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AN EXACT FUNCTIONAL-ANALYTIC REPRESENTATION OF SOLUTIONS TO A HAMILTON-JACOBI EQUATION OF RICCATI TYPE

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A generalized characteristic method [5, 7] preliminaries are described and used for studying functional-analytic solutions to the Cauchy problem of noncanonical Hamilton-Jacobi equations. If the equation is of the Riccati type solutions are obtained and investigated making use of the classical Leray-Schauder fixed point theory.

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Описаны основания обобщенного метода характеристик, использованого для изучения функционально-аналитических решений задачи Коши неканонических уравнений Гамильтона-Якоби. В случае уравнения типа Риккати решения получены и исследованы с помощью классической теории неподвижной точки Лере-Шаудера.

1. The problem setting. Consider a canonical Hamilton-Jacobi equation

$$u_t + ||u_x||^2/2 = 0, \tag{1.1}$$

where $||\cdot||$ is the standard norm in the Euclidean space $\mathbb{E}^n := (\mathbb{R}^n, <\cdot,\cdot>)$, and try to construct its exact functional-analytic [8, 6, 7] generalized solutions $u : \mathbb{E}^n \times \mathbb{R}_+ \to \mathbb{R}$, satisfying the Cauchy condition

$$u|_{t=+0} = u_0 \tag{1.2}$$

for a given function $u_0 : \mathbb{E}^n \to \mathbb{R}$. One can easily enough to state, making use of the characteristic method [8, 6, 10, 13], that equation (1.1) possesses for smooth Cauchy data $u_0 \in C^1(\mathbb{E}^n; \mathbb{R})$ an exact functional-analytic generalized solution in the form

$$u(x,t) = u_0(y) + \frac{1}{2t}||x-y||^2, \tag{1.3}$$

where a vector $y := y(x,t) \in \mathbb{E}^n$ for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$ satisfies the following determining equation

$$\partial u_0(y)/\partial y - (x-y)/t = 0. \tag{1.4}$$

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It was proved in [11, 8, 6] that in a more general case of convex and below semicontinuos Cauchy data $u_0 \in BSC_{(c)}(\mathbb{R}^n;\mathbb{R})$ the expression (1.3) allows the completely equivalent to (1.3) so called Hopf-Lax type represntation

$$u(x,t) = \inf_{y\in\mathbb{E}^n}\{u_0(y) + \frac{1}{2t}||x-y||^2\}, \tag{1.5}$$

being a generlaized [8] solution to the Hamilton-Jacobi equation (1.1). The solution (1.3) satisfies the following natural asymptotic "viscosity" property: $\lim_{t\to\infty} u(x,t) = \inf\{u_0(y) : y \in \mathbb{E}^n\}$ for almost all $x \in \mathbb{E}^n$. The Cauchy problem (1.1) and (1.2) for functions $u_0 \in BSC(\mathbb{E}^n;\mathbb{R})\cap C^1(\mathbb{E}^n;\mathbb{R})$ possesses a unique functional-analytic representation for its generalized solutions and satisfying the standard viscosity property. Below we consider a generalized geometric Monge charactersitic method of solving noncanonical Hamilton-Jacobi type equations in the general form $u_t + H(x,t;u,u_x) = 0$ wiith Cauchy data (1.2), and give two examples, where $H := H_1 = \frac{1}{2} < u_x, u_x >$ and $H := H_2 = \frac{1}{2}(< u_x, u_x > + u^2)$, an evolution Riccati type equation.

2. A generalized Monge characteristic method: short backgrounds.

2.1. A noncanonical Hamilton-Jacobi equation

$$u_t + H(x,t;u,u_x) = 0$$

within the geometric Monge approach [5, 10, 7, 1, 13] can be considered as a characteristic surface $S_H \subset \mathbb{E}^n\times\mathbb{R}_+\times\mathbb{E}^n\times\mathbb{R}^2$ in the following form:

$$\begin{aligned} S_H \ &: = \{(x,t;u,p,\sigma) \in \mathbb{E}^n\times\mathbb{R}_+\times\mathbb{E}^n\times\mathbb{R}^2 : \sigma + \bar{H}(x,t;u,p) = 0, \\ \bar{H}(x,t;u,p) \ &: = H(x,t;u,\pi)|_{\pi=\psi(x;u,p)}\}, \end{aligned} \tag{2.1}$$

where a related Monge cones parametrization is taken as $\pi = \psi(x;u,p) \in \mathbb{E}^n$, $(x;u,p) \in S_H$, for some nondegenerate mapping $\psi \in C^1(\mathbb{R}^{2n+1};\mathbb{E}^n)$, that is $det(\partial\psi/\partial p) \neq 0$. We denoted here $u_t := \sigma \in \mathbb{R}$, $u_x := \pi \in \mathbb{E}^n$ for $(x,t) \in \mathbb{E}^n\times\mathbb{R}_+$ and $< \cdot,\cdot >$ is the standard scalar product in the Euclidean vector space $\mathbb{E}^n := (\mathbb{E}^n, < \cdot,\cdot >)$. As an example of equation (1.1), we will put below

$$H_1(x,t;u,\pi) := \frac{1}{2} < \pi,\pi >, \quad H_2(x,t;u,\pi) := \frac{1}{2}(< \pi,\pi > + u^2). \tag{2.2}$$

Now one can construct [5, 10, 7] a solution surface $\bar{S}_H \subset \mathbb{E}^n\times\mathbb{R}_+\times\mathbb{R}$, which is compatible with the characteristic surface $S_H \subset \mathbb{E}^n\times\mathbb{R}_+\times\mathbb{E}^n\times\mathbb{R}^2$, and satisfying the following Cartan's compatibility relationships:

$$du = \sigma dt + < \psi, dx >, \qquad < d\sigma, \wedge dt > + < d\psi, \wedge dx > = 0, \tag{2.3}$$

holding upon $\bar{S}_H$ along any solution $u : \mathbb{E}^n\times\mathbb{R}_+ \to \mathbb{R}$.

2.2. Consider now a related characteristic vector field on the surface S_H in the form

$$\begin{aligned} dx/d\tau \ &= \ \mu^{(1|1)}\partial\bar{H}/\partial p, \qquad dp/d\tau = -\mu^{(1|1),*}(\partial\bar{H}/\partial x + \psi\partial\bar{H}/\partial u - \bar{H}\partial\psi/\partial u), \\ du/d\tau \ &= \ < p, \mu^{(1|1)}\partial\bar{H}/\partial p > - \bar{H}, \quad d\sigma/d\tau = \sigma\partial\bar{H}/\partial u, \quad dt/d\tau = 1, \end{aligned} \tag{2.4}$$

under the following mixed Cauchy data:

$$\begin{array}{rcl} x|_{\tau=0} & = & y := y(x,t), \qquad x|_{\tau=t} = x, \quad , \\ u|_{\tau=0} & = & u_0(y), \qquad p|_{\tau=0} = p_0(y), \\ \psi(x_0;u_0,p_0) & : & = \partial u_0(x_0)/\partial x, \end{array} \tag{2.5}$$

where, by definition, the tensor $\mu^{(1|1)} := (\partial\psi/\partial p)^{*,-1}$ and the condition $u|_{\tau=t} = u(x,t)$ for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$ is assumed, owing to the relationships (2.3), to be satisfied. The equations (2.4) fully ensure [3, 10, 5, 13, 7] the invariance of the characteristic S_H and fulfillment of the related Cartan's compatibility conditions (2.3). The problem (2.4) and (2.5) is, actually, an inverse one subject to the corresponding initial data at $\tau = 0 \in \mathbb{R}_+$, if the corresponding data at $\tau = t \in \mathbb{R}_+$ are *a priori* given. The general solution to this inverse problem gives rise [10, 5] to the following exact functional-analytic expression:

$$u(x,t) = (u_0(y) + \mathcal{P}(x,t;y))\big|_{y=x_0(x,t)}, \tag{2.6}$$

where a vector $y := x_0(x,t) \in \mathbb{E}^n$, defined by (2.5), must belong to the set of points $U(x) \subset \mathbb{E}^n$, reachable at $\tau = t \in \mathbb{R}_+$ by the vector field (2.4) starting at $x \in \mathbb{E}^n$, and the kernel

$$\mathcal{P}(x,t;y) := \int_0^t \mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u)d\tau, \tag{2.7}$$

$\dot{x} := dx/d\tau$, is defined by the Lagrangian function

$$\mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u) := \left(< \psi, \mu^{(1|1)}\partial\bar{H}/\partial p > -\bar{H}(x,\tau;,p)\right)\big|_{\dot{x}=\mu^{(1|1)}\partial\bar{H}/\partial p} \tag{2.8}$$

for all reachable points $(x(\tau),\tau) \in U(x) \times \mathbb{R}_+$. The functional analytic expression (2.6) for the Cauchy problem (1.1) and (1.2) gives rise [7] right away to its generalized solution in the Hopf-Lax type form, since the tensor field $\mu^{(1|1)} \in C^1(\mathbb{E}^n \times \mathbb{R}_+; \mathbb{E}^n \otimes \mathbb{E}^n)$ on the corresponding characteristic surface S_H is symplectic [2, 1, 4, 12]. This means, in particular, that the differential 2-form

$$\omega^{(2)} :=< d\psi, \wedge dx > |_{\Sigma_H} \tag{2.9}$$

is nondegenerate on the characteristic strip $\Sigma_H \subset S_H$. We can now easily obtain from (2.4), (2.5) and (2.6) that the Hamilton-Jacobi equation (1.1) with Cauchy data $u_0 \in C^1(\mathbb{E}^n;\mathbb{R})$ possesses solution (1.3) for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, defined by the vector $y = x_0(x,t) \in \mathbb{E}^n$, which solves the determining equation (1.4).

3. Examples.

Example 3.1. The canonical Hamiltonian function $H_1(x,t;u,\pi) := \frac{1}{2} < \pi, \pi >$.

3.1. Actually, the Lagrangian function (2.8) corresponding to this case, owing to the (2.2), equals the expression

$$\mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u) = ||\dot{x}||^2/2, \tag{3.1}$$

where, owing to equations (2.4) and conditions (2.5), the following relationships

$$x(\tau) - y = \psi(x_0; u_0, p_0)\tau, \quad \psi(x_0; u_0, p_0) := \partial u_0(y)/\partial y, \tag{3.2}$$

hold for all $\tau \in \mathbb{R}_+$. Therefore, the expressions (2.6) and (2.7) give rise to such an exact solution to the equation (1.1):

$$u(x,t) = u_0(y) + \frac{1}{2t}||x-y||^2\Big|_{y=x_0(x,t)}, \tag{3.3}$$

where a vector $y := x_0(x,t) \in \mathbb{E}^n$ for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$ satisfies the determining equation (1.4), easily following from (3.2), that is

$$\partial u_0(y)/\partial y - (x-y)/t = 0. \tag{3.4}$$

The result obtained one can interestly interpret making use of the Lagrangian variational principle: the system (2.4) of charactersitic Hamiltonian vector fields is completely equivalent to the variational equation $\delta\mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u)/\delta x = 0$, solving the extremum problem

$$\tilde{u}(x,t) = \inf_{\substack{x\in C^2([0,t];\mathbb{R}) \\ \{x(0)=y\in\mathbb{E}^n, x(t)=x\in\mathbb{E}^n\}}} (u_0(y) + \int_0^t \mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u)d\tau) \tag{3.5}$$

in the space of smooth functions $x \in C^2([0,t];\mathbb{R})$, $t \in \mathbb{R}_+$.

3.2. A very important fact concerning the constructed function (3.5) consists in that it satisfies [14] the Hamilton-Jacobi equation for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$ exactly the same as (1.1), that is

$$\partial\tilde{u}/\partial t + ||\tilde{u}_x||^2/2 = 0 \tag{3.6}$$

under the evident initial condition

$$\tilde{u}|_{t=0} = u_0. \tag{3.7}$$

Thereby, we can identify the obtained function (2.4) with our solution to the Hamilton-Jacobi equation (1.1) with Cauchy data (1.2), that is $u = \tilde{u}$. Since the infimum problem (3.5) is equivalent to that

$$u(x,t) = \inf_{y\in U(x)} \{u_0(y) + \int_0^t \mathcal{L}(\tau; x(\tau), \dot{x}(\tau)|u)d\tau\} \tag{3.8}$$

in the space of solutions to the equations (2.4) with Cauchy data (2.5), we obtain right away its well known [8, 6] the Hopf-Lax type representation

$$u(x,t) = \inf_{y\in\mathbb{R}^n)} \{u_0(y) + \frac{||x-y||^2}{2t}\}, \tag{3.9}$$

where we took into account that $U(x) = \mathbb{R}^n$ and the kernel (2.7) equals

$$\mathcal{P}(x,t;y) = \frac{||x-y||^2}{2t} \tag{3.10}$$

for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$. If the Cauchy data $u_0 \in BSC_{(c)}(\mathbb{E}^n;\mathbb{R}) \cap C^1(\mathbb{E}^n;\mathbb{R})$, a vector $y = y(x,t) \in \mathbb{E}^n$, solving the problem (3.9), satisfies evidently the equation (3.4), thereby confirming the result, obtained previously.

Example 3.2. The evolutionary Hamilton-Jacobi equation of the Riccati type: the Hamiltonian function $H := H_2 = \frac{1}{2}(<\pi,\pi> + u^2)$.

3.3. The corresponding characteristic vector fields equations on the surface S_H, parametrized as $\pi := \psi(x;u,p) \in \mathbb{E}^n$, $(x;u,p) \in S_H$, are given as

$$\begin{array}{c} dx/d\tau = \psi(x;u,p), \\ (\partial\psi/\partial p)dp/d\tau = -u\psi, \\ du/d\tau = \frac{1}{2}(<\psi,\psi> -u^2), \end{array} \tag{3.11}$$

where parameter $\tau \in \mathbb{R}$. It is easy to see that a suitable parametrization of S_H can be given by the mapping $\psi(x;u,p) := up/||p||$ for all $(x;u,p) \in S_H$. Then, the system of equations (3.11) transforms into

$$\begin{array}{c} dx/d\tau = up/||p||, \\ (\mathbf{I}-p\otimes p/||p||^2)dp/d\tau = -up/||p||, \\ du/d\tau = \frac{1}{2}(<\psi,\psi> -u^2) = 0, \end{array} \tag{3.12}$$

meaning that the solution $u : \mathbb{E}^n \times \mathbb{R}_+ \to \mathbb{R}$ along the corresponding charactersitic strip $\Sigma_H \subset S_H$ is exactly constant. Solve now the inverse Cauchy problem related with system (3.12):

$$\begin{array}{rcl} x|_{\tau=0} & = & x_0 := y(x,t), \quad x|_{\tau=t} = x, \\ p|_{\tau=0} & = & p_0(y) = ||p_0(y)||\partial \ln u_0(y)/\partial y \end{array} \tag{3.13}$$

for any fixed $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$. Before doing this, note that the latter condition of (3.12) imposes the natural constraint on the class of Cauchy data (1.2):

$$u_0(x) = c_0 \exp[\int_0^1 \frac{<p_0(x\lambda),x>}{||p_0(x\lambda)||} d\lambda] \tag{3.14}$$

for all $x \in \mathbb{E}^n$. Here $c_0 \in \mathbb{R}_+$ is arbitrary positive constant and a vector-function $p_0 \in C^1(\mathbb{E}^n;\mathbb{E}^n)$ satosfies the following compatibility condition:

$$p_0' - p_0^{\prime,*} = ||p_0||^{-2}[p^{\prime,*}\ (p_0 \otimes p_0) - (p_0 \otimes p_0)\ p_0'], \tag{3.15}$$

where $p' := \partial p_0(x)/\partial x \in \mathbb{E}^n \otimes \mathbb{E}^n$, $x \in \mathbb{E}^n$, is the usual Jacobi matrix. We will call further this class of Cauchy data as $Ricc(\mathbb{E}^n;\mathbb{R})$. Having assumed now that the Cauchy data satisfy conditions (3.14) and (3.15), one can easily solve the inverse Cauchy problem (3.13) for equations (3.12) as follows:

$$\begin{array}{rcl} y & = & x + \frac{p_0(y)}{||p_0(y)||}(1-e^{-ut})u^{-1}, \\ u & = & u_0(x + \frac{p_0(y)}{||p_0(y)||}(1-e^{-ut})u^{-1}), \end{array} \tag{3.16}$$

holding for all $(x,t) \in \mathbb{E}^n \times \mathbb{R}_+$. The system of functional-analytic equations (3.16) is, evidently, equivalent to the following fixed point problem

$$\mathcal{P}(y;u) = (y;u) \tag{3.17}$$

for $(y;u) \in C(\mathbb{E}^n;\mathbb{E}^n) \times C(\mathbb{E}^n;\mathbb{R})$ and any fixed parameter $t \in \mathbb{R}_+$, where the mapping

$$\mathcal{P}(y;u) := (x + \frac{p_0(y)}{||p_0(y)||}(1-e^{-ut})u^{-1}; u_0(y)) \tag{3.18}$$

with $u_0 \in Ricc(\mathbb{E}^n;\mathbb{R})$. The following lemma simply follows from the expressions (3.16) and (3.14).

Lemma 3.3. *The mapping (3.18) is continuous with respect to the norm*

$$||(y;u)||_{\mathbb{E}^n} := \sup_{x\in\mathbb{E}^n}(\exp(-||x||)|u(x)|) + \sup_{x\in\mathbb{E}^n}(||y||),$$

where $(y;u) \in C(\mathbb{E}^n;\mathbb{E}^n) \times C(\mathbb{E}^n;\mathbb{R})$.

It is easy to see, owing to (3.14) and (3.16), that the mapping (3.18) leaves the convex set $B_1(\mathbb{E}^n;\mathbb{E}^n) \times C_{\exp}(\mathbb{E}^n;\mathbb{R}_+)$ invariant for any fixed $t \in \mathbb{R}_+$, where we denoted

$$\begin{aligned} B_1(\mathbb{E}^n;\mathbb{E}^n) \ &: \ = \{y \in C(\mathbb{E}^n;\mathbb{E}^n) : \sup_{x\in\mathbb{E}^n} ||y(x) - x|| \le 1\}, \\ C_{\exp}(\mathbb{E}^n;\mathbb{R}) \ &: \ = \{u \in C(\mathbb{E}^n;\mathbb{R}) : \sup_{x\in\mathbb{E}^n} \exp(-||x||)|u(x)| \le e\}. \end{aligned} \tag{3.19}$$

Take now any compact $K \subset \mathbb{E}^n$ and consider the mapping (3.17) reduced to points $x \in K$. Then the mapping (3.17) also will leave the set $B_1(K;\mathbb{E}^n) \times C_{\exp}(K;\mathbb{R}_+)$ invariant for any fixed $t \in \mathbb{R}_+$. Concerning the constructed set $B_1(K;\mathbb{E}^n) \times C_{\exp}(K;\mathbb{R}_+)$ the following lemma, owing to the definitions (3.19), holds.

Lemma 3.4. *The set of functions* $B_1(K;\mathbb{E}^n) \times C_{\exp}(K;\mathbb{R}_+)$ *is bounded and uniformly continuous with respect to the induced norm* $||(y;u)||_K := \sup_{x\in K}(\exp(-||x||)|u(x)|) + \sup_{x\in K}(||y||)$ *for any compact* $K \subset \mathbb{E}^n$ *and all fixed* $t \in \mathbb{R}_+$.

From the above lemma 3.4 we find, owing to the standard Riesz criterium [9, 8], the next corollary.

Corollary 3.5. *The set of functions* $B_1(K;\mathbb{E}^n) \times C_{\exp}(K;\mathbb{R}_+)$ *is convex and compact for any compact* $K \subset \mathbb{E}^n$ *and all fixed* $t \in \mathbb{R}_+$.

Thereby, one can apply to the constructed above reduced mapping

$$\mathcal{P} : B_1(K;\mathbb{E}^n) \times C_{\exp}(K;\mathbb{R}_+) \to B_1(K;\mathbb{E}^n) \times C_{\exp}(K;\mathbb{R}_+) \tag{3.20}$$

the classical Leray-Schauder theorem [8, 9] about the existence of a fixed point $(\bar{y};\bar{u}) \in B_1(K;\mathbb{E}^n) \times C_{\exp}(K;\mathbb{R}_+)$, such that $\mathcal{P}(\bar{y};\bar{u}) = (\bar{y};\bar{u})$. Really, owing to Lemma 3.3 and Corollary 3.4, the following theorem holds.

Theorem 3.6. (Leray-Schauder type) *The mapping (3.18), being continuous in the Banach space* $C(K;\mathbb{E}^n) \times C(K;\mathbb{R})$, *leaves the convex and compact set of functions* $B_1(K;\mathbb{E}^n) \times C_{\exp}(K;\mathbb{R}_+)$ *invariant and possesses there a fixed point.*

The obtained fixed point of the mapping (3.20) solves, obviously, our Cauchy problem (1.2) for any given function $u_0 \in Ricc(\mathbb{E}^n;\mathbb{R})$, finishing our calculations. In particular, we showed that solutions to the canonical Hamilton-Jacobi equation (1.1) with Cauchy data (1.2) can be constructed effectively in the functional-analytic form, using the generalized charactersitic method. Within those exact functional-analytic solutions can exist such ones, whose asymptotic properties possess nontrivial asymptotic viscosity behavior, being of importance important for applications.

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