B}_1$ is a regular minimal element, $n \geqslant 2$, $k \geqslant 2$. Then among the sections $\{[Y]_0, [Y]_1, \dots, [Y]_{m-1}\}$ there are no sets of the types $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{E}$.*

Proof. Presence of $\mathcal{E} = \varnothing$ among the sections is impossible because of the form of $G\,\mathrm{pr}_{12}(\mathsf{B}_1)$.

Define for $n, k \geqslant 3$ a mapping $\phi\colon n \times k \to (n-1) \times k$ by the formula:

$$\phi(i,j) = \begin{cases} (i,j) \text{ if } i \leqslant n-2, j \leqslant k-2, \\ (i, j-1) \text{ if } i \leqslant n-2, j = k-1, \\ (i-1, j) \text{ if } i = n-1. \end{cases}$$

This mapping has the following property: if $A, B, A_0, B_0 \subset n \times k$, $A \sim A_0$, $B \sim B_0$, the sets A_0, B_0 are regular, and $\phi(A) \subset \phi(B)$, then $A_0 \subset B_0$. Put $\Phi\colon m \times n \times k \to m \times (n-1) \times k$, $\Phi = \mathbf{1}_m \times \phi$. Note that Φ is not a surjection, but $G^2\Phi(V_{\cdot,m,n,k}) = G^2 e(V_{\cdot,m,n-1,k-1})$, where $e\colon m \times (n-1) \times (k-1) \hookrightarrow m \times n \times k$ is the embedding. Thus

$$\begin{gathered} G\Phi(\mathsf{B}_1) = G\Phi \circ \mu(m \times n \times k)(V_{\cdot,m,n,k}) = \\ = \mu(m \times (n-1) \times k) \circ G^2\Phi(V_{\cdot,m,n,k}) = \mu(m \times (n-1) \times k) \circ G^2 e(V_{\cdot,m,n-1,k-1}). \end{gathered}$$

Therefore all minimal elements of $G\Phi(\mathsf{B}_1)$ are contained in $e(m \times (n-1) \times (k-1)) = m \times (n-1) \times (k-1)$.

Construct a set $Y' \subset m \times n \times k$ as follows: if $0 \leqslant i \leqslant m-1$, for $[Y]_i \neq \{(0,0)\}$ let $[Y']_i = [Y]_i$, and for $[Y]_i = \{(0,0)\}$ let $[Y']_i = \{(n-1, k-1)\}$. Then $Y' \sim Y$, therefore Y' also is minimal in B_1. If one of the sections $[Y]_i$ is equal to $\mathcal{A} = n \times k$ or $\mathcal{D} = \{(0,0)\}$, then the corresponding section $[Y']_i$ contains $\{(n-1, k-1)\}$, therefore $\Phi'(Y') \ni (i, n-2, k-1)$, and $\Phi(Y') \not\subset m \times (n-1) \times (k-1)$. Thus the element $\Phi(Y')$ is not minimal in $G\Phi(\mathsf{B}_1)$. Then there is a minimal element $Z' \in \mathsf{B}_1$, such that $\Phi(Z') \subset \Phi(Y') \cap m \times (n-1) \times (k-1)$. By the above Z' is equivalent to a regular set $Z \subset m \times n \times k$. For any $i \in \{0, 1, \dots, m-1\}$ we have $\Phi(Z' \cap \{i\} \times n \times k) \subset \Phi(Y' \cap \{i\} \times n \times k)$, therefore $\mathrm{pr}_{23}(\Phi(Z' \cap \{i\} \times n \times k)) = \phi([Z']_i) \subset \mathrm{pr}_{23}(\Phi(Y' \cap \{i\} \times n \times k)) = \phi([Y']_i)$, $[Y']_i \sim [Y]$, $[Z']_i \sim [Z]$, and due to the mentioned above property of ϕ we obtain $[Z]_i \subset [Y]_i$ for $0 \leqslant i \leqslant m-1$. Thus $Z \subsetneq Y$, $Z \in \mathsf{B}_1$, that contradicts to minimality of Y. Therefore for $n, k \geqslant 3$ a minimal set $Y \in \mathsf{B}_1$ can not have sections of the types $\mathcal{A}$ and $\mathcal{D}$. In order to extend the arguments onto the cases when $n = 2$ or $k = 2$, it is sufficient to use a regular surjection $m \times (n+1) \times (k+1) \to m \times n \times k$. □

Thus under $(++)$ the sets of sections $\{[Y]_0, [Y]_1, \dots, [Y]_{m-1}\}$ of any regular minimal element $Y \in \mathsf{B}_1$ are: $\{\mathcal{C}_1\}$ and one or several of the sets $\{\mathcal{C}_1, \mathcal{C}_2\}$, $\{\mathcal{B}, \mathcal{C}_2\}$, $\{\mathcal{B}, \mathcal{C}_1, \mathcal{C}_2\}$.

Let $\mathsf{B}_2 = \mu(m \times n \times k)(V_{m,\cdot,n,k})$. It is easy to prove analogously to the proof of Lemma 5 that there are no sets of the types A and D among the sections Y_i, $0 \leqslant i \leqslant m-1$, of any regular minimal element $Y \in \mathsf{B}_2$. Take into consideration that

$$\begin{gathered} G\,\mathrm{pr}_{23}(\mathsf{B}_2) = G\,\mathrm{pr}_{23} \circ \mu(m \times n \times k)(V_{m,\cdot,n,k}) = \\ = \mu(n \times k) \circ G^2\,\mathrm{pr}_{23}(V_{m,\cdot,n,k}) = \mu(n \times k)(V_{\cdot,\cdot,n,k}) = V_{n,k}, \end{gathered}$$

$$G\,\mathrm{pr}_{12}(\mathsf{B}_2) = G\,\mathrm{pr}_{12} \circ \mu(m\times n\times k)(V_{m,\cdot,n,k}) =$$
$$= \mu(m\times n)\circ G^2\,\mathrm{pr}_{12}(V_{m,\cdot,n,k}) = \mu(m\times n)(V_{m,\cdot,n,\cdot}) = \mathsf{A}_3,$$

$$G\,\mathrm{pr}_{13}(\mathsf{B}_2) = G\,\mathrm{pr}_{13} \circ \mu(m\times n\times k)(V_{m,\cdot,n,k}) =$$
$$= \mu(m\times k)\circ G^2\,\mathrm{pr}_{13}(V_{m,\cdot,n,k}) = \mu(m\times k)(V_{m,\cdot,\cdot,k}) = \mathsf{A}_1$$

(if we replace $n \leftarrow k$), which implies $G\,\mathrm{pr}_{13}(\mathsf{B}_2) = V_{m,k}$. If we assume (**), then all minimal elements of B_2 are of the form $\{i\}\times\{j\}\times k$, where $0\leqslant i\leqslant m-1$, $0\leqslant j\leqslant n-1$, i.e. these are sets equivalent to a regular set with sections of the types $\mathcal{C}_1$, $\mathcal{E}$. If (*) is true, we have regular minimal elements with the sets of sections $\{\mathcal{C}_1\}$, $\{\mathcal{B},\mathcal{E}\}$ and, probably, $\{\mathcal{C}_1,\mathcal{C}_2\}$ or $\{\mathcal{C}_1,\mathcal{C}_2,\mathcal{E}\}$.

Let $\mathsf{B}_3 = \mu(m\times n\times k)(V_{m,n,\cdot,k})$. We have

$$G\,\mathrm{pr}_{23}(\mathsf{B}_3) = G\,\mathrm{pr}_{23} \circ \mu(m\times n\times k)(V_{m,n,\cdot,k}) =$$
$$= \mu(n\times k)\circ G^2\,\mathrm{pr}_{23}(V_{m,n,\cdot,k}) = \mu(n\times k)(V_{\cdot,n,\cdot,k}) = \mathsf{A}_4$$

(if we replace $m\leftarrow n$, $n\leftarrow k$),

$$G\,\mathrm{pr}_{12}(\mathsf{B}_3) = G\,\mathrm{pr}_{12} \circ \mu(m\times n\times k)(V_{m,n,\cdot,k}) =$$
$$= \mu(m\times n)\circ G^2\,\mathrm{pr}_{12}(V_{m,n,\cdot,k}) = \mu(m\times n)(V_{m,n,\cdot,\cdot}) = V_{m,n},$$

$$G\,\mathrm{pr}_{13}(\mathsf{B}_3) = G\,\mathrm{pr}_{13} \circ \mu(m\times n\times k)(V_{m,n,\cdot,k}) =$$
$$= \mu(m\times k)\circ G^2\,\mathrm{pr}_{13}(V_{m,n,\cdot,k}) = \mu(m\times k)(V_{m,\cdot,\cdot,k}) = \mathsf{A}_1$$

(if we replace $n\leftarrow k$), thus $G\,\mathrm{pr}_{13}(\mathsf{B}_3) = V_{m,k}$.

This implies that B_3 contains all preimages of minimal elements of $V_{m,n}$ under pr_{12}, i.e. sets of the form ${\mathrm{pr}_{12}}^{-1}(\{i\}\times n) = \{i\}\times m\times k$, $0\leqslant i\leqslant m-1$. If we assume $(+)$, then the form of $G\,\mathrm{pr}_{23}(\mathsf{B}_3)$ implies that there are no other minimal elements. If $(++)$ holds, then in addition to $\{\mathcal{A},\mathcal{E}\}$ the following sets of sections of regular minimal elements are possible: $\{\mathcal{B}\}$, $\{\mathcal{B},\mathcal{D}\}$, $\{\mathcal{B},\mathcal{E}\}$, $\{\mathcal{C}_1,\mathcal{C}_2,\mathcal{E}\}$, $\{\mathcal{C}_1,\mathcal{C}_2\}$, $\{\mathcal{B},\mathcal{C}_1\}$, $\{\mathcal{B},\mathcal{C}_2\}$, $\{\mathcal{B},\mathcal{C}_1,\mathcal{C}_2\}$, $\{\mathcal{C}_1,\mathcal{C}_2,\mathcal{D}\}$, and at least one of them should be present.

Consider the set of inclusion hyperspaces

$$\mathsf{C}_1 = \mu(m\times n\times k)\circ G\mu(m\times n\times k)(V_{\cdot,m,n,\cdot,\cdot,k}) = \mu(m\times n\times k)\circ \mu G(m\times n\times k)(V_{\cdot,m,n,\cdot,\cdot,k}).$$

It is obvious that it also satisfies (!). On the one hand, $G\mu(m\times n\times k)(V_{\cdot,m,n,\cdot,\cdot,k}) = (V_{\cdot,m,n,k})$, thus $\mathsf{C}_1 = \mu(m\times n\times k)(V_{\cdot,m,n,k}) = \mathsf{B}_1$.

To calculate C_1 in a different way, we fix a bijection $t\colon n^m \to \underbrace{n\times n\times\dots\times n}_{m}$ and note that $V_{\cdot,m,n,\cdot,\cdot,k} = G^2\psi(V_{\cdot,m,n,\cdot})$, where $\psi\colon m\times n\to G(m\times n\times k)$ is the mapping that is defined by the formula $\psi(i,j) = r(m\times n\times k)(\{(i,j)\}\times k)$. Thus

$$\mu G(m\times n\times k)(V_{\cdot,m,n,\cdot,\cdot,k}) =$$
$$= \mu G(m\times n\times k)\circ G^2\psi(V_{\cdot,m,n,\cdot}) = G\psi\circ\mu(m\times n)(V_{\cdot,m,n,\cdot}) = G\psi(\mathsf{A}_2).$$

Besides, $G\psi(\mathsf{A}_2) = G^2\Psi(V_{n^m,m,\cdot,k})$, where $\Psi\colon n^m\times m\times k\to m\times n\times k$ is the surjection defined by the formula $\Psi(l,i,j) = (i,t(l)(i),j)$. Therefore

$$\mathsf{C}_1 = \mu(m\times n\times k)\circ G^2\Psi(V_{n^m,m,\cdot,k}) = G\Psi\circ\mu(n^m\times m\times k)(V_{n^m,m,\cdot,k}) = G\Psi(\mathsf{B}_3)$$

(if we replace $m\leftarrow n^m$, $n\leftarrow m$).

Investigate how Ψ acts on possible minimal sets of the inclusion hyperspace B_3.

Lemma 6. *A subset $Z \subset m \times n \times k$ contains the image $\Psi(X)$ of some set $X \subset n^m \times m \times k$, that is equivalent to a regular set X' with the set of sections $\mathcal{M}$, if and only if Z contains a set $Y \subset m \times n \times k$ that is equivalent to a regular set Y' with the set of sections $\mathcal{N}$ or (for items (1), (2)) $\mathcal{N}'$, for any of the following combinations: 1) $\mathcal{M} = \{\mathcal{C}_1, \mathcal{C}_2\}$, $\mathcal{N} = \{\mathcal{A}, \mathcal{D}\}$, $\mathcal{N}' = \{\mathcal{B}, \mathcal{C}_2\}$; 2) $\mathcal{M} = \{\mathcal{B}, \mathcal{C}_1, \mathcal{C}_2\}$, $\mathcal{N} = \{\mathcal{A}, \mathcal{D}\}$, $\mathcal{N}' = \{\mathcal{B}, \mathcal{C}_2\}$; 3) $\mathcal{M} = \{\mathcal{A}, \mathcal{E}\}$, $\mathcal{N} = \{\mathcal{C}_1\}$; 4) $\mathcal{M} = \{\mathcal{B}, \mathcal{E}\}$, $\mathcal{N} = \{\mathcal{C}_1, \mathcal{D}\}$; 5) $\mathcal{M} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{E}\}$, $\mathcal{N} = \{\mathcal{C}_1, \mathcal{D}\}$; 6) $\mathcal{M} = \{\mathcal{B}, \mathcal{C}_2\}$, $\mathcal{N} = \{\mathcal{B}, \mathcal{C}_2\}$; 7) $\mathcal{M} = \{\mathcal{B}, \mathcal{C}_1\}$, $\mathcal{N} = \{\mathcal{A}, \mathcal{D}\}$; 8) $\mathcal{M} = \{\mathcal{B}\}$, $\mathcal{N} = \{\mathcal{A}, \mathcal{C}_2\}$; 9) $\mathcal{M} = \{\mathcal{B}, \mathcal{D}\}$, $\mathcal{N} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}\}$; 10) $\mathcal{M} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}\}$, $\mathcal{N} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}\}$.*

Proof. Let us prove (1). Assume that $Z \supset \Psi(X)$, where X satisfies the condition of the lemma. Then the set n^m is the union of two nonempty disjoint subsets L_1 and L_2, and for any $l \in L_1$ we have $[X]_l = \{i_l\} \times k$ for some $i_l \in \{0, 1, \dots, n-1\}$. For any $l \in L_2$ the section $[X]_l$ has the form $\{(0, k_0), (1, k_1), \dots, (n-1, k_{n-1})\}$, where $0 \leqslant k_0, k_1, \dots, k_{n-1} \leqslant k-1$. Denote $P = m \times n \setminus \{(i_l, t(l)(i_l)) \mid l \in L_1\}$, $Q = \mathrm{pr}_{12}(\Psi(X)) = \{(i_l, t(l)(i_l)) \mid l \in L_1\} \cup \{(i, t(l)(i)) \mid l \in L_2\}$. If $\mathrm{pr}_1(P) = m$, we choose $(0, p_0), (1, p_1), \dots, (m-1, p_{m-1}) \in P$. Assume that $(i, j) \in m \times n$, $(i, j) \notin Q$, and let $l \in n^m$ be determined by the equality $t(l) = (p_0, p_1, \dots, p_{i-1}, j, p_{i+1}, \dots, p_{m-1}$. Then both $l \in L_1$ and $l \in L_2$ are impossible, that is a contradiction. Therefore for $\mathrm{pr}_1(P) = m$ we have $\mathrm{pr}_{12}(\Psi(X)) = m \times n$, and $L_1 \neq \varnothing$ implies that $\Psi(X) \supset \{(i, j)\} \times k$ for some $0 \leqslant i \leqslant m-1$, $0 \leqslant j \leqslant n-1$. Thus $\Psi(X)$ contains a subset Y that is equivalent to a regular set with sections $\{\mathcal{B}, \mathcal{C}_2\}$. If $\mathrm{pr}_1(P) \neq m$, we choose $i \in m \setminus \mathrm{pr}_1(P)$ and obtain $\Psi(X) \supset \{i\} \times n \times k$. Moreover, $L_2 \neq \varnothing$ implies $\Psi(X) \supset \{(0, j_0, k_0), (1, j_1, k_1), \dots, (m-1, j_{m-1}, k_{m-1})\}$, for some $0 \leqslant j_0, j_1, \dots, j_{m-1} \leqslant n-1$, $0 \leqslant k_0, k_1, \dots, k_{m-1} \leqslant k-1$. Therefore $\Psi(X)$ contains a subset Y that is equivalent to a regular set with sections $\{\mathcal{A}, \mathcal{D}\}$.

Now let Z contain a subset Y equivalent to a regular set with sections $\{\mathcal{B}, \mathcal{C}_2\}$. Then Y has the form $\{(i_0, j_0)\} \times k \cup \{(i, j, k_{i,j}) \mid 0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant n\}$. We put $L_1 = \{l \in n^m \mid t(l)(i_0) = j_0\}$, $L_2 = n^m \setminus L_1$, $X \subset n^m \times m \times k$, $X = L_1 \times \{i_0\} \times k \cup \{(l, i, k_{i,t(l)(i)})\}$. Then X is equivalent to a regular set with sections $\{\mathcal{C}_1, \mathcal{C}_2\}$, and $\Psi(X) = Y \subset Z$.

If Z contains a subset Y equivalent to a regular set with sections $\{\mathcal{A}, \mathcal{D}\}$, then $Y = \{i_0\} \times n \times k \cup \{(0, j_0, k_0), (1, j_1, k_1), \dots, (m-1, j_{m-1}, k_{m-1})\}$, for some $0 \leqslant i_0 \leqslant m-1$, $0 \leqslant j_0, j_1, \dots, j_{m-1} \leqslant n-1$, $0 \leqslant k_0, k_1, \dots, k_{m-1} \leqslant k-1$. We put $l_0 = t^{-1}((j_0, j_1, \dots, j_{m-1}))$, $L_1 = n^m \setminus \{l_0\}$, $L_2 = \{l_0\}$, $X = L_1 \times \{i_0\} \times k \cup \{(l, 0, k_0), (l, 1, k_1), \dots, (l, m-1, k_{m-1})\}$. Then X is equivalent to a regular set with sections $\{\mathcal{C}_1, \mathcal{C}_2\}$, and $\Psi(X) = Y \subset Z$.

Other items can be proved by analogous not very difficult, but cumbersome arguments. □

We compare the obtained sets of sections with possible sections of regular minimal elements of B_1. Since there cannot be sets of the types $\mathcal{D}$ and $\mathcal{E}$ among sections of regular elements of B_1, only three possible types of regular minimal elements of B_3 remain, namely with sections $\{\mathcal{A}, \mathcal{E}\}$, $\{\mathcal{B}, \mathcal{C}_2\}$ and $\{\mathcal{B}\}$. They correspond respectively to regular minimal elements with sections of the types $\{\mathcal{C}_1\}$, $\{\mathcal{B}, \mathcal{C}_2\}$ and $\{\mathcal{A}, \mathcal{C}_2\}$ of the inclusion hyperspace B_1. A minimal regular element with sections of the type $\{\mathcal{B}\}$ is also impossible because in this case a regular element with sections of the types $\{\mathcal{A}, \mathcal{C}_2\}$ is minimal in B_1 and has a section of the type $\mathcal{A}$, which is also impossible by Lemma 5. Therefore we can have only regular minimal elements with sections $\{\mathcal{A}, \mathcal{E}\}$ and $\{\mathcal{B}, \mathcal{C}_2\}$ in B_3 and regular minimal elements with sections $\{\mathcal{C}_1\}$ and $\{\mathcal{B}, \mathcal{C}_2\}$ in B_1.

Summing up, we have two possibilities (everywhere all parameters m, n, k are not less than 2) :

1) $(+)$ holds, the inclusion hyperspace B_1 has regular minimal elements with sections $\{\mathcal{C}_1\}$, and the inclusion hyperspace B_3 has regular minimal elements with sections $\{\mathcal{A},\mathcal{E}\}$;

2) $(++)$ holds, the inclusion hyperspace B_1 has regular minimal elements with sections $\{\mathcal{C}_1\}$ and $\{\mathcal{B},\mathcal{C}_2\}$, and the inclusion hyperspace B_3 has regular minimal elements with sections $\{\mathcal{A},\mathcal{E}\}$ and $\{\mathcal{B},\mathcal{C}_2\}$.

Consider the family of inclusion hyperspaces

$$\mathsf{C}_2=\mu(m\times n\times k)\circ G\mu(m\times n\times k)(V_{m,\cdot,n,\cdot,\cdot,k})=\mu(m\times n\times k)\circ \mu G(m\times n\times k)(V_{m,\cdot,\cdot,n,\cdot,k}).$$

It also satisfies condition (!). On the one hand, $\mathsf{C}_2=\mu(m\times n\times k)\circ G\mu(m\times n\times k)\circ G\widetilde{e}(V_{m,\cdot})$, where the mapping $\widetilde{e}\colon m\to G^2(m\times n\times k)$ sends each $i\in m$ to $G^2e_i(V_{n,\cdot,\cdot,k})$, where $e_i\colon n\times k\to m\times n\times k$, $e_i(j,l)=(i,j,l)$. We obtain

$$\begin{aligned}\mu(m\times n\times k)\circ\widetilde{e}(i)&=\mu(m\times n\times k)\circ G^2e_i(V_{n,\cdot,\cdot,k})=\\&=Ge_i\circ\mu(n\times k)(V_{n,\cdot,\cdot,k})=Ge_i(\mathsf{A}_1)=Ge_i(V_{n,k}).\end{aligned}$$

Therefore $\mathsf{C}_2=\mu(m\times n\times k)G\theta(V_{m,\cdot})$, where $\theta\colon m\to G(m\times n\times k)$, $\theta(i)=Ge_i(V_{n,k})$, thus $\mathsf{C}_2=\mu(m\times n\times k)(V_{m,\cdot,n,k})=\mathsf{B}_2$.

On the other hand, $\mathsf{C}_2=\mu(m\times n\times k)\circ\mu G(m\times n\times k)\circ G^2\psi(V_{m,\cdot,n,\cdot})$, where the mapping $\psi\colon m\times n\to G(m\times n\times k)$ is again defined as $\psi(i,j)=r(m\times n\times k)(\{(i,j)\}\times k)$. Here

$$\begin{aligned}\mu G(m\times n\times k)(V_{m,\cdot,n,\cdot,\cdot,k})&=\mu G(m\times n\times k)\circ G^2\psi(V_{m,\cdot,n,\cdot})=\\&=G\psi\circ\mu(m\times n)(V_{m,\cdot,n,\cdot})=G\psi(\mathsf{A}_3).\end{aligned}$$

Consider the both possibilities (*) and (**). If (**) holds, then $G\psi(\mathsf{A}_3)=V_{mn,\cdot,\cdot,k}$ (we identify $m\times n$ and mn), which implies $\mathsf{C}_2=\mu(m\times n\times k)(V_{mn,\cdot,\cdot,k})=V_{mn,k}$. It is the inclusion hyperspace in $m\times n\times k$ with minimal sets of the form $\{i\}\times\{j\}\times k$, where $0\leqslant i\leqslant m-1$, $0\leqslant j\leqslant n-1$. If (*) is true, we put $m'=m+n^m$, $n'=\max\{m,n\}$ and fix a bijection $p\colon n^m\to\underbrace{n\times n\times\cdots\times n}_{m}$. Then $G\psi(\mathsf{A}_3)=G^2\Theta(V_{m',n',\cdot,k})$, where the mapping $\Theta\colon m'\times n'\times k\to m\times n\times k$ is defined by the formula:

$$\Theta(i,j,l)=\begin{cases}(\min\{j,m\},p(i)(\min\{j,m\}),k)\text{ if }i<n^m;\\(i-n^m,\min\{j,n\},k)\text{ if }i\geqslant n^m.\end{cases}$$

Thus

$$\mathsf{C}_2=\mu(m\times n\times k)\circ G^2\Theta(V_{m',n',\cdot,k})=G\Theta\circ\mu(m'\times n'\times k)(V_{m',n',\cdot,k})=G\Theta(\mathsf{B}_3)$$

(if we replace $m\leftarrow m'$, $n\leftarrow n'$).

Assume (*) and consider the images under $G\Theta$ of possible minimal elements of B_3.

Lemma 7. *A subset $Z\subset m\times n\times k$ contains the image $\Theta(X)$ of some set $X\subset n^m\times m\times k$, that is equivalent to a regular set X' with the set of sections $\mathcal{M}$, if and only if Z contains a set $Y\subset m\times n\times k$ that is equivalent to a regular set Y' with the set of sections $\mathcal{N}$ or (for item (1)) $\mathcal{N}'$, for each of the following combinations:*

1. *$\mathcal{M}=\{\mathcal{A},\mathcal{E}\}$, $\mathcal{N}=\{\mathcal{A},\mathcal{E}\}$, $\mathcal{N}'=\{\mathcal{C}_1\}$;*
2. *$\mathcal{M}=\{\mathcal{B},\mathcal{C}_2\}$, $\mathcal{N}=\{\mathcal{B},\mathcal{C}_2\}$;*

We omit a straightforward proof. Since no subset $Y \subset m\times n\times k$ equivalent to a regular set Y' with sections $\{\mathcal{A},\mathcal{E}\}$, $\{\mathcal{C}_1\}$ or $\{\mathcal{B},\mathcal{C}_2\}$ can be a subset of a minimal set Z of the inclusion hyperspace B_2 with sections $\{\mathcal{B},\mathcal{E}\}$, we obtain that under (*) the equality $G\Theta(\mathsf{B}_3)=\mathsf{B}_2$ is impossible, which is a contradiction. Thus (**) holds, and the unique regular minimal element of B_2 has sections of the types $\mathcal{C}_1$, $\mathcal{E}$, i.e. minimal elements of B_2 have the forms $\{i\}\times\{j\}\times k$, where $0\leqslant i\leqslant m-1$, $0\leqslant j\leqslant n-1$.

Consider the family of inclusion hyperspaces

$$\mathsf{C}_3=\mu(m\times n\times k)\circ G\mu(m\times n\times k)(V_{m,\cdot,\cdot,n,\cdot,k})=\mu(m\times n\times k)\circ\mu G(m\times n\times k)(V_{m,\cdot,\cdot,n,\cdot,k}).$$

It satisfies (!) as well. On the one hand, $\mathsf{C}_3=\mu(m\times n\times k)\circ\mu G(m\times n\times k)\circ G^2\psi(V_{m,\cdot,\cdot,n})$, where we again define the mapping $\psi\colon m\times n\to G(m\times n\times k)$ by the formula $\psi(i,j)=r(m\times n\times k)(\{(i,j)\}\times k)$. Thus $\mu G(m\times n\times k)(V_{m,\cdot,\cdot,n,\cdot,k})=\mu G(m\times n\times k)\circ G^2\psi(V_{m,\cdot,\cdot,n})=G\psi\circ\mu(m\times n)(V_{m,\cdot,\cdot,n})=G\psi(\mathsf{A}_1)=G\psi(V_{m,n})=V_{m,n,\cdot,k}$. Therefore $\mathsf{C}_3=\mu(m\times n\times k)(V_{m,n,\cdot,k})=\mathsf{B}_3$.

On the other hand, $\mathsf{C}_3=\mu(m\times n\times k)\circ G\mu(m\times n\times k)\circ G\hat{e}(V_{m,\cdot})$, where the mapping $\hat{e}\colon m\to G^2(m\times n\times k)$ sends each $i\in m$ to $G^2e_i(V_{\cdot,n,\cdot,k})$, where $e_i\colon n\times k\to m\times n\times k$, $e_i(j,l)=(i,j,l)$. We have $\mu(m\times n\times k)\circ\hat{e}(i)=\mu(m\times n\times k)\circ G^2e_i(V_{\cdot,n,\cdot,k})=Ge_i\circ\mu(n\times k)(V_{\cdot,n,\cdot,k})=Ge_i(\mathsf{A}_4)$. Thus $\mathsf{C}_3=\mu(m\times n\times k)G\xi(V_{m,\cdot})$, where $\xi\colon m\to G(m\times n\times k)$, $\xi(l)=Ge_i(\mathsf{A}_4)$.

Under (+) we have $G\xi(V_{m,\cdot})=V_{m,\cdot,n,k}$.

If (++) is true, we put $n'=n\cdot k^n$, $k'=n+k$ and fix a bijection $q\colon n\cdot k^n\to\underbrace{k\times k\times\cdots\times k}_{n}\times n$. Then $G\xi(V_{m,\cdot})=G^2\Xi(V_{m,\cdot,n',k'})$, where $\Xi\colon m\times n'\times k'\to m\times n\times k$ is defined by the formula: $\Xi(i,j,l)=\begin{cases}(i,l,q(j)(l)) \text{ if } l<n;\\ (i,q(j)(n),l-n) \text{ if } l\geqslant n.\end{cases}$

Therefore $\mathsf{C}_3=\mu(m\times n\times k)\circ G^2\Xi(V_{m,\cdot,n',k'})=G\Xi\circ\mu(m\times n'\times k')(V_{m,\cdot,n',k'})=G\Xi(\mathsf{B}_2)$ (if we replace $n\leftarrow n'$, $k\leftarrow k'$). Thus we obtain $G\Xi(\mathsf{B}_2)=\mathsf{B}_3$. Choose a minimal element $Y\in\mathsf{B}_2$ equivalent to a regular set with sections $\mathcal{B}$, $\mathcal{E}$ and equal to $\{0\}\times n'\times\{0\}\cup\{0\}\times\{q^{-1}((0,0,\ldots,0))\}\times k'$. It is easy to verify that $\Xi(Y)=\{0\}\times n\times\{0\}\cup\{0\}\times\{0\}\times k$ is a regular set with sections of the types $\mathcal{B}$, $\mathcal{E}$, that cannot be in B_3. Thus the assumption (++) is false, and the only possible variant is (+), (**).

Now we know that the inclusion hyperspace B_1 has regular minimal elements with sections $\{\mathcal{C}_1\}$, the inclusion hyperspace B_2 has regular minimal elements with sections $\{\mathcal{C}_1,\mathcal{E}\}$, and the inclusion hyperspace B_3 has regular minimal elements with sections $\{\mathcal{A},\mathcal{E}\}$.

Consider the family of inclusion hyperspaces $\mathsf{D}=\mu(s\times m\times n\times k)(V_{s,m,n,k})$ that is parameterized by natural s,m,n,k. It is obvious that D satisfies (!) and $G\,\mathrm{pr}_{234}(\mathsf{D})=\mathsf{B}_1$, $G\,\mathrm{pr}_{134}(\mathsf{D})=\mathsf{B}_2$, $G\,\mathrm{pr}_{124}(\mathsf{D})=\mathsf{B}_3$. This implies that the only regular minimal element of D has the form $\{0\}\times m\times\{0\}\times k$.

Lemma 8. *If $\alpha,\beta\colon G^2\to G$ are natural transformations, and for all natural s,m,n,k the equality $\alpha(s\times m\times n\times k)(V_{s,m,n,k})=\beta(s\times m\times n\times k)(V_{s,m,n,k})$ holds, then $\alpha=\beta$.*

Proof. First we will prove that for any $\mathsf{F}\in G^2X$, where X is a finite compactum, there are natural s,m,n,k and a mapping $f\colon s\times m\times n\times k\to X$ such that $G^2f(V_{s,m,n,k})=\mathsf{F}$. Put $k=|X|$, $n=|\exp X|$, $m=|GX|$, $s=|\exp GX|$, and for any $F\in\exp X$ fix a surjection $\alpha_F\colon k\to F$. For each $\mathcal{F}\in GX$ fix a surjection $\beta_{\mathcal{F}}\colon n\to\mathcal{F}$, and put $h_{\mathcal{F}}\colon n\times k\to X$,

$h_{\mathcal{F}}(j,l) = \alpha_{\beta_{\mathcal{F}}(j)}(k)$. It is easy to see that $Gh_{\mathcal{F}}(V_{n,k}) = \mathcal{F}$. Now for each $H \in \exp GX$ fix a surjection $\gamma_H \colon m \to H$, and put $g_H \colon m \times n \times k$, $g_H(i,j,k) = h_{\gamma_H(i)}(j,l)$. Fix a surjection $\delta_{\mathsf{F}} \colon s \to \mathsf{F}$ and put $f \colon s \times m \times n \times k$, $f(p,i,j,l) = g_{\delta_{\mathcal{F}}(p)}(i,j,k)$. It is straightforward to verify that the mapping f is required. Thus we obtain $\alpha X(\mathsf{F}) = \alpha X \circ G^2 f(V_{s,m,n,k}) = Gf \circ \alpha(s \times m \times n \times k)(V_{s,m,n,k}) = Gf \circ \beta(s \times m \times n \times k)(V_{s,m,n,k}) = \beta X \circ G^2 f(V_{s,m,n,k}) = \beta X(\mathsf{F})$. This implies that the components of the natural transformations α and β coincide on finite compacta. Since any zero-dimensional compactum can be represented as the inverse limit of finite compacta [6], and the inclusion hyperspace functor G is continuous, i.e. it preserves limits of inverse spectra, we have $\alpha X = \beta X$ for each zero-dimensional compactum X. Each compactum is a continuous image of a zero-dimensional compactum, and the functor G is epimorphic, i.e. it preserves the class of continuous surjections, therefore $\alpha X = \beta X$ for *any* compactum X. □

This implies that the main result is true:

Theorem. *There exists a unique monad in the category of compacta with the inclusion hyperspace functor as the functorial part.*

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