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HÖLDER CONTINUITY OF WEAK SOLUTIONS TO NONDIAGONAL DEGENERATE PARABOLIC SYSTEM OF THREE EQUATIONS

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Hölder continuity of weak solutions is studied for a nondiagonal parabolic system of degenerate quasilinear differential equations with matrix of coefficients satisfying special structure conditions. A technique based on estimating the linear combinations of unknowns is employed to this end.

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Исследуется гельдерова непрерывность слабых решений для недиагональной вырожденной параболической системы квазилинейных дифференциальных уравнений с матрицей коэффициентов, удовлетворяющей специальным структурным условиям. Для этого используется техника, основанная на оценивании линейных комбинаций неизвестных.

1. Introduction. In the present paper we study the Hölder continuity of weak solutions to the quasilinear nondiagonal parabolic system of three equations in divergence form under special assumptions upon its structure.

It is well-known that the De Giorgi-Nash-Moser estimates are no longer valid in general for an elliptic system, the latter can be regarded as a special case of the parabolic version. An example of an unbounded solution to the linear elliptic system with bounded coefficients was built up by De Giorgi in [10]

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\sum_{\alpha=1}^{n}A_{ij}^{\alpha\beta}(x)\frac{\partial}{\partial x_j}u_\beta\right)=0,\quad \beta\in\{1,\dots,n\},\quad n\ge 3;$$

under the following condition on the matrix of coefficients

$$A_{ij}^{\alpha\beta}(x)=\delta_{\alpha\beta}\delta_{ij}+\left[(n-2)\delta_{\alpha i}+n\frac{x_i x_\alpha}{|x|^2}\right]\cdot\left[(n-2)\delta_{\beta j}+n\frac{x_j x_\beta}{|x|^2}\right]$$

$$\delta_{\alpha\beta}=\begin{cases}1, & \alpha=\beta,\\ 0, & \alpha\ne\beta;\end{cases}$$

$$\text{for all } \xi\in\mathbb{R}^{2n},\ x\in\mathbb{R}^n\quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j\ge|\xi|^2;$$

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the vector valued function $u(x) = x/|x|^\gamma$ with $\gamma = (n/2)\left(1 - [(2n-2)^2+1]^{-1/2}\right)$ is an unbounded solution in $\Omega = \{x \big| |x| \le 1\} \subset \mathbb{R}^n$ and $A_{ij}^{\alpha\beta} \in L^\infty(\Omega)$.

There is yet another example due to J. Nečas and J. Souček of a nonlinear elliptic system with the coefficients sufficiently smooth, but the weak solution not belonging to $W^{2,2}$

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\sum_{\alpha=1}^{n} A_{ij}^{\alpha\beta}(x,u)\frac{\partial}{\partial x_j}u_\beta\right) = 0, \quad \beta \in \{1,\ldots,n\}, \quad n \ge 5;$$

where the matrix of coefficients is

$$A_{ij}^{\alpha\beta}(x,u) = \delta_{\alpha\beta}\delta_{ij} + c^2\left[\delta_{\alpha i} + b\frac{u^i u^\alpha |x|^{2\gamma-2}}{1+|u|^2|x|^{2\gamma-2}}\right]\cdot\left[\delta_{\beta j} + b\frac{u^j u^\beta |x|^{2\gamma-2}}{1+|u|^2|x|^{2\gamma-2}}\right];$$

and, obviously, for all $\xi \in \mathbb{R}^{2n}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i\xi_\beta^j \ge |\xi|^2;$$

the vector valued function $u(x) = x/|x|^\gamma \in W^{1,2}(\Omega)$ with $\gamma \in [(n-2)/2, n/2)$, $b = 2n/(n-2)$, $c^2 = \gamma(n-\gamma)(n-2)^2/[(n-2\gamma)^2(n-1)^2]$ is a solution in $\Omega = \{x \big| |x| \le 1\} \subset \mathbb{R}^n$ which does not even belong to $W^{2,2}(\Omega)$ though $A_{ij}^{\alpha\beta}(x,u)$ are sufficiently smooth.

These two and many other examples illustrate that the regularity problem for elliptic systems proves to be far more complicated then that for second order elliptic equations.

Concerning systems of differential equations until now a priori estimates of De Giorgi type has been extended only to a special class of parabolic systems of equations, the so-called weakly coupled systems. The system is said to be weakly coupled if it is coupled only through the terms which are not differentiated, each equation containing derivatives of just one component.

There exists yet another approach to establishing a priori estimates for a parabolic system of second order differential equations [12]. It concerns not each component separately, but the sum of squares of the components of solution. This applies for diagonal systems which on freezing the leading coefficients and discarding the right-hand sides and lower order terms reduce to just one single equation rewritten several times in turn for all the unknown functions; see also [6, p. 27], [5, pp. 32-33], [4].

An extensive study of the existence and regularity properties of solutions to nonlinear strongly coupled systems of equations with coefficients depending on spacial coordinates and unknowns was given by H. Amann (see [1], [2], [3] and vast bibliography therein). The technique used there relates to semigroups and is not extendable to equations with coefficients depending upon derivatives.

On the other hand, in the theory of scalar second order PDE's the most known and powerful tool for obtaining estimates is the maximum principle. The extension of the maximum principle to a system of two equations has been set forth in our previous paper [13].

The technique we are utilizing has been employed earlier in [14] for semilinear systems (see also [9], [15] and [11]), and consists in switching to new functions, for each of which the estimate is established in a conventional way, whence the final conclusion about each component of the vector function solution follows. This technique allows to tackle nonlinear nondiagonal systems.

The main idea of our approach is as follows instead of trying to establish estimates for each component of solution (u, v, w) rather to introduce some linear combinations of components of the solution

$$\begin{aligned} H_1 &= \alpha_1 u + \beta_1 v + w, \\ H_2 &= \alpha_2 u + \beta_2 v + w, \\ H_3 &= \alpha_3 u + \beta_3 v + w, \end{aligned} \tag{1.1}$$

in general some functions H of t, x, u, v, w, for each of which the estimates hold and from whose estimates we shall be able to derive the estimates for the components of solution (u, v, w).

In the present paper we further develop this method. Namely, although restricting ourselves to systems of second order equations in divergence form possessing special structure, we demonstrate Hölder continuity of solution to quasilinear degenerate parabolic systems of equations in which coupling occurs in the leading derivatives and whose leading coefficients depend on x, u, v, w and u_x, v_x, w_x,.

2. Basic notations and hypotheses. We shall be concerned with a system of three equations of the form

$$\begin{cases} u_t - \dfrac{\partial}{\partial x_i}\left(A_i^{(1)}(x, u, v, w, u_x, v_x, w_x)\right) = B^{(1)}(x, u, v, w, u_x, v_x, w_x), \\ v_t - \dfrac{\partial}{\partial x_i}\left(A_i^{(2)}(x, u, v, w, u_x, v_x, w_x)\right) = B^{(2)}(x, u, v, w, u_x, v_x, w_x), \\ w_t - \dfrac{\partial}{\partial x_i}\left(A_i^{(3)}(x, u, v, w, u_x, v_x, w_x)\right) = B^{(3)}(x, u, v, w, u_x, v_x, w_x), \qquad x \in Q, \end{cases} \tag{2.1}$$

for which the model is a system of the following structure

$$\begin{cases} u_t - \dfrac{\partial}{\partial x_i}(\mathrm{a}_1 \nabla u + \mathrm{b}_1 \nabla v + \mathrm{c}_1 \nabla w) = f_1 - \dfrac{\partial}{\partial x_i} F_i^1, \\ v_t - \dfrac{\partial}{\partial x_i}(\mathrm{a}_2 \nabla u + \mathrm{b}_2 \nabla v + \mathrm{c}_2 \nabla w) = f_2 - \dfrac{\partial}{\partial x_i} F_i^2, \\ w_t - \dfrac{\partial}{\partial x_i}(\mathrm{a}_2 \nabla u + \mathrm{b}_2 \nabla v + \mathrm{c}_2 \nabla w) = f_3 - \dfrac{\partial}{\partial x_i} F_i^3, \qquad x \in Q, \end{cases} \tag{2.2}$$

whose coefficients in the leading part are

$$\begin{aligned} \mathrm{a}_1 &= [\lambda_1 \alpha_1 (\beta_2 - \beta_3) + \lambda_2 \alpha_2 (\beta_3 - \beta_1) + \lambda_3 \alpha_3 (\beta_1 - \beta_2)]\, \Delta^{-1}, \\ \mathrm{b}_1 &= [\lambda_1 \beta_1 (\beta_2 - \beta_3) + \lambda_2 \beta_2 (\beta_3 - \beta_1) + \lambda_3 \beta_3 (\beta_1 - \beta_2)]\, \Delta^{-1}, \\ \mathrm{c}_1 &= [\lambda_1 (\beta_2 - \beta_3) + \lambda_2 (\beta_3 - \beta_1) + \lambda_3 (\beta_1 - \beta_2)]\, \Delta^{-1}, \\ \mathrm{a}_2 &= [\lambda_1 \alpha_1 (\alpha_3 - \alpha_2) + \lambda_2 \alpha_2 (\alpha_1 - \alpha_3) + \lambda_3 \alpha_3 (\alpha_2 - \alpha_1)]\, \Delta^{-1}, \\ \mathrm{b}_2 &= [\lambda_1 \beta_1 (\alpha_3 - \alpha_2) + \lambda_2 \beta_2 (\alpha_1 - \alpha_3) + \lambda_3 \beta_3 (\alpha_2 - \alpha_1)]\, \Delta^{-1}, \\ \mathrm{c}_2 &= [\lambda_1 (\alpha_3 - \alpha_2) + \lambda_2 (\alpha_1 - \alpha_3) + \lambda_3 (\alpha_2 - \alpha_1)]\, \Delta^{-1}, \\ \mathrm{a}_3 &= [\lambda_1 \alpha_1 (\alpha_2 \beta_3 - \beta_2 \alpha_3) + \lambda_2 \alpha_2 (\alpha_3 \beta_1 - \beta_3 \alpha_1) + \lambda_3 \alpha_3 (\alpha_1 \beta_2 - \beta_1 \alpha_2)]\, \Delta^{-1}, \\ \mathrm{b}_3 &= [\lambda_1 \beta_1 (\alpha_2 \beta_3 - \beta_2 \alpha_3) + \lambda_2 \beta_2 (\alpha_3 \beta_1 - \beta_3 \alpha_1) + \lambda_3 \beta_3 (\alpha_1 \beta_2 - \beta_1 \alpha_2)]\, \Delta^{-1}, \\ \mathrm{c}_3 &= [\lambda_1 (\alpha_2 \beta_3 - \beta_2 \alpha_3) + \lambda_2 (\alpha_3 \beta_1 - \beta_3 \alpha_1) + \lambda_3 (\alpha_1 \beta_2 - \beta_1 \alpha_2)]\, \Delta^{-1}, \end{aligned}$$

where α_j, β_j are some real numbers, $\lambda_j = \lambda_j(x, u, v, w, u_x, v_x, w_x) > 0$ are some bounded measurable $\Omega \times \mathbb{R}^3 \times \mathbb{R}^{3n} \to \mathbb{R}$ functions of x, u, v, w, u_x, v_x, w_x; $\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$; the right-hand sides are measurable functions such that

$$f_j(x,t) \in L^\tau(Q), \quad \tau = \frac{(p+n)}{p(1-\kappa_1)}; \tag{2.3}$$

$$F_i^j(x,t) \in L^\theta(Q), \quad \theta = \frac{(p+n)}{(p-1)(1-\kappa_1)}, \quad \kappa_1 \in (0,1). \tag{2.4}$$

The boundary conditions of the Dirichlet type are assigned

$$\begin{cases} (u - g_1, v - g_2, w - g_3)(x,t) \in W_0^{1,p}(\Omega) \qquad \text{a.e.} \quad t \in (0,T), \\ \qquad (u, v, w)(x, 0) = (u_0, v_0, w_0)(x). \end{cases} \tag{2.5}$$

A solution to system (2.1) with Dirichlet data (2.5) is understood in the weak sense, as in [8]. A measurable vector function $(u^1, u^2, u^3) = (u, v, w)$ is called *a weak solution* of problem (2.1)-(2.5) if $u^j \in C(0,T; L^2(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega))$ and for all $t \in (0,T]$

$$\int\limits_\Omega u^j \varphi_j(x,t)dx + \iint\limits_{\Omega\times(0,t]} \{-u^j \varphi_{j\,t} + A_i^j \varphi_{j\,x_i}\} dx d\tau = \int\limits_\Omega u_0^j \varphi_j(x,0)dx + \iint\limits_{\Omega\times(0,t]} B^j \varphi_j dx d\tau$$

for all testing functions $\varphi \in W^{1,2}(0,T; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega))$, $\varphi \geq 0$. The boundary condition in (2.5) is meant in the weak sense.

Let us also define the boundary norms of functions that will come useful in the follow-up considerations. Let Ω be a domain in $\mathbb{R}^n$ (here n is any natural number) and $\partial\Omega$ a portion of its boundary; $W(\Omega)$ be any Sobolev space. For a function $u(x)$ specified on $\partial\Omega$ we define

$$\|u\|_{W(\partial\Omega)} = \inf_\psi \|\psi\|_{W(\Omega)},$$

where the infimum is taken in all functions $\psi \in W(\Omega)$ such that $\psi(x) = u(x)$ a. e. on $\partial\Omega$. We shall denote by $W(\partial\Omega)$ a functional space for which the aforementioned norm is finite.

Let us describe the notions, quantities and functions entering systems (2.1)-(2.2) that will appear in this paper. Here and onward we accept the following notations $Q = (0,T] \times \Omega$; $S = \partial\Omega \times (0,T]$; $\partial Q \equiv \{\Omega \times \{0\}\} \cup \{\partial\Omega \times (0,T]\}$; Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary; $x \in \Omega$; $T > 0$; $t \in (0,T]$; $n > p > 2$; $i \in \{1, \ldots, n\}$; $j \in \{1,2,3\}$ and summation convention over repeated indices is assumed; $u, v, w \in C(0,T; L^2(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega))$; $W_0^{1,p}(\Omega)$ is a space of functions in $W^{1,p}(\Omega)$ vanishing on $\partial\Omega$ in the sense of traces for a.e $t \in (0,T]$.Throughout the paper, for the sake of brevity, by $|s|$ and $|s_i|$ is denoted the distance in $3n$-dimensional and n-dimensional Euclidean space respectively, i.e $|s| = \left(\sum_{j=1}^3 \sum_{i=1}^n (s_i^j)^2\right)^{1/2}$, $|s_i| = \left(\sum_{i=1}^n (s_i^j)^2\right)^{1/2}$, where s_i^j stands for a $3n$-component vector.

By parabolicity of system (2.1) it is meant that the part without derivatives with respect to time is elliptic. The notion of ellipticity of a system of differential equations is understood in the following sense, as it is introduced in [7]

$$(\exists\lambda > 0,\ 0 < F = F(x) \in L^{p/(p-1)}(Q))\ (\forall s_j^i \in \mathbb{R}^{3n})\ (\forall r_j \in \mathbb{R}^3)\ (\forall x \in \mathbb{R}^n):$$
$$A_i^j(x, r, s) s_j^i \geq \lambda |s|^p - F. \tag{2.6}$$

It should be emphasized that we impose neither the Legendre nor the Legendre-Hadamard condition. The Legendre condition stems from the calculus of variations, the problem of the minimization of functional, as a sufficient condition for the existence of extremal. Since it is to be calculated on the extremal it bears no relation to the set-up of a problem. Its usage as the ellipticity condition in the theory of systems of equations is entirely technically motivated. The Legendre-Hadamard condition, being a weakened version of the Legendre one, has been regarded by many authors as a more natural ellipticity condition for systems. Both conditions produce an obstacle from the technical point of view in the approach used by ours, and that is why we dispense with them and accept the ellipticity condition (2.6) for quasilinear system as the most appropriate to our ends.

About $A_i^j(x,r,s)$ it is assumed that they are measurable $\Omega\times\mathbb{R}^3\times\mathbb{R}^{3n}\to\mathbb{R}$ functions that satisfy the ellipticity condition and are subject to the growth conditions

$$(\exists\,\Lambda_2>0)\quad(\forall s_i^j\in\mathbb{R}^{3n})\;(\forall r^j\in\mathbb{R}^3)\;(\forall x\in\mathbb{R}^n):\quad |A_i^j(x,r,s)|\le\Lambda_2|s|^{p-1}. \tag{2.7}$$

and to the following structure conditions

$$\left(\exists\,\alpha_j,\beta_j\in\mathbb{R},\quad \begin{vmatrix}\alpha_1&\alpha_2&\alpha_3\\ \beta_1&\beta_2&\beta_3\\ 1&1&1\end{vmatrix}\ne 0\right)\quad(\forall s_i^j\in\mathbb{R}^{3n})\quad(\forall r^j\in\mathbb{R}^3)\quad(\forall x\in\mathbb{R}^n):$$

$$\begin{aligned}&\left|\alpha_1A_i^{(1)}(x,r,s)+\beta_1A_i^{(2)}(x,r,s)+A_i^{(3)}(x,r,s)-\lambda_1(x,r,s)(\alpha_1s_i^1+\beta_1s_i^2+s_i^3)\right|\le\\ &\qquad\le\eta_1(x,r,s)|\alpha_1s_i^1+\beta_1s_i^2+s_i^3|+\xi_1(x,r,s)+F_1,\end{aligned} \tag{2.8a}$$

$$\begin{aligned}&\left|\alpha_2A_i^{(1)}(x,r,s)+\beta_2A_i^{(2)}(x,r,s)+A_i^{(3)}(x,r,s)-\lambda_2(x,r,s)(\alpha_2s_i^1+\beta_2s_i^2+s_i^3)\right|\le\\ &\qquad\le\eta_2(x,r,s)|\alpha_2s_i^1+\beta_2s_i^2+s_i^3|+\xi_2(x,r,s)+F_2,\end{aligned} \tag{2.8b}$$

$$\begin{aligned}&\left|\alpha_3A_i^{(1)}(x,r,s)+\beta_3A_i^{(2)}(x,r,s)+A_i^{(3)}(x,r,s)-\lambda_3(x,r,s)(\alpha_3s_i^1+\beta_3s_i^2+s_i^3)\right|\le\\ &\qquad\le\eta_3(x,r,s)|\alpha_3s_i^1+\beta_3s_i^2+s_i^3|+\xi_3(x,r,s)+F_3,\end{aligned} \tag{2.8c}$$

$\lambda_j=\lambda_j(x,r,s)>0$, $\eta_j=\eta_j(x,r,s)>0$, $\xi_j=\xi_j(x,r,s)>0$ are some measurable $\Omega\times\mathbb{R}^3\times\mathbb{R}^{3n}\to\mathbb{R}$ functions of x, u, v, w, u_x, v_x, w_x on which the following growth conditions are imposed

$$(\exists\,\Lambda_1,\Lambda_2>0)\quad(\forall s_i^j\in\mathbb{R}^{3n})\quad(\forall r^j\in\mathbb{R}^3)\quad(\forall x\in\mathbb{R}^n):$$

$$0<\Lambda_1\left|\alpha_js_i^1+\beta_js_i^2+s_i^3\right|^{p-2}\le\lambda_j(x,r,s)\le\Lambda_2\left|\alpha_js_i^1+\beta_js_i^2+s_i^3\right|^{p-2}; \tag{2.9}$$

$$\eta_j(x,r,s)\le\eta_0|s|^{\mu},\quad \mu=\frac{p(p-2)(1-\kappa_1)}{(n+p)}>0; \tag{2.10}$$

$$\xi_j(x,r,s)\le\xi_0|s|^{\nu},\quad \nu=\frac{p(p-1)(1-\kappa_1)}{(n+p)}>0; \tag{2.11}$$

with F_j and κ_1 as in (2.4), ξ_0 and η_0 are positive numbers, moreover

$$\alpha_1,\ \beta_2>1; \tag{2.12}$$

$$\alpha_2,\ \alpha_3,\ \beta_1\,\beta_3<1; \tag{2.13}$$

$$3\max[1/p,\Lambda_2]\max[\alpha_1^{-1},\beta_2^{-1},\alpha_3,\beta_3]\le\Lambda_1/(2^pp); \tag{2.14}$$

$$9\max[\eta_0,\xi_0]\le\Lambda_1/(2^{p+1}p). \tag{2.15}$$

Remark 1. It is not difficult to check by direct calculation, taking into account the fact that $F_j \in L^{\frac{(p+n)}{(p-1)(1-\kappa_1)}}$, that structure conditions (2.8a)-(2.8c) along with (2.9) and (2.10)-(2.15) imply the ellipticity condition (2.6) with $\lambda = \Lambda_1/(2^{p+1}p)$ and $F \equiv C_1(|F_1| + |F_2| + |F_3|)^{p/p-1} + C_2$, $C_{1,2}$ are numbers depending only on the data.

About right-hand sides $B^j(x,r,s)$ it is assumed that they are measurable $\Omega\times\mathbb{R}^3\times\mathbb{R}^{3n} \to \mathbb{R}$ functions satisfying

$$\Big(\exists\,\varepsilon \in \Big(0, \frac{p^2(1-\kappa_1)}{(n+p)}\Big]\Big)\ (\exists \Lambda_3 > 0) \quad (\forall s_i^j \in \mathbb{R}^{3n}) \quad (\forall r_j \in \mathbb{R}^3) \quad (\forall x \in \mathbb{R}^n):$$

$$|B^j(x,r,s)| \le \Lambda_3 |s|^{\varepsilon} + f_j, \tag{2.16}$$

with f_j satisfying (2.3).

In what follows for the sake of conciseness we shall use the notations

$$\widetilde{u_0} = \begin{cases} u_0(x), & x\in\Omega, \quad t=0,\\ g_1(x,t), & x\in\partial\Omega, \quad t\in(0,T); \end{cases} \qquad \widetilde{v_0} = \begin{cases} v_0(x), & x\in\Omega, \quad t=0,\\ g_2(x,t), & x\in\partial\Omega, \quad t\in(0,T); \end{cases}$$

$$\widetilde{w_0} = \begin{cases} w_0(x), & x\in\Omega, \quad t=0,\\ g_3(x,t), & x\in\partial\Omega, \quad t\in(0,T). \end{cases}$$

Let us introduce in addition the following functional space

$$\widetilde{W}(Q) = L^{p'}(W^{1,p'}(0,T);\Omega) \cap L^p(0,T;W^{1,p}(\Omega)), \quad p' = \frac{p}{p-1};$$

i. e. *the function u belongs to $\widetilde{W}(Q)$* if the integral

$$\int_0^T\int_\Omega \left(|u_t|^{p'} + |\nabla u|^p + |u|^p + |u|^{p'}\right)$$

is finite.

On the functions $g_j(x,t)$, $(u_0,v_0,w_0)(x)$ in boundary data (2.5) we assume to be fulfilled the following assumptions

$$\widetilde{u_0} \in \widetilde{W}(\partial Q), \quad \widetilde{v_0} \in \widetilde{W}(\partial Q), \quad \widetilde{w_0} \in \widetilde{W}(\partial Q);$$

and, in addition, with $\alpha_g \in (0,1)$ and $\alpha_0 \in (0,1)$ their values on the ∂Q satisfy

$$g_j(x,t) \in H^{\alpha_g,\alpha_g/p}(S), \quad (u_0,v_0,w_0)(x) \in H^{\alpha_0}(\overline{\Omega}\times\{0\}),$$

where $H^{\alpha_g,\alpha_g/p}$ and H^{α_0} denotes Hölder spaces with exponents α_g and α_0 respectively.

3. Estimate for the sum of squares. For the ongoing considerations we need to estimate the integral of the sum of squares of the spacial derivatives of the components of the solution of problem (2.1)-(2.5).

Our goal in this section is to prove the following statement.

Theorem 3.1. *Let* (u, v) *be a solution to problem* (2.1)-(2.5)*and the hypotheses* (2.8a)-(2.8c),(2.9), (2.10)-(2.15) *and* (2.16) *are satisfied,, then there hold the estimates*

$$\sup_{0<t<T}\int\limits_{\Omega}|u-\widetilde{u_0}|^2+\sup_{0<t<T}\int\limits_{\Omega}|v-\widetilde{v_0}|^2+\sup_{0<t<T}\int\limits_{\Omega}|w-\widetilde{u_0}|^2+$$
$$+\int\limits_0^T\int\limits_{\Omega}(|\nabla(u-\widetilde{u_0})|^p+|\nabla(v-\widetilde{v_0})|^p+|\nabla(w-\widetilde{w_0})|^p)\le C$$

and

$$\int\limits_0^T\int\limits_{\Omega}(|\nabla u|^p+|\nabla v|^p+|\nabla w|^p)\le C$$

with constant C *depending only on the data* f^j, F^j, $||\widetilde{u_0}||_{\widetilde{W}(\partial Q)}$, $||\widetilde{v_0}||_{\widetilde{W}(\partial Q)}$, $||\widetilde{w_0}||_{\widetilde{W}(\partial Q)}$, p, n, Λ_1, Λ_2, ξ_0, η_0, κ_1, α^j, β^j, ε, $\operatorname{mes} Q$, *and independent of* u, v *and* w.

Remark 2. By $\widetilde{u_0}$, $\widetilde{v_0}$ and $\widetilde{w_0}$ in the formulation of the theorem and in the follow-up proof is meant any function from $\widetilde{W}(Q)$ assuming the values of either $\widetilde{u_0}$ or $\widetilde{v_0}$, $\widetilde{w_0}$ on the parabolic boundary. Therefore the final statement remains valid with the boundary norms.

Proof. Multiply the first equation of (2.1) by $(u-\widetilde{u_0})$, the second one by $(v-\widetilde{v_0})$, and the third by $(w-\widetilde{w_0})$. After adding all three together and integrating over the domain $\Omega\times(0,t)$ this results in

$$\int\limits_{\Omega(t)}\frac{1}{2}(u-\widetilde{u_0})^2+\int\limits_{\Omega(t)}\frac{1}{2}(v-\widetilde{v_0})^2+\int\limits_{\Omega(t)}\frac{1}{2}(w-\widetilde{w_0})^2+$$
$$+\int\limits_0^t\int\limits_{\Omega}\vec{A}^{(1)}\nabla(u-\widetilde{u_0})+\int\limits_0^t\int\limits_{\Omega}\vec{A}^{(2)}\nabla(v-\widetilde{v_0})+\int\limits_0^t\int\limits_{\Omega}\vec{A}^{(3)}\nabla(w-\widetilde{w_0})\le$$
$$\le\int\limits_0^t\int\limits_{\Omega}|B^{(1)}||u-\widetilde{u_0}|+\int\limits_0^t\int\limits_{\Omega}|B^{(2)}||v-\widetilde{v_0}|+\int\limits_0^t\int\limits_{\Omega}|B^{(3)}||w-\widetilde{w_0}|+$$
$$+\int\limits_0^t\int\limits_{\Omega}(|f_1|+|\widetilde{u_{0t}}|)|u-\widetilde{u_0}|+\int\limits_0^t\int\limits_{\Omega}(|f_2|+|\widetilde{v_{0t}}|)|v-\widetilde{v_0}|+\int\limits_0^t\int\limits_{\Omega}(|f_3|+|\widetilde{w_{0t}}|)|w-\widetilde{w_0}|,\quad (3.1)$$

where the integration by parts with respect to time variable in the first two terms and the initial condition is taken into account. On the strength of ellipticity condition (2.6) and growth conditions on $A^{(1),(2),(3)}$ (2.7), the second group of terms on the left permits the estimation

$$\int\limits_0^t\int\limits_{\Omega}\Big(\vec{A}^{(1)}\nabla(u-\widetilde{u_0})+\vec{A}^{(2)}\nabla(v-\widetilde{v_0})+\vec{A}^{(3)}\nabla(w-\widetilde{w_0})\Big)=$$
$$=\int\limits_0^t\int\limits_{\Omega}\Big(\vec{A}^{(1)}\nabla u+\vec{A}^{(2)}\nabla v+\vec{A}^{(3)}\nabla w-\vec{A}^{(1)}\nabla\widetilde{u_0}-\vec{A}^{(2)}\nabla\widetilde{v_0}-\vec{A}^{(3)}\nabla\widetilde{w_0}\Big)\ge$$

$$\geq \int_0^t \int_\Omega \lambda\left(|\nabla u|^p + |\nabla v|^p + |\nabla w|^p\right) - \int_0^t \int_\Omega \lambda\left(|\nabla u|^{p-1} + |\nabla v|^{p-1} + |\nabla w|^{p-1}\right) \times$$

$$\times(|\nabla u_0| + |\nabla v_0| + |\nabla w_0|) - \int_0^t \int_\Omega |F_1| + |F_2| + |F_3| \geq \int_0^t \int_\Omega \frac{1}{2}\lambda\left(|\nabla u|^p + |\nabla v|^p + |\nabla w|^p\right) -$$

$$-C(p,\lambda)\int_0^t \int_\Omega \left(|\nabla u_0|^p + |\nabla v_0|^p + |\nabla w_0|^p\right) - C \geq \int_0^t \int_\Omega \frac{1}{2}\lambda\left(|\nabla(u - \widetilde{u_0})|^p + |\nabla(v - \widetilde{v_0})|^p +\right.$$

$$\left.+|\nabla(w - \widetilde{w_0})|^p\right) - \int_0^t \int_\Omega \widetilde{C}(p,\lambda)\left(|\nabla u_0|^p + |\nabla v_0|^p + |\nabla w_0|^p\right) - C.$$

Here the use is also made of Young's inequality and the inequality

$$|a+b|^p \leq C(p)(|a|^p + |b|^p), \quad \forall a, b \in \mathbb{R}. \tag{3.2}$$

The first three terms on the right of (3.1) in virtue of Young's inequality, the Sobolev inequality and growth condition (2.16) can be estimated like that

$$\int_0^t \int_\Omega |B^{(1)}||u - \widetilde{u_0}| + \int_0^t \int_\Omega |B^{(2)}||v - \widetilde{v_0}| + \int_0^t \int_\Omega |B^{(3)}||w - \widetilde{w_0}| \leq$$

$$\leq \int_0^t \int_\Omega \left(|\nabla u| + |\nabla v| + |\nabla w|\right)^\varepsilon \left(|u - \widetilde{u_0}| + |v - \widetilde{v_0}| + |w - \widetilde{w_0}|\right) \leq$$

$$\leq \delta_1 C_1(\varepsilon, p)\int_0^t \int_\Omega \left(|\nabla(u - \widetilde{u_0})| + |\nabla(v - \widetilde{v_0})| + |\nabla(w - \widetilde{w_0})|\right)^p +$$

$$+\delta_2 C_2(p)\int_0^t \int_\Omega \left(|u - \widetilde{u_0}| + |v - \widetilde{v_0}| + |w - \widetilde{w_0}|\right)^p +$$

$$+C\left(C_{1,2}, \delta_{1,2}, \widetilde{u_0}, \widetilde{v_0}, \widetilde{w_0}, \operatorname{mes} Q\right) \leq \delta_3 \int_0^t \int_\Omega \left(|\nabla(u - \widetilde{u_0})| + |\nabla(v - \widetilde{v_0})| + |\nabla(w - \widetilde{w_0})|\right)^p + C_3.$$

Here it has been taken into account that $\varepsilon/p + 1/p < 1/2 + 1/p \leq 1$. Taking into account our hypotheses upon functions in much the same way can be estimated the last three integrals in the right-hand side of (3.1)

$$\int_0^t \int_\Omega (|f_1| + |\widetilde{u_{0t}}|)|u - \widetilde{u_0}| + \int_0^t \int_\Omega (|f_2| + |\widetilde{v_{0t}}|)|v - \widetilde{v_0}| + \int_0^t \int_\Omega (|f_3| + |\widetilde{w_{0t}}|)|w - \widetilde{w_0}| \leq$$

$$\leq \int_0^t \int_\Omega \left(|f_1| + |f_2| + |f_3| + |\widetilde{u_{0t}}| + |\widetilde{v_{0t}}| + |\widetilde{w_{0t}}|\right)\left(|u - \widetilde{u_0}| + |v - \widetilde{v_0}| + |w - \widetilde{w_0}|\right) \leq$$

$$\leq \Big\| |f_1| + |f_2| + |f_3| + |\widetilde{u_{0t}}| + |\widetilde{v_{0t}}| + |\widetilde{w_{0t}}| \Big\|_{p',Q} \left(\int_0^t \int_\Omega (|u-\widetilde{u_0}| + |v-\widetilde{v_0}| + |w-\widetilde{w_0}|)^p \right)^{1/p}$$

$$\leq \delta_4 \int_0^t \int_\Omega (|\nabla(u-\widetilde{u_0})| + |\nabla(v-\widetilde{v_0})| + |\nabla(w-\widetilde{w_0})|)^p + C_4 \left(\operatorname{mes} Q, f_{1,2,3}, \delta_4, \widetilde{u_0}, \widetilde{v_0}, \widetilde{w_0}\right).$$

Collecting the above estimates, from (3.1) we get

$$\int_{\Omega(t)} \frac{1}{2} \left[(u-\widetilde{u_0})^2 + (v-\widetilde{v_0})^2 + (w-\widetilde{w_0})^2 \right] +$$

$$+ \int_0^t \int_\Omega \frac{1}{2} \lambda \left(|\nabla(u-\widetilde{u_0})|^p + |\nabla(v-\widetilde{v_0})|^p + |\nabla(w-\widetilde{w_0})|^p \right) \leq$$

$$\leq \delta_5 \int_0^t \int_\Omega \left(|\nabla(u-\widetilde{u_0})|^p + |\nabla(v-\widetilde{v_0})|^p + |\nabla(w-\widetilde{w_0})|^p \right) + C_5 \left(\operatorname{mes} Q, F, f_{1,2,3}, \delta_5, \widetilde{u_0}, \widetilde{v_0}, \widetilde{w_0}\right).$$

Let us choose $\delta_5 = \frac{1}{4}\lambda$,which yields the inequality

$$\int_{\Omega(t)} \frac{1}{2} \left[(u-\widetilde{u_0})^2 + (v-\widetilde{v_0})^2 + (w-\widetilde{w_0})^2 \right] +$$

$$+ \int_0^t \int_\Omega \frac{1}{4} \lambda \left(|\nabla(u-\widetilde{u_0})|^p + |\nabla(v-\widetilde{v_0})|^p + |\nabla(w-\widetilde{w_0})|^p \right) \leq$$

$$\leq C_4 \left(\operatorname{mes} Q, F, f_{1,2,3}, \delta_5, \widetilde{u_0}, \widetilde{v_0}, \widetilde{w_0}\right). \tag{3.3}$$

On this step we take the supremum in t in the left-hand side of (3.3) and obtain the estimate

$$\sup_{0<t<T} \int_\Omega |u-\widetilde{u_0}|^2 + \sup_{0<t<T} \int_\Omega |v-\widetilde{v_0}|^2 + \sup_{0<t<T} \int_\Omega |w-\widetilde{w_0}|^2 +$$

$$+ \int_0^T \int_\Omega \left(|\nabla(u-\widetilde{u_0})|^p + |\nabla(v-\widetilde{v_0})|^p + |\nabla(w-\widetilde{w_0})|^p \right) \leq C_5$$

with constant C_5 depending on n, p, ε, λ, F^j, f^j, p, n, Λ_1, Λ_2, ξ_0, η_0, κ_1, α^j, β^j, ε, $\operatorname{mes} Q$ and, on the strength of Remark 2, the boundary norms $||\widetilde{u_0}||_{\widetilde{W}(\partial Q)}$, $||\widetilde{v_0}||_{\widetilde{W}(\partial Q)}$ and $||\widetilde{w_0}||_{\widetilde{W}(\partial Q)}$ of functions in the boundary conditions only. Hence the second statement of the Theorem is self-evident. □

4. Estimates of L^∞-norms. Let us now turn our attention to the question of boundedness of weak solutions to a system with whose coefficients satisfy assumptions (2.8a)-(2.8c). Our main result in this section is the following

Theorem 4.1. *Let (u, v, w) be a solution to system (2.1). If there exist such numbers α_j, β_j, $\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$ satisfying assumptions (2.8a)-(2.8c), then for the three linearly independent functions H_1, H_2, and H_3 defined by (1.1) the following estimates hold*

$$||H_1||_{L_\infty(Q)} \leq C_1, \quad ||H_2||_{L_\infty(Q)} \leq C_2, \quad ||H_3||_{L_\infty(Q)} \leq C_3.$$

Hence it is easily seen that the same estimates take place for the components of the solution themselves

$$||u||_{L_\infty(Q)} \leq C_1, \quad ||v||_{L_\infty(Q)} \leq C_2, \quad ||w||_{L_\infty(Q)} \leq C_3,$$

where constants $C_{1,2,3}$ depend only on the data n, p, ε, λ, F^j, f^j, p, n, Λ_1, Λ_2, ξ_0, η_0, κ_1, α^j, β^j, ε, $\operatorname{mes} Q$ $|g_{1,2,3}|_{\infty,(S)}$, $|u_0, v_0, w_0|_{\infty,(\Omega)}$; constants in the embedding theorems and is independent of u, v, and w.

To prove the Theorem we need the well-known Stampacchia's lemma

Lemma 4.2. *Let $\psi(y)$ be a nonnegative nondecreasing function defined on $[l_0, \infty)$ which satisfies*

$$\psi(m) \leq \frac{C}{(m-l)^\vartheta} \{\psi(l)\}^\delta \qquad \textit{for} \quad m > l \geq l_0,$$

with $\vartheta > 0$ and $\delta > 1$. Then $\psi(l_0 + d) = 0$, where $d = C^{1/\vartheta}\{\psi(k_0)\}^{(\delta-1)/\vartheta} 2^{\delta/(\delta-1)}$.

For proof see [7, Lemma 4.1, p. 8]. We make also use of the following lemma (see [8, Prop. 3.1, p. 7]).

Lemma 4.3. *If $u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ then there holds the inequality*

$$\int_0^T \int_\Omega u^q \leq C \left(\int_0^T \int_\Omega |\nabla u|^p \right) \left(ess \sup_{0<t<T} \int_\Omega |u|^2 \right)^{p/n}$$

with $q = p\frac{n+2}{n}$ and constant C depending only on p and n.

Proof of Theorem 4.1. Let α_1, β_1 be from hypotheses (2.8a). Multiply the first equation of (2.1) by α_1, add the second one multiplied by β_1, and the third one. Choose $H_1 \equiv \operatorname{sign}(\alpha_1 u + \beta_1 v + w)(|\alpha_1 u + \beta_1 v + w| - l)_+$ as a testing function with $l \geq l_0 = \max[||\alpha_1 g_1 + \beta_1 g_2 + g_3||_{L^\infty(S)}, ||\alpha_1 u_0 + \beta_1 v_0 + w_0||_{L^\infty(\Omega)}]$. After integrating in t from 0 to t, $t \leq T$, and in x over the domain Ω, this results in

$$\frac{1}{2} \int_{\Omega(t)} H_1^2 + \int_0^t \int_\Omega \left\langle \alpha_1 \vec{A}^{(1)} + \beta_1 \vec{A}^{(2)} + \vec{A}^{(3)}, \nabla H_1 \right\rangle = \int_0^t \int_\Omega \left(\alpha_1 B^{(1)} + \beta_1 B^{(2)} + B^{(3)} \right) H_1,$$

Making use of hypotheses (2.8a)-(2.8c) we have

$$\int_0^t\int_\Omega \left\langle \alpha_1\vec{A}^{(1)} + \beta_1\vec{A}^{(2)} + \vec{A}^{(3)}, \nabla H_1 \right\rangle =$$

$$= \int_0^t\int_\Omega \Big\langle [\alpha_1\vec{A}^{(1)} + \beta_1\vec{A}^{(2)} + \vec{A}^{(3)} - \lambda_1(\alpha_1\nabla u + \beta_1\nabla v + \nabla w)] +$$

$$+ \lambda_1(\alpha_1\nabla u + \beta_1\nabla v + \nabla w), \nabla H_1 \Big\rangle \geq \int_0^t\int_\Omega \langle \lambda_1(\alpha_1\nabla u + \beta_1\nabla v + \nabla w), \nabla H_1\rangle -$$

$$- \int_0^t\int_\Omega |\alpha_1\vec{A}^{(1)} + \beta_1\vec{A}^{(2)} + \vec{A}^{(3)} - \lambda_1(\alpha_1\nabla u + \beta_1\nabla v + \nabla w)||\nabla H_1| \geq$$

$$\geq \int_0^t\int_\Omega \lambda_1 \langle \alpha_1\nabla u + \beta_1\nabla v + \nabla w, \nabla H_1\rangle - \int_0^t\int_\Omega \left(\eta_1|\nabla H_1|^2 + \xi_1|\nabla H_1| + |F_1||\nabla H_1|\right);$$

which hence yields the inequality

$$\frac{1}{2}\int_{\Omega(t)} H_1^2 + \int_0^t\int_\Omega \lambda_1 \langle \alpha_1\nabla u + \beta_1\nabla v + \nabla w, \nabla H_1\rangle \leq$$

$$\leq \int_0^t\int_\Omega \eta_1|\nabla H_1|^2 + \int_0^t\int_\Omega (|F_1| + \xi_1)|\nabla H_1| + C\int_0^t\int_\Omega BH_1,$$

where it is denoted $B = \alpha_1 B^1 + \beta_1 B^2 + B^3 - f$; $f = \alpha_1 f_1 + \beta_1 f_2 + f_3$; λ, η, ξ are functions from (2.8a)-(2.8c); $C = C(\alpha_1, \beta_1, p, n)$ is a constant. Since $t \in (0, T]$ is arbitrary, then taking the supremum we have

$$\sup_{0<t<T}\int_\Omega H_1^2 + C_1\int_0^T\int_\Omega |\nabla H_1|^p \leq \int_0^T\int_\Omega \eta_1|\nabla H_1|^2 + \int_0^T\int_\Omega (|F_1| + \xi_1)|\nabla H_1| +$$

$$+ C\int_0^T\int_\Omega BH_1 + \int_0^T\int_\Omega fH_1, \tag{4.1}$$

where $C_1 = C_1(\Lambda_1, \alpha_1, \beta_1, p, n)$ and the use has been made of the assumption upon function λ_1 (2.9). Applying generalized Hölder's inequality consequently to the terms on the right yields

$$\int_0^T\int_\Omega \eta_1|\nabla H_1|^2 \le \left\||\nabla H_1|^2\right\|_{p/2,Q}\|\eta_1\|_{p/\mu,Q}\left(\int_0^T\int_\Omega \chi_{A(l)}\right)^{1-2/p-\mu/p}; \tag{4.2a}$$

$$\int_0^T\int_\Omega |F_1||\nabla H_1| \le \|\nabla H_1\|_{p,Q}\|F_1\|_{\theta,Q}\left(\int_0^T\int_\Omega \chi_{A(l)}\right)^{1-1/p-/\theta}; \tag{4.2b}$$

$$\int_0^T\int_\Omega \xi_1|\nabla H_1| \le \|\nabla H_1\|_{p,Q}\|\xi_1\|_{p/\nu,Q}\left(\int_0^T\int_\Omega \chi_{A(l)}\right)^{1-1/p-\nu/p}; \tag{4.2c}$$

$$\int_0^T\int_\Omega BH_1 \le \|H_1\|_{q,Q}\|B\|_{p/\varepsilon,Q}\left(\int_0^T\int_\Omega \chi_{A(l)}\right)^{1-1/q-\varepsilon/p}; \tag{4.2d}$$

$$\int_0^T\int_\Omega fH_1 \le \|H_1\|_{q,Q}\|f\|_{r,Q}\left(\int_0^T\int_\Omega \chi_{A(l)}\right)^{1-1/q-1/r}; \tag{4.2e}$$

where $\chi_{A(l)}$ is a characteristic function of the set $A(l)$; and r has been selected such that $\tau > r > (p+n)/p > p/(p-1)$, since it is not difficult to check that the later inequality holds. From condition upon η (2.10) and the second statement of Theorem 3.1 it follows that

$$\|\eta_1\|_{p/\mu,Q} \le C_1. \tag{4.3}$$

Analogously, (2.4), (2.11), (2.16), (2.3) and Theorem 3.1 imply that

$$\|F_1\|_{\theta,Q} \le C_2; \quad \|\xi_1\|_{p/\nu,Q} \le C_3; \quad \|B\|_{p/\varepsilon,Q} \le C_4; \quad \|f\|_{r,Q} \le C_5. \tag{4.4}$$

Collecting (4.2a)-(4.2e) and taking account of (4.3)-(4.4) from (4.1) we obtain the inequality

$$\begin{aligned}\sup_{0<t<T}\int_\Omega H_1^2 + \int_0^T\int_\Omega |\nabla H_1|^p \le C_1\left\||\nabla H_1|^2\right\|_{p/2,Q}\{\psi(l)\}^{1-2/p-\mu/p} +\\ +C_2\|\nabla H_1\|_{p,Q}\{\psi(l)\}^{1-1/p-1/\theta} + C_3\|\nabla H_1\|_{p,Q}\{\psi(l)\}^{1-1/p-\nu/p} +\\ +C_4\|H_1\|_{q,Q}\{\psi(l)\}^{1-1/q-\varepsilon/p} + C_5\|H_1\|_{q,Q}\{\psi(l)\}^{1-1/q-1/r},\end{aligned} \tag{4.5}$$

here we've denoted

$$\psi(l) = \int_0^T \mathrm{mes}A\{H_1 \ge l\}(l,t)dt.$$

>From Lemma 4.3 it follows

$$||H_1||_{q,Q} \le \left(\sup_{0<t<T}\int_\Omega H_1^2 + \int_0^T\int_\Omega |\nabla H_1|^p\right)^{\frac{p+n}{qn}}. \tag{4.6}$$

>From relation (4.5) and this inequality we get

$$\sup_{0<t<T}\int_\Omega H_1^2+\int_0^T\int_\Omega|\nabla H_1|^p\le C_1\left(\sup_{0<t<T}\int_\Omega H_1^2+\int_0^T\int_\Omega|\nabla H_1|^p\right)^{2/p}\{\psi(l)\}^{1-2/p-\mu/p}+$$
$$+C_2\left(\sup_{0<t<T}\int_\Omega H_1^2+\int_0^T\int_\Omega|\nabla H_1|^p\right)^{1/p}\left(\{\psi(l)\}^{1-1/p-1/\theta}+\{\psi(l)\}^{1-1/p-\nu/p}\right)+$$
$$+C_3\left(\sup_{0<t<T}\int_\Omega H_1^2+\int_0^T\int_\Omega|\nabla H_1|^p\right)^{(n+p)/nq}\left(\{\psi(l)\}^{1-1/q-\varepsilon/p}+\{\psi(l)\}^{1-1/q-1/r}\right). \tag{4.7}$$

Applying Young's inequality to the right-hand side of (4.7) gives

$$\sup_{0<t<T}\int_\Omega H_1^2+\int_0^T\int_\Omega|\nabla H_1|^p\le C_1\{\psi(l)\}^{(1-2/p-\mu/p)\left(\frac{p}{p-2}\right)}+C_2\{\psi(l)\}^{(1-1/p-1/\theta)\left(\frac{p}{p-1}\right)}+$$
$$+C_3\{\psi(l)\}^{(1-1/p-\nu/p)\left(\frac{p}{p-1}\right)}+C_4\{\psi(l)\}^{(1-1/q-\varepsilon/p)\left(\frac{nq}{n+p}\right)^\#}+C_5\{\psi(l)\}^{(1-1/q-1/r)\left(\frac{nq}{n+p}\right)^\#};$$

with $C_{1,2,3,4,5} = C_{1,2,3,4,5}(\widetilde{u_0},\widetilde{v_0},\widetilde{w_0},F,f,\mu,\nu,\theta,\tau,\Lambda_1,\Lambda_2,p,n)$ and $\left(\frac{nq}{n+p}\right)^\#$ such that $\left(\left(\frac{nq}{n+p}\right)^\#\right)^{-1}+\frac{n+p}{nq}=1$. Resorting again to (4.6) implies

$$\begin{aligned}(\|H_1\|_{q,Q})^{nq/(n+p)}&\le C_1\{\psi(l)\}^{(1-2/p-\mu/p)\left(\frac{p}{p-2}\right)}\\ &+C_2\{\psi(l)\}^{(1-1/p-1/\theta)\left(\frac{p}{p-1}\right)}+C_3\{\psi(l)\}^{(1-1/p-\nu/p)\left(\frac{p}{p-1}\right)}+\\ &+C_4\{\psi(l)\}^{(1-1/q-\varepsilon/p)\left(\frac{nq}{n+p}\right)^\#}+C_5\{\psi(l)\}^{(1-1/q-1/r)\left(\frac{nq}{n+p}\right)^\#}.\end{aligned} \tag{4.8}$$

Let us estimate

$$(m-l)\{\psi(m)\}^{1/q}=(m-l)\left(\int_0^T\int_\Omega\chi_{A(m)}\right)^{1/q}<\left(\int_0^T\int_\Omega H_1^q\chi_{A(m)}\right)^{1/q}<||H_1||_{q,Q},$$

where $m>l\ge l_0$. Substituting this into (4.8) we come down to

$$\begin{aligned}(m-l)^q\psi(m)&\le C_1\{\psi(l)\}^{(1-2/p-\mu/p)\left(\frac{p(n+p)}{n(p-2)}\right)}+\\ &+C_2\{\psi(l)\}^{(1-1/p-1/\theta)\left(\frac{p(n+p)}{n(p-1)}\right)}+C_3\{\psi(l)\}^{(1-1/p-\nu/p)\left(\frac{p(n+p)}{n(p-1)}\right)}+\\ &+C_4\{\psi(l)\}^{(1-1/q-\varepsilon/p)\left(\frac{nq}{n+p}\right)^\#\left(\frac{n+p}{n}\right)}+C_5\{\psi(l)\}^{(1-1/q-1/r)\left(\frac{nq}{n+p}\right)^\#\left(\frac{n+p}{n}\right)}\end{aligned}$$

or, succinctly

$$\begin{aligned}\psi(m)&\le\frac{C_1}{(m-l)^q}\{\psi(l)\}^{\delta_1}+\frac{C_2}{(m-l)^q}\{\psi(l)\}^{\delta_2}+\\ &+\frac{C_3}{(m-l)^q}\{\psi(l)\}^{\delta_3}+\frac{C_4}{(m-l)^q}\{\psi(l)\}^{\delta_4}+\frac{C_5}{(m-l)^q}\{\psi(l)\}^{\delta_5}\end{aligned} \tag{4.9}$$

with $\delta_1 = \left(1-\frac{2}{p}-\frac{\mu}{p}\right)\left(\frac{p(n+p)}{n(p-2)}\right)$, $\delta_2 = \left(1-\frac{1}{p}-\frac{1}{\theta}\right)\left(\frac{p(n+p)}{n(p-1)}\right)$, $\delta_3 = \left(1-\frac{1}{p}-\frac{\nu}{p}\right)\left(\frac{p(n+p)}{n(p-1)}\right)$, $\delta_4 = \left(1-\frac{n}{p(n+2)}-\frac{\varepsilon}{p}\right)/\left(\frac{n}{n+p}-\frac{n}{p(n+2)}\right)$, and $\delta_5 = \left(1-\frac{n}{p(n+2)}-\frac{1}{r}\right)/\left(\frac{n}{n+p}-\frac{n}{p(n+2)}\right)$. >From the assumption upon η_j

$$0 < \mu < p(p-2)/(n+p); \tag{2.10}$$

it is not difficult to check that

$$1-\frac{2}{p}-\frac{\mu}{p} > \frac{n(p-2)}{p(n+p)}; \qquad \text{and hence} \quad \delta_1 > 1.$$

In much the same way, from the hypotheses on F_j, (2.4), it follows that

$$1-\frac{1}{p}-\frac{1}{\theta} > \frac{n(p-1)}{p(n+p)}; \qquad \text{and thus} \quad \delta_2 > 1;$$

from the hypotheses on ξ_j

$$0 < \nu < p(p-1)/(n+p), \tag{2.11}$$

hence

$$1-\frac{1}{p}-\frac{\nu}{p} > \frac{n(p-1)}{p(n+p)}; \qquad \text{and thus} \quad \delta_3 > 1;$$

from the hypotheses on B^j

$$0 < \varepsilon < p^2/(n+p), \tag{2.16}$$

hence

$$1-\frac{n}{p(n+2)}-\frac{\varepsilon}{p} > \frac{n}{n+p}-\frac{n}{p(n+2)}; \qquad \text{and thus} \quad \delta_4 > 1;$$

from the hypotheses on f_j and by the choice of r

$$\tau > r > (p+n)/p \tag{2.3}$$

hence

$$1-\frac{n}{p(n+2)}-\frac{1}{r} > \frac{n}{n+p}-\frac{n}{p(n+2)}; \qquad \text{and thus} \quad \delta_5 > 1.$$

Without loss of generality we may assume that $\psi(l) < 1$. In fact, from the first statement of Theorem 3.1 and (4.6) it follows that

$$(l-l_0)\{\psi(l)\}^{1/q} = (l-l_0)\left(\int_0^T\int_\Omega \chi_{A(l)}\right)^{1/q} < \left(\int_0^T\int_\Omega (H_1-l_0)^q\chi_{A(l)}\right)^{1/q} <$$

$$< ||H_1-l_0||_{q,Q} \le \left(\sup_{0<t<T}\int_\Omega (H_1-l_0)^2 + \int_0^T\int_\Omega |\nabla(H_1-l_0)|^p\right)^{\frac{p+n}{qn}} \le \widetilde{C};$$

where $l \ge l_0$; and hence

$$\psi(l) \le \widetilde{C}^q/(l-l_0)^q; \quad \text{and it's easy to see that} \quad \psi(l) < 1 \quad \text{whenever} \quad l > \widetilde{C}+l_0.$$

Since $\psi(l)$ is non-increasing function, $\psi(l) < 1$ is true for all $l > \widetilde{C}+l_0$. Due to this, (4.9) yields

$$\psi(m) \le \frac{C}{(m-l)^q}\{\psi(l)\}^{\delta}, \tag{4.10}$$

with $\delta = \min[\delta_1, \delta_2, \delta_3, \delta_4, \delta_5]$ and $C = \max[C_1, C_2, C_3, C_4, C_5]$. On the strength of Lemma 4.2 from relation (4.10) we can conclude that

$$\psi(l_0 + d) = 0$$

for some d sufficiently large, but finite, depending only on the data n, p, ε, λ, F^j, f^j, p, n, Λ_1, Λ_2, ξ_0, η_0, κ_1, α^j, β^j, ε, $\mathrm{mes}\, Q$ $|g_{1,2,3}|_{\infty,(S)}$, $|u_0, v_0, w_0|_{\infty,(\Omega)}$; constants in the embedding theorems and is independent of u, v, and w. Analogously is done for $H_2 = \alpha_2 u + \beta_2 v + w$ and $H_3 = \alpha_3 u + \beta_3 v + w$, where $\alpha_{2,3}$ and $\beta_{2,3}$ are from (2.8b)-(2.8c).

It is not difficult to see from the previous considerations that the same estimates hold for the components (u, v, w) of solution themselves. In fact,

$$\begin{aligned} ||u||_\infty = ||u\Delta||_\infty/|\Delta| = ||(\alpha_1 u + \beta_1 v + w)(\beta_2 - \beta_3) - (\alpha_2 u + \beta_2 v + w)(\beta_1 - \beta_3) + \\ +(\alpha_3 u + \beta_3 v + w)(\beta_1 - \beta_2)||_\infty/|\Delta| = \\ = ||(\beta_2 - \beta_3)H_1 - (\beta_1 - \beta_3)H_2 + (\beta_1 - \beta_2)H_3||_\infty/|\Delta| \le \\ \le (|\beta_2 - \beta_3|C_1 + |\beta_1 - \beta_3|C_2 + |\beta_1 - \beta_2|C_3)/|\Delta|; \end{aligned}$$

$$\begin{aligned} ||v||_\infty = ||v\Delta||_\infty/|\Delta| = ||(\alpha_1 u + \beta_1 v + w)(\alpha_2 - \alpha_3) - (\alpha_2 u + \beta_2 v + w)(\alpha_1 - \alpha_3) + \\ +(\alpha_3 u + \beta_3 v + w)(\alpha_1 - \alpha_2)||_\infty/|\Delta| = \\ = ||(\alpha_2 - \alpha_3)H_1 - (\alpha_1 - \alpha_3)H_2 + (\alpha_1 - \alpha_2)H_3||_\infty/|\Delta| \le \\ \le (|\alpha_2 - \alpha_3|C_1 + |\alpha_1 - \alpha_3|C_2 + |\alpha_1 - \alpha_2|C_3)/|\Delta|; \end{aligned}$$

$$\begin{aligned} ||w||_\infty = ||(\alpha_1 u + \beta_1 v + w) - \alpha_1 u - \beta_1 v||_\infty \le ||H_1 - \alpha_1 u - \beta_1 v||_\infty \le \\ \le ||H_1||_\infty + |\alpha_1|||u||_\infty + |\beta_1|||v||_\infty, \end{aligned}$$

where $||\cdot||_\infty$ stands for L_∞ norm. Hence follows the statement. □

5. Hölder continuity of weak solutions.

Interior regularity. Let the coefficients and right-hand sides in the equations be measurable functions satisfying conditions (2.8a)-(2.8c) and (2.16). We retain as far as possible the notations of [8]. Introduce the number $\hat{q} \ge 1$ such that

$$\hat{q} = \frac{(p+n)}{p(1-\kappa_1)}, \quad \kappa_1 \in (0,1).$$

Let us introduce also the numbers q and κ satisfying

$$q = \frac{\hat{q}p(1+\kappa)}{\hat{q}-1}, \quad \kappa = \frac{p\kappa_1}{n}; \tag{5.1}$$

Conventional notations are used (see e. g. [8])
K_ρ be the n-dimensional cube centered at the origin with wedge 2ρ

$$K_\rho \equiv \left\{ x \in \mathbb{R}^n |\, \max_i |x_i| < \rho \right\};$$

$[x_0 + K_\rho]$ be an n-dimensional cube with center x_0 and wedge 2ρ congruent to K_ρ

$$[x_0 + K_\rho] \equiv \left\{ x \in \mathbb{R}^n |\, \max_i |x_i - x_{0i}| < \rho \right\};$$

$Q(\theta,\rho)$ be a cylinder of hight θ built up upon a cube K_ρ

$$Q(\theta,\rho)\equiv K_\rho\times\{-\theta,0\};$$

$[(x_0,t_0)+Q(\theta,\rho)]$ be a cylinder congruent to $Q(\theta,\rho)$

$$[(x_0,t_0)+Q(\theta,\rho)]\equiv[x_0+K_\rho]\times\{t_0-\theta,t_0\};$$

$A^{\pm}_{k,\rho}$ be a subset of $[x_0+K_\rho]$ such that

$$A^{\pm}_{k,\rho}\equiv\{x\in[x_0+K_\rho]|(H(x,\tau)-k)_\pm>0\};$$

where ρ and θ are some positive numbers so small that $[(x_0,t_0)+Q(\theta,\rho)]\subset Q$;

$$M^{\pm}_k\equiv\operatorname{ess}\sup_{[(x_0,t_0)+Q(\theta,\rho)]}|(H-k)_\pm|\le\delta\le\delta_0, \tag{5.2}$$

$\delta_0=\dfrac{\Lambda_1}{4\Lambda_2}$ is a positive parameter, Λ_1 and Λ_2 are from (2.9);
let $\zeta(x,t)$ be a piecewise smooth cutoff function in $[(x_0,t_0)+Q(\theta,\rho)]$ with the properties

$$\begin{aligned}\zeta(x,t)\in[0,1],\quad |D\zeta|<\infty\quad &\text{for}\quad x\in[x_0+K_\rho],\\ \text{and}\quad \zeta(x,t)\equiv0\quad &\text{for}\quad x\ni[x_0+K_\rho].\end{aligned} \tag{5.3}$$

For every function $\varphi\in L^1(Q)$ and for $0<h<T$ let us introduce the Steklov averages

$$\varphi(x,t)_h\equiv\begin{cases}\dfrac{1}{h}\int\limits_t^{t+h}\varphi(x,\tau)d\tau, & t\in(0,T-h],\\ 0, & t>T-h;\end{cases}$$

for all $0<t<T$. Remind that for $\varphi\in L^q(\Omega\times(0,T))$ the Steklov average $\varphi_h\xrightarrow[h\to0]{}\varphi$ in $L^q(\Omega\times(0,T-\epsilon))$ for every $\epsilon\in(0,T)$; and for $\varphi\in C(0,T;L^q(\Omega))$ $\varphi_h(t)\xrightarrow[h\to0]{}\varphi(t)$ in $L^q(\Omega)$ for every $t\in(0,T-\epsilon)$, $\forall\epsilon\in(0,T)$.

According to the methodology set forth in monograph [8], for Hölder continuity of weak solutions (Theorems 5.3 and 5.6 below) it is necessary to show the following propositions, Theorems 5.1-5.5

Theorem 5.1 (Local energy estimates). *Let (u,v,w) be a solution to the system and $H=H_j$ be like in Theorem 4.1. There exist constants $\widetilde{C}$ and δ_0 that can be determined a priori only in terms of the data such that for every cylinder $[(x_0,t_0)+Q(\theta,\rho)]\subset Q$ and for every level k satisfying (5.2) there holds the estimate*

$$\begin{aligned}\sup_{t_0-\theta<t<t_0}\int\limits_{[x_0+K_\rho]}(H-k)^2_\pm\zeta^p(x,t)dx+\widetilde{C}^{-1}\iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]}|\nabla(H-k)_\pm\zeta|^p dxd\tau\le\\ \le\int\limits_{[x_0+K_\rho]}(H-k)^2_\pm\zeta^p(x,t_0-\theta)dx+\widetilde{C}\iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]}(H-k)^p_\pm|\nabla\zeta|^p dxd\tau+\\ +\widetilde{C}\iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]}(H-k)^2_\pm\zeta^{p-1}\zeta_t dxd\tau+\widetilde{C}\left\{\int\limits_{t_0-\theta}^{t_0}|A^{\pm}_{k,\rho}(\tau)|d\tau\right\}^{p(1+\kappa)/q}.\end{aligned} \tag{5.4}$$

Proof. It suffices to prove (5.4) for the cube $Q(\theta,\rho)$ since (x_0,t_0) without loss of generality may be assumed to coincide with the origin. Acting as in the previous section, multiply the first equation of system (2.1) by α ($\alpha=\alpha_1,\alpha_2,\alpha_3$), the second one by β ($\beta=\beta_1,\beta_2,\beta_3$), then add all three together and choose as testing functions

$$\varphi=\pm(H_h-k)_\pm\zeta^p,$$

with $\zeta(x,t)$ satisfying (5.3). After integrating in τ over $(-\theta,t)$, $t\in(-\theta,0)$, sending h to zero, and making use of structure conditions (2.8a)-(2.8c) for the leading terms we get

$$\begin{aligned}
&\iint\limits_{K_\rho\times(-\theta,t)} \left\langle \alpha A^{(1)}+\beta A^{(2)}+A^{(3)}, \pm\nabla(H-k)_\pm\zeta^p\pm p(H-k)_\pm\zeta^{p-1}\nabla\zeta\right\rangle dxd\tau\geq\\
&\geq\Lambda_1\iint\limits_{K_\rho\times(-\theta,t)}|\nabla H|^p\zeta^p dxd\tau-\Lambda_2\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^{p(p-2)(1-\kappa_1)/(n+p)}|\nabla H|^2\zeta^p dxd\tau-\\
&-\Lambda_2\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^{p(p-1)(1-\kappa_1)/(n+p)}|\nabla H|\zeta^p dxd\tau-\iint\limits_{K_\rho\times(-\theta,t)}F|\nabla H|\zeta^p dxd\tau-\\
&-p\Lambda_2\iint\limits_{K_\rho\times(-\theta,t)}|\nabla H|^{p-1}(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau-\\
&-p\Lambda_2\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^{p(p-2)(1-\kappa_1)/(n+p)}|\nabla H|(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau-\\
&-p\Lambda_2\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^{p(p-1)(1-\kappa_1)/(n+p)}(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau-\\
&-p\iint\limits_{K_\rho\times(-\theta,t)}F(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau. \qquad (5.5)
\end{aligned}$$

Here and onward for brevity it is denoted $|\nabla u,\nabla v,\nabla w|=(|\nabla u|^2+|\nabla v|^2+|\nabla w|^2)^{1/2}$. Let us estimate various terms of the latter expression by Young's inequality. Considering the first group of terms in (5.5), we get the following

$$\begin{aligned}
&\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^{p(p-2)(1-\kappa_1)/(n+p)}|\nabla H|^2\zeta^p dxd\tau\leq\\
&\leq C_1(p,\Lambda_1,\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)}\left(|\nabla u,\nabla v,\nabla w|^{\frac{p^2(1-\kappa_1)}{(n+p)}}\right)\chi[(H-k)_\pm>0]dxd\tau+\\
&+(\Lambda_1/10\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)}|\nabla H|^p\zeta^p dxd\tau; \qquad (5.6)
\end{aligned}$$

$$\iint\limits_{K_\rho\times(-\theta,t)} |\nabla u,\nabla v,\nabla w|^{p(p-1)(1-\kappa_1)/(n+p)}|\nabla H|\zeta^p dxd\tau \leq$$
$$\leq C_2(p,\Lambda_1,\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)}\left(|\nabla u,\nabla v,\nabla w|^{\frac{p^2(1-\kappa_1)}{(n+p)}}\right)\chi[(H-k)_\pm>0]dxd\tau+$$
$$+(\Lambda_1/10\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)}|\nabla H|^p\zeta^p dxd\tau; \tag{5.7}$$

$$\iint\limits_{K_\rho\times(-\theta,t)} F|\nabla H|\zeta^p dxd\tau \leq C_3(p,\Lambda_1,\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)} F^{\frac{p}{p-1}}\chi[(H-k)_\pm>0]dxd\tau+$$
$$+(\Lambda_1/10)\iint\limits_{K_\rho\times(-\theta,t)}|\nabla H|^p\zeta^p dxd\tau. \tag{5.8}$$

For the second group of terms in (5.5) we obtain the following estimates

$$p\iint\limits_{K_\rho\times(-\theta,t)}|\nabla H|^{p-1}(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau\leq$$
$$\leq C_4(p,\Lambda_1,\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)}(H-k)_\pm^p|\nabla\zeta|^p dxd\tau+(\Lambda_1/10\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)}|\nabla H|^p\zeta^p dxd\tau; \tag{5.9}$$

$$p\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^{p(p-2)(1-\kappa_1)/(n+p)}|\nabla H|(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau\leq$$
$$\leq C_5(p,\Lambda_1,\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)}(H-k)_\pm^p|\nabla\zeta|^p dxd\tau+(\Lambda_1/10\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)}|\nabla H|^p\zeta^p dxd\tau+$$
$$+C_6(p,\Lambda_1,\Lambda_2)\iint\limits_{K_\rho\times(-\theta,t)}\left(|\nabla u,\nabla v,\nabla w|^{\frac{p^2(1-\kappa_1)}{(n+p)}}\right)\chi[(H-k)_\pm>0]dxd\tau; \tag{5.10}$$

$$p\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^{p(p-1)(1-\kappa_1)/(n+p)}(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau\leq$$
$$\leq(p-1)\iint\limits_{K_\rho\times(-\theta,t)}\left(|\nabla u,\nabla v,\nabla w|^{\frac{p^2(1-\kappa_1)}{(n+p)}}\right)\chi[(H-k)_\pm>0]dxd\tau+$$
$$+\iint\limits_{K_\rho\times(-\theta,t)}(H-k)_\pm^p|\nabla\zeta|^p dxd\tau; \tag{5.11}$$

$$p \iint\limits_{K_\rho\times(-\theta,t)} F(H-k)_\pm \zeta^{p-1}|\nabla\zeta| dxd\tau \le$$

$$\le \iint\limits_{K_\rho\times(-\theta,t)} (H-k)_\pm^p |\nabla\zeta|^p dxd\tau + (p-1) \iint\limits_{K_\rho\times(-\theta,t)} F^{\frac{p}{p-1}} \chi[(H-k)_\pm > 0] dxd\tau. \tag{5.12}$$

Now turn our attention to the right-hand sides of the system, i.e. terms containing B's. Making use of the growth assumptions upon B^j (2.16) yields the following terms in the integral inequality

$$\pm \iint\limits_{K_\rho\times(-\theta,t)} \left(\alpha B^{(1)} + \beta B^{(2)} + B^{(3)}\right)(H-k)_\pm \zeta^p dxd\tau \le$$

$$\le \Lambda_2 (|\alpha| + |\beta| + 1) \iint\limits_{K_\rho\times(-\theta,t)} (|\nabla u, \nabla v, \nabla w|^\varepsilon)(H-k)_\pm \zeta^p dxd\tau +$$

$$+ \max[1, |\alpha|, |\beta|] \iint\limits_{K_\rho\times(-\theta,t)} (|f_1| + |f_2| + |f_3|)(H-k)_\pm \zeta^p dxd\tau.$$

Taking notice of the restriction put upon the set of levels in (5.2) we can estimate the latter like this

$$\pm \iint\limits_{K_\rho\times(-\theta,t)} \left(\alpha B^{(1)} + \beta B^{(2)} + B^{(3)}\right)(H-k)_\pm \zeta^p dxd\tau \le$$

$$\le C_7(p, \alpha, \beta, \Lambda_1, \Lambda_2) \iint\limits_{K_\rho\times(-\theta,t)} (|\nabla u, \nabla v, \nabla w|^\varepsilon) \chi[(H-k)_\pm > 0] dxd\tau +$$

$$+ C_8(p, \alpha, \beta, \Lambda_1, \Lambda_2) \iint\limits_{K_\rho\times(-\theta,t)} |f| \chi[(H-k)_\pm > 0] dxd\tau, \tag{5.13}$$

for brevity we've denoted $|f| \equiv (|f_1| + |f_2| + |f_3|)$. Collecting all the above estimates, i.e. (5.6)-(5.8) and (5.9)- (5.13) we obtain the following inequality

$$\frac{1}{2} \int\limits_{K_\rho} (H-k)_\pm^2 \zeta^p(x,t) dx + (\Lambda_1/2) \iint\limits_{K_\rho\times(-\theta,t)} |\nabla(H-k)_\pm|^p \zeta^p dxd\tau \le$$

$$\le \frac{1}{2} \int\limits_{K_\rho} (H-k)_\pm^2 \zeta^p(x,-\theta) dx + \frac{p}{2} \int\limits_{-\theta}^{t} \int\limits_{K_\rho} (H-k)_\pm^2 \zeta^{p-1} \zeta_t dxd\tau + \widetilde{C}_1 \iint\limits_{K_\rho\times(-\theta,t)} (H-k)_\pm^p |\nabla\zeta|^p dxd\tau$$

$$+ \widetilde{C}_2 \iint\limits_{K_\rho\times(-\theta,t)} \left(|\nabla u, \nabla v, \nabla w|^{\frac{p^2(1-\kappa_1)}{(n+p)}}\right) \chi[(H-k)_\pm > 0] dxd\tau +$$

$$+ \widetilde{C}_3 \iint\limits_{K_\rho\times(-\theta,t)} \left(|F|^{\frac{p}{p-1}}\right) \chi[(H-k)_\pm > 0] dxd\tau +$$

$$+ \widetilde{C}_4 \iint\limits_{K_\rho\times(-\theta,t)} (|\nabla u, \nabla v, \nabla w|^\varepsilon) \chi[(H-k)_\pm > 0] dxd\tau + \widetilde{C}_5 \iint\limits_{K_\rho\times(-\theta,t)} |f| \chi[(H-k)_\pm > 0] dxd\tau.$$

Recalling that $t \in (-\theta, 0)$ is arbitrary, after taking the supremum in t hence follows the inequality

$$\sup_{-\theta<t<0} \int_{K_\rho} (H-k)_\pm^2 \zeta^p(x,t)dx + (\Lambda_1/2) \iint_{Q(\theta,\rho)} |\nabla(H-k)_\pm|^p \zeta^p dx d\tau \le$$

$$\le \int_{K_\rho} (H-k)_\pm^2 \zeta^p(x,-\theta) dx d\tau + C \iint_{Q(\theta,\rho)} (H-k)_\pm^p |\nabla\zeta|^p dx d\tau + C \iint_{Q(\theta,\rho)} (H-k)_\pm^2 \zeta^{p-1}\zeta_t dx d\tau +$$

$$+C \iint_{Q(\theta,\rho)} \left(|\nabla u, \nabla v, \nabla w|^{\frac{p^2(1-\kappa_1)}{(n+p)}} \right) \chi[(H-k)_\pm > 0] dx d\tau +$$

$$+C \iint_{Q(\theta,\rho)} \left(|F|^{\frac{p}{p-1}} \right) \chi[(H-k)_\pm > 0] dx d\tau +$$

$$+C \iint_{Q(\theta,\rho)} \left(|\nabla u, \nabla v, \nabla w|^{\varepsilon} \right) \chi[(H-k)_\pm > 0] dx d\tau + C \iint_{Q(\theta,\rho)} |f| \chi[(H-k)_\pm > 0] dx d\tau. \quad (5.14)$$

Estimating the last four terms on the right by Hölder's inequality and taking into account hypotheses (2.10), (2.11), (2.16), and Theorem 3.1 we arrive at

$$\iint_{Q(\theta,\rho)} \left(|\nabla u, \nabla v, \nabla w|^{\frac{p^2(1-\kappa_1)}{(n+p)}} \right) \chi[(H-k)_\pm > 0] dx d\tau + \iint_{Q(\theta,\rho)} \left(|F|^{\frac{p}{p-1}} \right) \chi[(H-k)_\pm > 0] dx d\tau +$$

$$(5.15)$$

$$+ \iint_{Q(\theta,\rho)} \left(|\nabla u, \nabla v, \nabla w|^{\varepsilon} \right) \chi[(H-k)_\pm > 0] dx d\tau + \iint_{Q(\theta,\rho)} |f| \chi[(H-k)_\pm > 0] dx d\tau \le$$

$$\le \left(\|\nabla u, \nabla v, \nabla w\|_{\frac{\hat{q}p^2(1-\kappa_1)}{(n+p)}, Q} \right)^{\frac{p^2(1-\kappa_1)}{(n+p)}} \left\{ \int_{-\theta}^{0} \left| A_{k,\rho}^\pm(\tau) \right| d\tau \right\}^{(\hat{q}-1)/\hat{q}} +$$

$$+ \left(\|\nabla u, \nabla v, \nabla w\|_{\hat{q}\varepsilon, Q} \right)^{\varepsilon} \left\{ \int_{-\theta}^{0} \left| A_{k,\rho}^\pm(\tau) \right| d\tau \right\}^{(\hat{q}-1)/\hat{q}} +$$

$$+ \left(\|F\|_{\frac{\hat{q}p}{p-1}, Q} \right)^{\frac{p}{p-1}} \left\{ \int_{-\theta}^{0} \left| A_{k,\rho}^\pm(\tau) \right| d\tau \right\}^{(\hat{q}-1)/\hat{q}} + \|f\|_{\hat{q},Q} \left\{ \int_{-\theta}^{0} \left| A_{k,\rho}^\pm(\tau) \right| d\tau \right\}^{(\hat{q}-1)/\hat{q}}.$$

Applying the estimate

$$\iint_{Q(\theta,\rho)} |\nabla(H-k)_\pm|^p \zeta^p dx d\tau \le \iint_{Q(\theta,\rho)} |\nabla(H-k)_\pm \zeta|^p dx d\tau + \iint_{Q(\theta,\rho)} (H-k)_\pm^p |\nabla\zeta|^p dx d\tau$$

to the second integral in the left of (5.14) we finally obtain (5.4). □

Theorem 5.2 (Local logarithmic estimates). *Let (u, v, w) be a solution to the system and $H = H_j$ be like in Theorem 4.1. There exist constants C and δ_0 that can be determined* *a priori only in terms of the data such that for every cylinder* $[(x_0,t_0)+Q(\theta,\rho)]\subset Q$ *and for every level* k *satisfying* (5.2) *there holds the following estimate*

$$\sup_{t_0-\theta<t<t_0}\int\limits_{[x_0+K_\rho]}\Psi^2(M_k^\pm,(H-k)_\pm,c)(x,t)\zeta^p(x)dx\le$$
$$\le\int\limits_{[x_0+K_\rho]}\Psi^2(M_k^\pm,(H-k)_\pm,c)(x,t_0-\theta)\zeta^p(x)dx+$$
$$+C\iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]}\Psi|\Psi_H(M_k^\pm,(H-k)_\pm,c)|^{2-p}|\nabla\zeta|^p dxd\tau+$$
$$+(C/c^2)\left(1+\ln\frac{M_k^\pm}{c}\right)\left\{\int\limits_{t_0-\theta}^{t_0}|A_{k,\rho}^\pm(\tau)|d\tau\right\}^{p(1+\kappa)/q},\tag{5.16}$$

where the following notations are used

$$\Psi(M_k^\pm,(H-k)_\pm,c)\equiv\ln^+\left\{\frac{M_k^\pm}{M_k^\pm-(H-k)_\pm+c}\right\},\quad 0<c<M_k^\pm;\tag{5.17}$$
$$for s>0\quad \ln^+s\equiv\max\{\ln s,0\};\tag{5.18}$$

and about ζ *is additionally assumed that it is independent of* t.

Proof. It suffices to prove (5.16) for the cube $Q(\theta,\rho)$ since (x_0,t_0) without loss of generality may be assumed to coincide with the origin. Choose as testing functions

$$\varphi=\left[\Psi^2(H_h)\right]'\zeta^p,$$

prime denotes differentiation with respect to H. By direct calculation it's easy to verify that $[\Psi^2(H_h)]''=2(1+\Psi)\Psi'^2$ and φ is admissible. As before, multiply the first equation of system (2.1) by α ($\alpha=\alpha_1,\alpha_2,\alpha_3$), the second one by β ($\beta=\beta_1,\beta_2,\beta_3$), then add all three together. After integrating in τ over $(-\theta,t)$, $t\in(-\theta,0)$ with testing function φ, sending h to zero, and making use of hypotheses (2.8a)-(2.8c) for the leading terms we get

$$\iint\limits_{K_\rho\times(-\theta,t)}\left\langle\alpha A^{(1)}+\beta A^{(2)}+A^{(3)},2(1+\Psi)\Psi'^2\nabla H\zeta^p+2p\Psi\Psi'\zeta^{p-1}\nabla\zeta\right\rangle dxd\tau\ge$$
$$\ge 2\Lambda_1\iint\limits_{K_\rho\times(-\theta,t)}(1+\Psi)\Psi'^2|\nabla H|^p\zeta^p-$$
$$-2\Lambda_2\iint\limits_{K_\rho\times(-\theta,t)}(1+\Psi)\Psi'^2|\nabla u,\nabla v,\nabla w|^{p(p-2)(1-\kappa_1)/(n+p)}|\nabla H|^2\zeta^p dxd\tau-$$
$$-2\Lambda_2\iint\limits_{K_\rho\times(-\theta,t)}(1+\Psi)\Psi'^2|\nabla u,\nabla v,\nabla w|^{p(p-1)(1-\kappa_1)/(n+p)}|\nabla H|\zeta^p dxd\tau-$$
$$-2\iint\limits_{K_\rho\times(-\theta,t)}(1+\Psi)\Psi'^2F|\nabla H|\zeta^p dxd\tau-$$

$$-2p\Lambda_2 \iint\limits_{K_\rho\times(-\theta,t)} \Psi\Psi'|\nabla H|^{p-1}(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau-$$

$$-2p\Lambda_2 \iint\limits_{K_\rho\times(-\theta,t)} \Psi\Psi'|\nabla u,\nabla v,\nabla w|^{p(p-2)(1-\kappa_1)/(n+p)}|\nabla H|(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau-$$

$$-2p\Lambda_2 \iint\limits_{K_\rho\times(-\theta,t)} \Psi\Psi'|\nabla u,\nabla v,\nabla w|^{p(p-1)(1-\kappa_1)/(n+p)}(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau-$$

$$-2p \iint\limits_{K_\rho\times(-\theta,t)} \Psi\Psi' F(H-k)_\pm\zeta^{p-1}|\nabla\zeta|dxd\tau. \tag{5.19}$$

Let's estimate each term of the latter expression by Young's inequality. Considering the first group of terms in (5.19), we get the following

$$2\Lambda_2 \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2|\nabla u,\nabla v,\nabla w|^{p(p-2)(1-\kappa_1)/(n+p)}|\nabla H|^2\zeta^p dxd\tau \le$$

$$\le (\Lambda_1/5) \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2|\nabla H|^p\zeta^p dxd\tau+$$

$$+C_1(p,\Lambda_1,\Lambda_2) \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2|\nabla u,\nabla v,\nabla w|^{p^2(1-\kappa_1)/(n+p)}\zeta^p dxd\tau; \tag{5.20}$$

$$2\Lambda_2 \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2|\nabla u,\nabla v,\nabla w|^{p(p-1)(1-\kappa_1)/(n+p)}|\nabla H|\zeta^p dxd\tau \le$$

$$\le (\Lambda_1/5) \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2|\nabla H|^p\zeta^p dxd\tau+$$

$$+C_2(p,\Lambda_1,\Lambda_2) \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2|\nabla u,\nabla v,\nabla w|^{p^2(1-\kappa_1)/(n+p)}\zeta^p dxd\tau; \tag{5.21}$$

$$2 \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2 F|\nabla H|\zeta^p dxd\tau \le$$

$$\le (\Lambda_1/5) \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2|\nabla H|^p\zeta^p dxd\tau + C_3(p,\Lambda_1,\Lambda_2) \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2 F^{\frac{p}{p-1}}\zeta^p dxd\tau. \tag{5.22}$$

For the second group of terms in (5.19) the following estimates are valid

$$2p\Lambda_2 \iint\limits_{K_\rho\times(-\theta,t)} \Psi\Psi'|\nabla H|^{p-1}\zeta^{p-1}|\nabla\zeta|dxd\tau \le$$

$$\le C_4(p,\Lambda_1,\Lambda_2) \iint\limits_{K_\rho\times(-\theta,t)} \Psi(\Psi')^{2-p}|\nabla\zeta|^p dxd\tau + (\Lambda_1/5) \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)\Psi'^2|\nabla H|^p\zeta^p dxd\tau; \tag{5.23}$$

$$2p\Lambda_2 \iint\limits_{K_\rho\times(-\theta,t)} \Psi\Psi'|\nabla u,\nabla v,\nabla w|^{p(p-2)(1-\kappa_1)/(n+p)}|\nabla H|\zeta^{p-1}|\nabla\zeta|dxd\tau \leq$$

$$\leq C_5(p,\Lambda_1,\Lambda_2) \iint\limits_{K_\rho\times(-\theta,t)} \Psi(\Psi')^{2-p}|\nabla\zeta|^p dxd\tau + (\Lambda_1/5) \iint\limits_{K_\rho\times(-\theta,t)} (1+\Psi)(\Psi')^2|\nabla H|^p\zeta^p dxd\tau +$$

$$+C_6(p,\Lambda_1,\Lambda_2) \iint\limits_{K_\rho\times(-\theta,t)} \Psi(\Psi')^2|\nabla u,\nabla v,\nabla w|^{p^2(1-\kappa_1)/(n+p)}\zeta^p dxd\tau; \tag{5.24}$$

$$2p \iint\limits_{K_\rho\times(-\theta,t)} \Psi\Psi'|\nabla u,\nabla v,\nabla w|^{p(p-1)(1-\kappa_1)/(n+p)}\zeta^{p-1}|\nabla\zeta|dxd\tau \leq 2 \iint\limits_{K_\rho\times(-\theta,t)} \Psi(\Psi')^{2-p}|\nabla\zeta|^p dxd\tau +$$

$$+2(p-1) \iint\limits_{K_\rho\times(-\theta,t)} \Psi(\Psi')^2|\nabla u,\nabla v,\nabla w|^{p^2(1-\kappa_1)/(n+p)}\zeta^p dxd\tau; \tag{5.25}$$

$$2p \iint\limits_{K_\rho\times(-\theta,t)} \Psi\Psi' F\zeta^{p-1}|\nabla\zeta|dxd\tau \leq$$

$$\leq 2 \iint\limits_{K_\rho\times(-\theta,t)} \Psi(\Psi')^{2-p}|\nabla\zeta|^p dxd\tau + 2(p-1) \iint\limits_{K_\rho\times(-\theta,t)} \Psi(\Psi')^2 F^{\frac{p}{p-1}}\zeta^p dxd\tau. \tag{5.26}$$

Now turn our attention to the right-hand sides of the system, the terms containing B's. Making use of the growth assumptions upon B^j (2.16) yields the following terms in the integral inequality

$$2 \iint\limits_{K_\rho\times(-\theta,t)} \left(\alpha B^{(1)} + \beta B^{(2)} + B^{(3)}\right) \Psi\Psi'\zeta^p dxd\tau \leq$$

$$\leq 2\Lambda_2\left(|\alpha|+|\beta|+1\right) \iint\limits_{K_\rho\times(-\theta,t)} \left(|\nabla u,\nabla v,\nabla w|^\varepsilon\right)\Psi\Psi'\zeta^p dxd\tau +$$

$$+2\max\left[1,|\alpha|,|\beta|\right] \iint\limits_{K_\rho\times(-\theta,t)} \left(|f_1|+|f_2|+|f_3|\right)\Psi\Psi'\zeta^p dxd\tau. \tag{5.27}$$

From (5.2) and the restriction upon the set of levels k, taking into account the definition of Ψ such estimations follow

$$\Psi'^{-1} = M_k^\pm - (H-k)_\pm + c < 2\delta; \tag{5.28}$$

$$\Psi \leq \ln\left(\frac{M_k^\pm}{c}\right), \quad \Psi' \leq 1/c. \tag{5.29}$$

With (5.28) and (5.29) (5.27) can be rewritten like that

$$2 \iint\limits_{K_\rho\times(-\theta,t)} \left(\alpha B^{(1)} + \beta B^{(2)} + B^{(3)}\right) \Psi\Psi'\zeta^p dxd\tau \leq$$

$$\leq C_7(c,|\alpha|,|\beta|,\Lambda_2)\left(1+\ln\left(\frac{M_k^\pm}{c}\right)\right)\iint\limits_{K_\rho\times(-\theta,t)}(|\nabla u,\nabla v,\nabla w|^\varepsilon)\,\chi[(H-k)_\pm>0]dxd\tau+$$

$$+C_8(c,|\alpha|,|\beta|,\Lambda_2)\left(1+\ln\left(\frac{M_k^\pm}{c}\right)\right)\iint\limits_{K_\rho\times(-\theta,t)}|f|\chi[(H-k)_\pm>0]dxd\tau. \qquad (5.30)$$

Making use of (5.28) and (5.29) in (5.20)-(5.22), and in (5.24)-(5.26) implies the following estimates

$$\iint\limits_{K_\rho\times(-\theta,t)}(1+\Psi)\Psi'^2|\nabla u,\nabla v,\nabla w|^{p^2(1-\kappa_1)/(n+p)}\zeta^p dxd\tau\leq$$

$$\leq(1/c^2)\left(1+\ln\left(\frac{M_k^\pm}{c}\right)\right)\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^{p^2(1-\kappa_1)/(n+p)}\chi[(H-k)_\pm>0]dxd\tau; \quad (5.31)$$

$$\iint\limits_{K_\rho\times(-\theta,t)}\Psi\Psi'^2|F|^{\frac{p}{p-1}}\zeta^p dxd\tau\leq\iint\limits_{K_\rho\times(-\theta,t)}(1+\Psi)\Psi'^2F^{\frac{p}{p-1}}\zeta^p dxd\tau\leq$$

$$\leq(1/c^2)\left(1+\ln\left(\frac{M_k^\pm}{c}\right)\right)\iint\limits_{K_\rho\times(-\theta,t)}|F|^{\frac{p}{p-1}}\chi[(H-k)_\pm>0]dxd\tau; \qquad (5.32)$$

Apply inequalities (5.31) and (5.32) to the appropriate terms in estimates (5.20)-(5.22), and in (5.24)-(5.26). After collecting together (5.20)-(5.22), (5.23)-(5.26), and (5.30) this yields the following integral inequality

$$\int\limits_{K_\rho\times\{t\}}\Psi^2\zeta^p dx\leq\int\limits_{K_\rho\times\{-\theta\}}\Psi^2\zeta^p dx+C\iint\limits_{K_\rho\times(-\theta,t)}\Psi|\Psi'|^{2-p}|\nabla\zeta|^p dxd\tau+$$

$$+(C/c^2)\left(1+\ln\left(\frac{M_k^\pm}{c}\right)\right)\Bigg\{\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^{\frac{p^2(1-\kappa_1)}{(n+p)}}\chi[(H-k)_\pm>0]dxd\tau+$$

$$+\iint\limits_{K_\rho\times(-\theta,t)}|F|^{\frac{p}{p-1}}\chi[(H-k)_\pm>0]dxd\tau+\iint\limits_{K_\rho\times(-\theta,t)}|f|\chi[(H-k)_\pm>0]dxd\tau+$$

$$+\iint\limits_{K_\rho\times(-\theta,t)}|\nabla u,\nabla v,\nabla w|^\varepsilon\chi[(H-k)_\pm>0]dxd\tau\Bigg\}.$$

Since $t\in(-\theta,0)$ is arbitrary, after taking the supremum in t the latter inequality implies

$$\sup_{-\theta<t<0}\int\limits_{K_\rho\times\{t\}}\Psi^2\zeta^p dx\leq\int\limits_{K_\rho\times\{-\theta\}}\Psi^2\zeta^p dx+\widetilde{C}\iint\limits_{Q(\theta,\rho)}\Psi|\Psi'|^{2-p}|\nabla\zeta|^p dxd\tau+$$

$$+(\widetilde{C}/c^2)\left(1+\ln\left(\frac{M_k^\pm}{c}\right)\right)\Bigg\{\iint\limits_{Q(\theta,\rho)}|\nabla u,\nabla v,\nabla w|^{\frac{p^2(1-\kappa_1)}{(n+p)}}\chi[(H-k)_\pm>0]dxd\tau+$$

$$+\iint\limits_{Q(\theta,\rho)}|F|^{\frac{p}{p-1}}\chi[(H-k)_\pm>0]dxd\tau+\iint\limits_{Q(\theta,\rho)}|f|\chi[(H-k)_\pm>0]dxd\tau+$$

$$+\iint\limits_{Q(\theta,\rho)} |\nabla u, \nabla v, \nabla w|^{\varepsilon}\chi[(H-k)_{\pm} > 0]dxd\tau\Big\}. \tag{5.33}$$

Estimating the last four term in the right of (5.33) by Hölder's inequality, as in (5.15), we come down to (5.16). □

From Theorem 5.1 and Theorem 5.2, with the help of Lemma 2.2, Corollary 3.1, Proposition 3.3, and Lemma 4.2 of Chapter I of [8], it follows that a weak solution to system (2.1) is Hölder continuous within the interior of the domain Q. Namely, the following statement is valid

Theorem 5.3 (Interior Hölder continuity). *If function $H(x,t) \in C(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$ is bounded and satisfies inequalities (5.4) of Theorem 5.1 and (5.16) of Theorem 5.2 then there exist constants C and $\alpha \in (0,1)$ depending only upon the data, such that for all subdomains $Q' \subset Q$, for every pair of points $(x_1,t_1),(x_2,t_2) \in Q'$ holds*

$$|H(x_1,t_1) - H(x_2,t_2)| \leq C\left(\frac{|x_1-x_2| + |t_1-t_2|^{1/p}}{\operatorname{dist}(Q',\partial Q,p)}\right)^{\alpha},$$

with $\operatorname{dist}(Q',\partial Q,p) \equiv \inf\limits_{\substack{(x,t)\in Q'\\(y,s)\in\partial Q}} \left(|x-y| + |t-s|^{1/p}\right).$

For the proof of this theorem see [8, p. 41, Theorem 1.1].

Regularity up to the boundary. Let us introduce some additional notations. Introduce the quantities $D_k^{\pm}$ and put upon the set of levels the restrictions

$$D_k^{\pm} \equiv \operatorname{ess}\sup_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap Q} |(H-k)_{\pm}| \leq \delta \leq \delta_0, \tag{5.34a}$$

$$\begin{cases} k \geq \sup\limits_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap S} [\alpha g_1 + \beta g_2 + g_3] & \text{for testing function } (H-k)_+\zeta^p, \\ k \leq \sup\limits_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap S} [\alpha g_1 + \beta g_2 + g_3] & \text{for testing function } -(H-k)_-\zeta^p. \end{cases} \tag{5.34b}$$

δ_0 is a positive parameter from (5.2); define the logarithmic function

$$\Psi(D_k^{\pm},(H-k)_{\pm},c) \equiv \ln^+\left\{\frac{D_k^{\pm}}{D_k^{\pm} - (H-k)_{\pm} + c}\right\}, \quad c < D_k^{\pm}; \tag{5.35}$$

and introduce the subsets of $[x_0+K_\rho]\cap\Omega$, $B_{k,\rho}^{\pm}$ such that

$$B_{k,\rho}^{\pm}(\tau) \equiv \{x \in [x_0+K_\rho]\cap\Omega | (H(x,\tau)-k)_{\pm} > 0\}.$$

Theorem 5.4. *There are constants C and δ_0 determined only by the data such that for fixed $(x_0,t_0) \in S$ for every cylinder $[(x_0,t_0)+Q(\theta,\rho)]$ with θ so small that $t_0-\theta > 0$, and* *for every level k satisfying* (5.34a)-(5.34b) *the following inequalities hold*

$$\sup_{t_0-\theta<t<t_0} \int\limits_{[x_0+K_\rho]\bigcap\Omega} (H-k)^2_\pm \zeta^p(x,t)dx + C^{-1} \iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap Q} |\nabla(H-k)_\pm\zeta|^p dx d\tau \leq$$

$$\leq \int\limits_{[x_0+K_\rho]\bigcap\Omega} (H-k)^2_\pm \zeta^p(x,t_0-\theta)dx + C \iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap Q} (H-k)^p_\pm |\nabla\zeta|^p dx d\tau +$$

$$+C \iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap Q} (H-k)^2_\pm \zeta^{p-1}\zeta_t dx d\tau + C\left\{ \int\limits_{t_0-\theta}^{t_0} |B^\pm_{k,\rho}(\tau)|^{r/q} d\tau \right\}^{p(1+\kappa)/r}, \tag{5.36}$$

and, provided that the cut-off function is independent of t for $t \in (t_0-\theta, t_0)$,

$$\sup_{t_0-\theta<t<t_0} \int\limits_{[x_0+K_\rho]\bigcap\Omega} \Psi^2(D^\pm_k, (H-k)_\pm, c)(x,t)\zeta^p(x)dx \leq$$

$$\leq \int\limits_{[x_0+K_\rho]\bigcap\Omega} \Psi^2(D^\pm_k, (H-k)_\pm, c)(x,t_0-\theta)\zeta^p(x)dx +$$

$$+C \iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap Q} \Psi|\Psi_H(D^\pm_k, (H-k)_\pm, c)|^{2-p}|\nabla\zeta|^p dx d\tau +$$

$$+(C/c^2)\left(1+\ln\frac{D^\pm_k}{c}\right)\left\{ \int\limits_{t_0-\theta}^{t_0} |B^\pm_{k,\rho}(\tau)| d\tau \right\}^{p(1+\kappa)/q}. \tag{5.37}$$

The proof is a literary repeating of that to Theorem 5.1 and Theorem 5.2 with the only difference that we had to consider $D^\pm_k$ instead of $M^\pm_k$, and $B^\pm_{k,\rho}(\tau)$ instead of $A^\pm_{k,\rho}(\tau)$.

Initial regularity.

Theorem 5.5. *There are constants C and δ_0 determined but by the data such that for every $(x_0,t_0) \in Q$ and for every cylinder $[(x_0,t_0)+Q(\theta,\rho)]$ with $t_0-\theta \equiv 0$, if the cut-off function ζ is independent of t on $t \in (0,t_0)$, the following inequalities hold*

$$\sup_{0<t<t_0} \int\limits_{[x_0+K_\rho]} (H-k)^2_\pm \zeta^p(x,t)dx + \iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap Q} |\nabla(H-k)_\pm\zeta|^p dx d\tau \leq$$

$$\leq C \iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap Q} (H-k)^p_\pm |\nabla\zeta|^p dx d\tau + C\left\{ \int\limits_{0}^{t_0} |B^\pm_{k,\rho}(\tau)|^{r/q} d\tau \right\}^{p(1+\kappa)/r}, \tag{5.38}$$

and, moreover

$$\sup_{0<t<t_0} \int\limits_{[x_0+K_\rho]\bigcap\Omega} \Psi^2(D^\pm_k, (H-k)_\pm, c)(x,t)\zeta^p(x)dx \leq$$

$$\leq C \iint\limits_{[(x_0,t_0)+Q(\theta,\rho)]\bigcap Q} \Psi|\Psi_H(D^\pm_k, (H-k)_\pm, c)|^{2-p}|\nabla\zeta|^p dx d\tau +$$

$$+(C/c^2)\left(1+\ln\frac{D_k^{\pm}}{c}\right)\left\{\int_0^{t_0}|B_{k,\rho}^{\pm}(\tau)|d\tau\right\}^{p(1+\kappa)/q}, \tag{5.39}$$

where q,κ *satisfy* (5.1), k *fulfills* (5.34a) *and in addition the following restrictions are assumed* $k\geq\sup_{[x_0+K_\rho]\bigcap\Omega}H_0$ *for* $(H-k)_+$, $k\leq\sup_{[x_0+K_\rho]\bigcap\Omega}H_0$ *for* $(H-k)_-$.

The proof is analogous to that of Theorems 5.1 and Theorems 5.2.

Thus, summing up, from Theorem 5.4 and Theorem 5.5 we come down to the statement

Theorem 5.6 (Hölder continuity up to the boundary). *If* $H(x,t)$ *is from Theorem 5.3, satisfies inequalities* (5.36), (5.37) *from Theorem 5.4, and* (5.38), (5.39) *from Theorem 5.5; boundary data are Hölder continuous on* S *with exponent* $\widetilde{\alpha}_g^j$, *and initial data Hölder continuous in* $\overline{\Omega}$ *with exponent* $\widetilde{\alpha}_0^j$ *then there exist constants* c *and* $\alpha\in(0,1)$ *depending only upon the data of the problem, such that for every pair of points* $(x_1,t_1),(x_2,t_2)\in\bar{Q}$ *there holds*

$$|H(x_1,t_1)-H(x_2,t_2)|\leq c\left(|x_1-x_2|+|t_1-t_2|^{1/p}\right)^{\alpha}.$$

For the proof of this theorem see [8, p 42, Theorem 1.2]. Hence the Hölder continuity of the components of solution themselves immediately follows

$$\begin{aligned}||u||_{H^{\alpha,\alpha/p}}&=||u\Delta||_{H^{\alpha,\alpha/p}}/|\Delta|=||(\alpha_1u+\beta_1v+w)(\beta_2-\beta_3)-(\alpha_2u+\beta_2v+w)(\beta_1-\beta_3)+\\&+(\alpha_3u+\beta_3v+w)(\beta_1-\beta_2)||_{H^{\alpha,\alpha/p}}/|\Delta|=\\&=||(\beta_2-\beta_3)H_1-(\beta_1-\beta_3)H_2+(\beta_1-\beta_2)H_3||_{H^{\alpha,\alpha/p}}/|\Delta|\leq\\&\leq(|\beta_2-\beta_3|C_1+|\beta_1-\beta_3|C_2+|\beta_1-\beta_2|C_3)/|\Delta|;\end{aligned}$$

$$\begin{aligned}||v||_{H^{\alpha,\alpha/p}}&=||v\Delta||_{H^{\alpha,\alpha/p}}/|\Delta|=||(\alpha_1u+\beta_1v+w)(\alpha_2-\alpha_3)-(\alpha_2u+\beta_2v+w)(\alpha_1-\alpha_3)+\\&+(\alpha_3u+\beta_3v+w)(\alpha_1-\alpha_2)||_{H^{\alpha,\alpha/p}}/|\Delta|=\\&=||(\alpha_2-\alpha_3)H_1-(\alpha_1-\alpha_3)H_2+(\alpha_1-\alpha_2)H_3||_{H^{\alpha,\alpha/p}}/|\Delta|\leq\\&\leq(|\alpha_2-\alpha_3|C_1+|\alpha_1-\alpha_3|C_2+|\alpha_1-\alpha_2|C_3)/|\Delta|;\end{aligned}$$

$$\begin{aligned}||w||_{H^{\alpha,\alpha/p}}&=||(\alpha_1u+\beta_1v+w)-\alpha_1u-\beta_1v||_\infty\leq||H_1-\alpha_1u-\beta_1v||_{H^{\alpha,\alpha/p}}\leq\\&\leq||H_1||_{H^{\alpha,\alpha/p}}+|\alpha_1|||u||_{H^{\alpha,\alpha/p}}+|\beta_1|||v||_{H^{\alpha,\alpha/p}},\end{aligned}$$

where $||\cdot||_{H^{\alpha,\alpha/p}}$ denotes the Hölder norm

$$||u||_{H^{\alpha,\alpha/p}}=\sup_{(x_1,t_1),(x_2,t_2)\in Q}\frac{|u(x_1,t_1)-u(x_2,t_2)|}{\left(|x_1-x_2|+|t_1-t_2|^{1/p}\right)^{\alpha}}.$$

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