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ON MULTIVALUED IMPULSIVE DIFFERENTIAL EQUATIONS*

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Consider the following boundary value problem for multivalued differential equations with the interface (impulsive) conditions:

$$\frac{dx(t)}{dt} - A(t)x(t) \in F(t, x(t), \quad Lx(t) = r,$$

$$x(t_i^+) - B_i x(t_i^-) = C_i, \quad i \in \{1, 2, \cdots, \ell\}.$$

Under suitable hypotheses and the use of a fixed point theorem for multivalued mappings, the existence of solutions for the above boundary value problem is proved. An application to the obtained result to a periodic problem is given.

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Рассматривается следующая краевая задача для дифференциальных включений с импульсными воздействиями:

$$\frac{dx(t)}{dt} - A(t)x(t) \in F(t, x(t), \quad Lx(t) = r,$$

$$x(t_i^+) - B_i x(t_i^-) = C_i, \quad i \in \{1, 2, \cdots, \ell\}.$$

Доказано существование решения указанной граничной задачи. При этом используются соответствующие гипотезы и теорема о неподвижной точке для многозначных отображений. Приведено применение полученных результатов к периодической задаче.

1. Introduction. This paper is concerned with the existence of the solutions to the following boundary value problem for multivalued differential equations with interface conditions:

$$\frac{dx(t)}{dt} - A(t)x(t) \in F(t, x(t)) \tag{1}$$

$$x(t_i^+) - B_i x(t_i^-) = C_i, \quad i \in \{1, 2, \dots, \ell\}, \tag{2}$$

$$Lx(t) = r, \tag{3}$$

where A is a measurable and integrable operator on $\triangle = [a, b] \subset R$, $F(t, x)$ is a mapping on $\triangle \times R^n$, B_i are real $n \times n$ nonsingular matrices and $C_i \in R_n$, L is a linear and continuous

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mapping defined on a subspace $C^n - C(\triangle, R^n)$, the space of all continuous mappings of $\triangle$ into R^n, and $r \in R^m$. The internal interface points $t_1, t_2, \dots, t_\ell$ are such that

$$a < t_1 < t_2 < \cdots < t_\ell < b,$$

and

$$x(t_i^+) = \lim_{t \to t_i^+} x(t), \quad x(t_i^-) = \lim_{t \to t_i^-} x(t).$$

We apply the fixed point theorem for multivalued mappings due to Ky Fan [7], to prove the existence of solutions for the problem $((1),(2),(3))$ under suitable hypotheses.

The analogous problem for ordinary differential equations has been treated by Gonnelli [4] and by Shahin [9] in Banach spaces. The multivalued differential problem $((1),(3))$ has been studied under a variety of hypotheses on the function $F(t,x)$ and the operator L (see for example, Lasota-Opial [8], Grandolfi [5], and Cellina [2]). For further information on multivalued differential equations refer to [1], [3], [6], and [10].

An application of of our main result to a periodic problem is given.

2. Notations and Hypotheses. Let $\triangle = [a,b] \subset R$ and let $G = \{t_i \in \triangle, i \in \{1,\dots,\ell\}\} \subset \triangle$ with $a < t_1 < t_2 < \cdots < t_\ell < b$. Then the subintervals are denoted by $\delta_0 = [a,t_1), \delta_1 = (t_1,t_2), \dots, \delta_i = (t_i, t_{i-1}), \delta_\ell = (t_\ell, b)$. The closure of these subintervals are denoted by $\bar{\delta}_i$, for $i \in \{0,\dots,\ell\}$ and $\delta = \bigcup_i \delta_i = [a,b] \setminus G$.

Let $C^n(G)$ be the space of all continuous functions from δ into R^n, such that $x(t_i^+)$, and $x(t_i^-)$ exist and are finite. Then $C^n(G)$ is a Banach space under the norm

$$\|x\|_G = \sup_{t \in \delta} \|x(t)\|,$$

where $\|\cdot\|$ denotes the norm in R^n [11].

Further, we employ the following notation.

a) $A\colon \triangle \to A^\circ$ (the algebra of all real $n \times n$ matrices) is a measurable and integrable mapping;

b) $F\colon \triangle \times R^n \to cf(R^n)$ (the set of all closed convex nonempty subsets of R^n)

c) $B_i \in A^\circ, \quad C_i \in R^n, \quad i \in \{1,2,\dots,\ell\}$;

d) $L\colon C^n(G) \to R^m$ $(m \le n)$ is a linear and continuous operator, $r \in R^m$.

e) $L^{p,n}$ is the space of all p–times measurable and integrable mappings of $\triangle$ into R^n.

f) $\mathcal{F}(x)$ is the set of all measurable functions $f\colon \triangle \to R^n$ such that for every $x \in C^n$, $f(t) \in F(t, x(t))$ a.e., on $\triangle$.

We say that $x \in C^n(G)$ is a solution of the interface problem $((1),(2),(3))$ if it is absolutely continuous on δ and it satisfies (1) a.e. on $\triangle$, and satisfies the conditions (2) and (3).

Further, we suppose that the following hypotheses hold:

(H_1) The function $F(t,x)$ satisfies the following conditions:

(i) for each $x \in R^n$ the functions $F(\cdot, x)$ is measurable on $\triangle$;

(ii) for each $t \in \triangle$ the function $F(t,\cdot)$ is upper semicontinuous (u.s.c.) on R^n in the sense that if $\{x_i\} \to x_0$, $\{y_i\} \to y_0$, and $y_i \in F(t, x_i)$, then $y_0 \in F(t, x_0)$;

(iii) there exist measurable and integrable functions $\alpha(t)$, $\beta(t)$, defined on $\triangle$ such that

$$\|f(t,x)\| \le \alpha(t) + \beta(t)\|x\|.$$

(H_2) A, L, r are such that the problem

$$\dot{x} - A(t)x = f(t) \tag{1*}$$

$$x(t_i^+) - B_i x(t_i^-) = C_i, \quad i \in \{1, 2, \dots, \ell\}; \tag{2}$$

$$Lx = r \tag{3}$$

has solutions for all $f \in \mathcal{L} \cap \mathcal{F}(x)$ where $\mathcal{L}$ is a linear manifold of $L^{1,n}$.
(H_3) $\mathcal{F}(\bar{x}) \in cf(\mathcal{L})$ for all $\bar{x} \in C^n(G)$ such that $L\bar{x} = r$.
(H_4) Ker $L = \{x \in C^n(G) : Lx = 0\} \neq \{0\}$.

Now we introduce the matrix valued function [4].

$$V(t,a) = U(t,a)\Lambda(t)$$

with $\Lambda(t) = I$, Λ_1, $\Lambda_2\Lambda_1$, $\dots$, $\Lambda_\ell \dots \Lambda_1$ for $t \in \delta_0, \delta_1, \delta_2, \dots, \delta_\ell$ respectively, and

$$\Lambda_i = U(a.t_i)B_iU(t_i,a), \quad U(t,a) = I + \int_a^t A(\tau)U(\tau,a)d\tau.$$

Put $\psi_i = U(t_i,a)C_i$ with

$$\psi(t) = 0,\ \psi_1,\ \Lambda_2\psi_1 + \psi_2,\ \Lambda_3\Lambda_2\psi_1 + \Lambda_3\psi_2 + \psi_3, \dots$$

for $t \in \delta_0, \delta_1, \delta_2, \delta_3, \dots, \delta_\ell$ respectively.

Let L_0 be the restriction of L to the subspace of $C^n(G)$ of solutions of the homogeneous system

$$\dot{x} - A(t)x(t) = 0,$$
$$x(t^+) - B_i x(t^-) = 0, \quad i \in \{1, 2, \dots, \ell\}.$$

Let $L_V = L_0V$, so that L_V has the generalized inverse L_V^* such that $L_V L_V^* L_V = L_V$, $L_V^* \colon R^m \to R^n$ is a linear operator.

In addition to the hypotheses $(H_1), (H_2), (H_3)$, and (H_4) we assume that
(H_5) $Lx = r \Rightarrow (I_m - L_V L_V^*)(r - LU(t,a)\psi(t) - L\int_a^t V(t,s)F(s,x(s))\,ds) = 0.$

3. Preliminaries. The following lemma will be needed for our main result.

Lemma 1. *Let the hypotheses $(H_1), (H_2), (H_3), (H_4)$ and (H_5) hold. Then the problem*

$$\dot{x} - A(t)x(t) = f(t), \quad f \in \mathcal{F}(x),$$

has a solution given by

$$x(t) = \Gamma f(t) + Hr,$$

where

$$\Gamma \colon f \to \Gamma f = V(t,a)L_V^* L \int_a^t V(t,s)f(s)\,ds + \int_a^t V(t,s)f(s)\,ds,$$

and

$$Hr = V(t,a)[x_0 + L_V^* r] + V(t,a)L_V^* LU(t,a)\psi(t) + U(t,a)\psi(t),$$

with $x_0 \in \mathrm{Ker}(L_V)$.

Proof. The proof can be found in [4] and [9]. □

Let T_0 be the operator

$$T_0\colon x \to T_0(x) = \Gamma f(t) + Hr.$$

Lemma 2. *The operator T_0 is a compact operator.*

Proof. From our assumptions it is easy to see that the operator T_0 transforms the bounded set into uniformly bounded set. Now we show that the sequence $\{T_0 x_n(t)\}$ defined on $\bar{\delta}_i$ by

$$T_0 x_n(t) = T_0 x_n(t), \quad t \in \delta,$$
$$T_0 x_n(t_i) = T_0 x_n(t_i^+),$$
$$T_0 x_n(t_{i+1}) = T_0 x_n(t_{i+1}^-),$$

is equicontinuous on $\bar{\delta}_i$. In fact, for t', $t'' \in \bar{\delta}_i$, $t' < t''$, we have

$$\left\| \int_a^{t''} V(t'', s) f(s)\, ds - \int_a^{t'} V(t', s) f(s)\, ds \right\|_G \leq$$
$$\|V(t'', t') - I\|_G \int_a^{t'} \|V(t', s)\|_G \|f(s)\|\, ds + \int_{t'}^{t''} \|V(t'', s)\|_G \|f(s)\|\, ds.$$

Since $\|V(t, a)\|_G \leq \|U(t, a)\|_G \|\Lambda(t)\|_G$, L_V^* is a linear operator and from our assumptions on L, A, and F, the equicontinuity of $\{T_0 x_n(t)\}$ on $\bar{\delta}_i$ follows. Applying the criteria for compactness in $C^n(G)$ by Weiss and Pham [11], analogous to Ascoli's theorem, it follows that T_0 is a compact operator. □

Lemma 3. *The correspondence $x \to \mathcal{F}(x)$ defines a bounded mapping of $C^n(G)$ into $cf(L^{1,n})$. If Γ is a linear and continuous mapping of $L^{1,n}$ into $C^n(G)$, then $\Gamma\mathcal{F}$ maps $C^n(G)$ into $cf(c^n(G))$ and is upper semi-continuous.*

Proof. For the proof the reader is referred to the proof of Theorem 1 and Theorem 2 of Lasota and Opial [8]. □

Let $T\colon\ C^n(G) \to cf(C^n(G))$ defined by $T(x) = \Gamma\mathcal{F}(x) + \{Hr\}$.

Remark. From Lemma 3, the operator T is upper semi-continuous.

4. Main Result. The main result is contained in the following:

Theorem 1. *Let the hypotheses (H_i), $i \in \{1, \ldots, 5\}$, hold. Then the problem $((1), (2), (3))$ admits at least one solution.*

Proof. We show that there exist $B_\rho = \{x \in C^n(G) : \|x\|_g \leq \rho\}$ such that $T(B_\rho) \subset B_\rho$. In fact,

$$\|Tx\|_G = \|Hr + \Gamma\mathcal{F}(x)\|_G \leq \|Hr\|_G + \|\Gamma\|(\alpha_0 + \beta_0\|X\|_G),$$

where $\alpha_0 = \int_\triangle \alpha(t)\, dt$, $\beta_0 = \int_\triangle \beta(t)\, dt$, and $\|\Gamma\|$ denotes the norm of Γ. Assuming that $\rho = (\|Hr\|_G + \alpha_0\|\Gamma\|)(1 - \beta_0\|\Gamma\|)^{-1}$, with $\beta_0\|\Gamma\| < 1$, it follows that $T(x) \subset B_\rho$ for any $x \in B_\rho$, i.e., $T(B_\rho) \subset b_\rho$, where $T(B_\rho) = \bigcup_{x \in B_\rho} T(x)$.

Now we prove that

$$T(\overline{\text{co } T(B_\rho)}) \subset \overline{\text{co } T(B_\rho)},$$

where co D denotes the convex hull of the set D. In fact, we have

$$\overline{\text{co } T(B_\rho)} \subset \overline{\text{co } (B_\rho)} = B_\rho$$

and then

$$T(\overline{\text{co } T(B_\rho)}) \subset T(B_\rho) \subset \overline{\text{co } T(B_\rho)}.$$

From Lemma 3, $\mathcal{F}(B_\rho)$ is bounded and since Γ is a compact mapping, it follows that $T(B_\rho)$ is relatively compact. Since T is an upper semi-continuous and transforms into itself the convex and compact set $\overline{\text{co } T(B_\rho)}$, then applying Ky Fan fixed point theorem [7], it follows that there exists $x \in \overline{\text{co } T(B_\rho)}$ and then $x \in C^n(G)$ satisfies (2)and (3) so that $x \in T(x)$ and this proves the theorem. □

As an application we give the following example:

Example. We search for the T-periodic solution $x(t)$ of the system

$$\dot{x} - A(T)x \in F(t,x) \tag{1.p}$$

$$x(\tau^+) - Bx(\tau^-) = C, \quad \tau = \nu T, \quad 0 < \nu < 1, \tag{2.p}$$

$$Lx = x(T) - x(0) = 0, (3.p)$$

where A is a $n \times n$ periodic function of period T, $F(t,x)\colon R \times R^n \to cf(R^n)$, such that $F(t+T,x) = F(t,x)$, and $F(t,x)$ satisfies the assumption (H_1), and $\tau = \nu T$ is the unique discontinuity point. In this case $\triangle = [0,T]$, $\delta_0 = [0,\tau)$, $\delta_1 = (\tau, T]$, $m = n$, $r = 0 \in R^n$. Now if we consider the system

$$\begin{gathered} \dot{x} - A(T)x \in f(t) \\ x(\tau^+) - Bx(\tau^-) = C, \\ Lx = x(T) - x(0) = 0, \end{gathered}$$

corresponding to the system ((1*), (2), (3)), then we have

$$V(t,0) = \begin{cases} U(t,0); & t \in \delta_0 \\ U(t,\tau)BU(\tau,0); & t \in \delta_1, \end{cases}$$

and

$$\psi(t) = \begin{cases} 0; & t \in \delta_0 \\ U(t,0)C; & t \in \delta_1. \end{cases}$$

Further the operator Γ is given by

$$\Gamma f = V(t,0)L_V^* \left[\int_0^\tau U(T,\tau)BU(\tau,s)f(s)\,ds + \int_\tau^T U(T,s)f(s)\,ds \right] + \int_0^t V(t,s)f(s)\,ds,$$

$$\Gamma f = \begin{cases} U(T,0)L_V^* \big[\int_0^\tau U(T,\tau)BU(\tau,s)f(s)\,ds + \int_\tau^T U(T,s)f(s)\,ds \big] + \\ \qquad + \int_0^t U(t,s)f(s)\,ds; & t \in \delta_0 \\ U(t,\tau)BU(\tau,0)L_V^* \big[\int_0^\tau U(T,\tau)BU(\tau,s)f(s)\,ds + \int_\tau^T U(T,s)f(s)\,ds \big] + \\ \qquad + \int_0^\tau U(T,s)f(s)\,ds + \int_\tau^t U(t,\tau)BU(\tau,s)f(s)\,ds; & t \in \delta_1, \end{cases}$$

and the operator H is given by

$$Hr = \begin{cases} U(t,0)x_0 + U(t,0)L_V^* U(T,0)U(\tau,0)C; & t \in \delta_0, \\ U(t,\tau)BU(\tau,0)\big[x_0 + L_V^* U(T,0)U(\tau,0)C\big] + U(t,0)U(\tau,0)C; & t \in \delta_1, \end{cases}$$

where $x_0 \in \operatorname{Ker}[U(T,\tau)BU(\tau,0) - I]$.

Let C_0 be the space of T-periodic continuous functions from R into R^n and Q be the space of periodic solutions of the homogeneous system

$$\dot{x} - A(t)x = 0,$$
$$x(\tau^+) - Bx(\tau^-) = o,$$
$$x(T) - x(0) = 0.$$

Assume that $F(t, v(t)) \in cf(Z_0)$ for every $v \in C_0$, where $Z_0 = \{f \in L^{1,n} : \int_0^T q^*(t)f(t)\,dt = 0,\ q \in Q\}$. All the conditions of Theorem 1 are satisfied here, thus the problem $((1.p), (2.p), (3.p))$ has at least one T-periodic solution $x(t)$.

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