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ON CLASSIFICATION OF SIGMA HEREDITARY DISCONNECTED SPACES

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We introduce and investigate a transfinite dimension function p with the property: $\mathrm{p}(X) = 0$ iff X is a punctiform space. We prove that the dimension p classify sigma hereditary disconnected spaces. Using this fact we show that each complete sigma hereditary disconnected space is a C-space in the sense of Haver.

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В статье определяется и исследуется размерностная функция p, которая владеет следующим свойством: $\mathrm{p}(X) = 0$ тогда и только тогда, когда X является вполне разрывным пространством. Доказано, что размерность p определена для всех сигма наследственно несвязных пространств. Используя этот факт, доказано, что каждое полное сигма наследственно несвязное пространство является C-пространством в смысле Хэйвера.

1. Introduction. All spaces under the discussion are assumed to be metrizable and separable.

A subset L of a space X is a *partition* in X if there exist two disjoint non-empty open in X subsets U and V such that $L = X \setminus (U \cup V)$.

The transfinite dimension function trt was introduced in [1] for finite numbers and in [2] in general case as follows:

$\mathrm{trt}(\varnothing) = -1$; $\mathrm{trt}(X) = 0$ if X is a point.

Let $\mathrm{card}(X) \geq 2$ and α be an ordinal number ≥ 0; if for any closed subset M of X with $\mathrm{card}(M) \geq 2$, there exists a partition T in M with $\mathrm{trt}(T) < \alpha$ then we say $\mathrm{trt}(X) \leq \alpha$.

If $\mathrm{trt}(X) \leq \alpha$ is true and $\mathrm{trt}(X) \leq \beta$ is false for all $\beta < \alpha$ then $\mathrm{trt}(X) = \alpha$.

If $\mathrm{trt}(X) \leq \alpha$ is false for all α then $\mathrm{trt}(X) = \infty$

Let us remark that $\mathrm{trt}(X) = 0$ iff X is hereditary disconnected.

In this paper we introduce a transfinite dimension coinciding with trt on compacta.

The paper is organized as follows: in Section 2 we give neccesary terminology and notation, in Section 3 we introduce the dimension p and study its connection with trt. In 4 we prove that for each sigma hereditary disconnected space X an ordinal number $\mathrm{p}(X)$ is defined. Using this fact we prove that each complete sigma hereditary disconnected space is C-space in the sense of Haver. In section 5 we ask some questions.

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2.Terminology and notation. Let us remind that the space X is called *hereditary disconnected (punctiform)* if X does not contain any connected subspace (continuum) of cardinality larger then one [3, Section 1.4].

A space X is called *countable dimensional* (in the following more shortly c.d.) when $X = \bigcup_{i=1}^{\infty} X_i$ where each X_i is 0-dimensional [4]. Analogously we obtain the notion of *sigma hereditary disconnected* (in the following s.h.d.) space replacing in the previous definition 0-dimensionality to hereditary disconnectedness. S.h.d. spaces were considered in [5] and [6].

A space X has property C(briefly is a C-space) if for each sequence $\{\alpha_n, n \in \mathbb{N}\}$ of open coverings of X there exists a sequence $\{\beta_n, n \in \mathbb{N}\}$ of open disjoint families such that each family β_n refines α_n and $\bigcup_{i=1}^{\infty} \beta_n$ covers X [7].

For a family β of subsets of a metric space (X, d) we put $\operatorname{mesh}_d(\beta) = \operatorname{Sup}\{d(x, y) \mid x, y \in B, B \in \beta\}$. A space X is called a C-space in the sense of Haver, abbreviated C-Ha, iff there exists an admissible metric d on X such that for each sequence $\{\epsilon_n > 0, n \in \mathbb{N}\}$ there exists a sequence $\{\beta_n, n \in \mathbb{N}\}$ of open disjoint families such that $\operatorname{mesh}_d(\beta_n) < \epsilon_n$ and $\bigcup_{i=1}^{\infty} \beta_n$ covers X [8].

Let us remark that each C-space is a C-Ha-space and both versions coincide in compact case. In Section 4 we will show that each complete s.h.d. space is a C-Ha-space.

3. Dimension p**.** In this section we introduce a transfinite dimension p and study its properties.

Definition 3.1. We define the *dimension* p by transfinite induction as follows:

1. $\mathrm{p}(\varnothing) = -1$; $\mathrm{p}(X) = 0$ if X is a point.
2. let $\operatorname{card}(X) \geq 2$ and α be an ordinal number ≥ 0; if for any compact subset M of X with $\operatorname{card}(M) \geq 2$, there exists a partition T in M with $\mathrm{p}(T) < \alpha$ then we say $\mathrm{p}(X) \leq \alpha$.
3. $\mathrm{p}(X) = \alpha$ if and only if $\mathrm{p}(X) \leq \alpha$ and is false that $\mathrm{p}(X) \leq \beta$ for any ordinal number $\beta < \alpha$.
4. $\mathrm{p}(X) = \infty$ if and only if for every ordinal number α we have $\mathrm{p}(X) > \alpha$, and then we say that $\mathrm{p}(X)$ does not exist.

Let us remark that $\mathrm{p}(X) = 0$ iff X is punctiform. The next proposition follows from definition.

Proposition 3.2. *Let X be a space and $A \subseteq X$. Then* $\mathrm{p}(A) \leq \mathrm{p}(X)$.

Consider a continuous mapping $f\colon X \to Y$ from a non-empty space X to a non-empty space Y. Denote $\mathrm{p}(f) = \operatorname{Sup}\{\mathrm{p}(f^{-1}(y)) \mid Y\}$.

Proposition 3.3. $\mathrm{p}(X) \leq \mathrm{p}(f) + \mathrm{p}(Y)$.

Proof. We use induction by $\mathrm{p}(Y)$. If $\mathrm{p}(Y) = -1$ the proof is trivial. Let α be an ordinal number ≥ 1. Suppose the proposition is true for each ordinal number $\beta < \alpha$ and consider a space Y with $\mathrm{p}(Y) \leq \alpha$. Let $A \subset X$ be a compact subset. If $f(A) = \{y_0\}$ then there exists a partition L in A with $\mathrm{p}(L) < \mathrm{p}(f) \leq \mathrm{p}(f) + \mathrm{p}(Y)$. If $\operatorname{card} f(A) \geq 2$ then there exists a partition M in $f(A)$ with $\mathrm{p}(M) < \mathrm{p}(Y)$. Then $S = f^{-1}(M) \cap A$ is a partition in A and by induction assumption we have $\mathrm{p}(S) \leq \mathrm{p}(f|A) + \mathrm{p}(M) < \mathrm{p}(f) + \mathrm{p}(Y)$. The proposition is proved. □

Corollary 3.4. $\mathrm{p}(X \times W) \leq \min\{\mathrm{p}(X) + \mathrm{p}(W), \mathrm{p}(W) + \mathrm{p}(X)\}$.

Proposition 3.5. *Let X be a space. Then*

1. $\mathrm{p}(X) \leq \mathrm{trt}(X)$;
2. $\mathrm{p}(X) = \mathrm{Sup}\{\mathrm{trt}\, K \mid K$ *is a compact subset of* $X\}$;
3. *if X is a complete space and $\mathrm{p}(X) < \infty$ then X is a C-Ha space.*

Proof. The parts 1 and 2 one can obtain using induction and the fact that hereditary disconnectedness and punctiformness are equivalent in the realm of compact spaces.

Let us prove the last assertion of the proposition. Let X be a complete space with $\mathrm{p}(X) < \infty$. Then for each compact subset $K \subset X$ we have $\mathrm{trt}\, K < \infty$.Then K is a C-compactum [2] and X is a C-Ha-space [9]. The proposition is proved. □

4. Sigma hereditary disconnected spaces. In this section we investigate relation between the dimension p and the class of s.h.d. space.

Lemma 4.1. *Let X be a space with cardinality ≥ 2 and $A \subset X$ is a hereditary disconnected subspace. Then there exists a partition L of X with $L \cap A = \varnothing$.*

Proof. The case when A ia a one-point set is trivial. Let $\mathrm{card}\, A \geq 2$. Then $A = L_1 \cup L_2$, where L_1 and L_2 are nonempty disjoint and open in A sets. Then $\mathrm{Cl}_X L_1 \cap L_2 = L_1 \cap \mathrm{Cl}_X L_2 = \varnothing$ and by [3, 1.2.8 Lemma] we can choose open in X sets U and W such that $L_1 \subset U$, $L_2 \subset W$ and $U \cap W = \varnothing$. Then $L = X \setminus (U \cup W)$ is the desired partition. □

Theorem 4.2. *For each s.h.d. space X we have $\mathrm{p}(X) < \infty$.*

Proof. Let X be a s.h.d. space. Represent $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is hereditary disconnected. Assume the contrarty $\mathrm{p}(X) = \infty$. Then there exists a compactum K_0 such that for each partition K_1 in K_0 we have $\mathrm{p}(K_1) = \infty$. By the Lemma 4.1 we can choose K_1 with $K_1 \cap X_1 = \varnothing$. Since for the compactum K_1 we have $\mathrm{p}(K_1) = \infty$ and $K_1 \cap X_2 = \varnothing$, we can repeat the above argument and, by induction, obtain a decreasing sequence $K_1 \supset K_2 \supset \ldots$ of compacta K_i such that $K_i \cap X_i = \varnothing$ and $\mathrm{p}(K_i) = \infty$. As the sets K_i are nonempty we have $\bigcap_{i=1}^{\infty} K_i \neq \varnothing$. On the other hand $\bigcap_{i=1}^{\infty} K_i \subset \bigcap_{i=1}^{\infty} (X \setminus X_i) = X \setminus \bigcup_{i=1}^{\infty} X_i = \varnothing$.

This contradiction shows that $\mathrm{p}(X) < \infty$. □

The following corrolary follows from the above theorem and Proposition 3.5.

Corollary 4.3. *Each complete s.h.d. space is C-Ha. In particularly each compact s.h.d. space is a C-compactum.*

Let us remark that it was proved in [5] that each s.h.d. compact space is weakly infinite dimensional.

5. Questions. We begin this section with a question stated by V.Chatyrko in [2]:

Question 5.1. Does there exists a compact space X_α such that $\mathrm{trt}\, X_\alpha = \alpha$ for each ordinal number α?

Let us remark that by Theorem 4.2 there exists an ordinal number α_0 such that $\mathrm{p}\,(N_\omega) = \alpha_0$ where N_ω is the Nagata space universal for countable dimensional spaces [10]. It follows from Proposition 3.5 that for each countable dimensional compactum K we have $\mathrm{trt}\,(K) \leq \alpha_0$. Thus starting from $\alpha_0 + 1$ all compacta X_α must be uncountable dimensional.

R.Pol has proved that there exists a weakly infinite dimensional compactum X which is uncountable dimensional [11]. Let us remark that X is in fact a s.h.d. compactum.

Question 5.2. Does there exist a space which distinquish the classes of C-spaces and s.h.d. spaces?

Question 5.3. Does there exist a compact C-space X such that X is not s.h.d. and $\mathrm{p}\,(X) < \infty$?

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