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BINARY DARBOUX TRANSFORMATIONS AND SCATTERING OPERATOR FOR NONSTATIONARY SYSTEM OF DIRAC EQUATIONS

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We construct a scattering operator for a nonstationary system of Dirac equations using Darboux binary transformations.

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Мы строим оператор рассеяния для нестационарной системы уравнений Дирака, используя бинарные преобразования Дарбу.

Let us consider Dirac system of the form

$$LY = 0,\ \ Y = \begin{pmatrix} Y_1(x,y) \\ Y_2(x,y) \end{pmatrix},\ \ L = \begin{pmatrix} \alpha\partial_y - \partial_x & u_1(x,y) \\ u_2(x,y) & \alpha\partial_y + \partial_x \end{pmatrix}, \tag{1}$$

where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$, α – arbitrary constant, $u_1(x,y)$, $u_1(x,y)$ be decreasing functions in infinity.

The direct and inverse scattering problems for the Dirac system (1), where variable $y \equiv t$ (time) and constant $\alpha = 1$, were studied by L. Nizhnik in [1]. In that paper L.Nizhnik described the properties of the scattering operator S which was defined by the equation

$$b = Sa, \tag{2}$$

where

$$b = \begin{pmatrix} b_1(y+\alpha x) \\ b_2(y-\alpha x) \end{pmatrix},\ a = \begin{pmatrix} a_1(y+\alpha x) \\ a_2(y-\alpha x) \end{pmatrix},$$

were the asymptotics of the solution $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ of the system $LY = 0$ (1):

$$Y_1(x,y) = a_1(y+\alpha x) + 0(1),$$

$$Y_2(x,y) = b_2(y-\alpha x) + 0(1),\ \ x \to +\infty;$$

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$$Y_1(x,y) = b_1(y+\alpha x) + 0(1),$$
$$Y_2(x,y) = a_2(y-\alpha x) + 0(1), \quad x \to -\infty.$$

The result of the given work is the explicit construction of the scattering operator S (2) for the system (1) using the binary Darboux transformations [2] for system (1).

Let us expose some results of [2].

Let 1. $Y = \begin{pmatrix} Y_1(x,y) \\ Y_2(x,y) \end{pmatrix}$ be an arbitrary and

$$\varphi = \begin{pmatrix} \varphi_1(x,y) \\ \varphi_2(x,y) \end{pmatrix} := \begin{pmatrix} \varphi_{11} & \dots & \varphi_{1K} \\ \varphi_{21} & \dots & \varphi_{2K} \end{pmatrix}$$

some fixed $(2\times K)$-matrix solutions of system (1).

2. $\psi = \begin{pmatrix} \psi_1(x,y) \\ \psi_2(x,y) \end{pmatrix} := \begin{pmatrix} \psi_{11} & \dots & \psi_{1K} \\ \psi_{21} & \dots & \psi_{2K} \end{pmatrix}$ is some fixed $(2\times K)$-matrix solutions of transposed system

$$L^{\tau}\psi = 0, \quad L^{\tau} = \begin{pmatrix} -\alpha\partial_y + \partial_x & u_2 \\ u_1 & -\alpha\partial_y - \partial_x \end{pmatrix}.$$

Easy to prove that $(K\times K)$-matrix functions

$$P[\psi,\varphi] := \psi^{\top}\varphi, \quad Q = \alpha^{-1}\psi^{\top}\sigma_3\varphi, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfy the relation

$$P_y = Q_x. \tag{3}$$

In consequence of (3) follows the existence (up to arbitrary constant) of the matrix potential $\Omega := \Omega[\psi,\varphi]$:

$$d\Omega[\psi,\varphi] = P\,dx + Q\,dy.$$

By the direct computation the following proposition is proved.

Proposition 1. *The integral operator of binary Darboux transformations W defined on the space of solutions of system (1) by the formula*

$$WY = Y - \varphi(C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,Y] = \hat{Y}, \tag{4}$$

where

$$\Omega[\psi,\varphi] := \int_{M_0}^{M} \psi^{\top}\varphi\,dx + \alpha^{-1}\psi^{\top}\sigma_3\varphi\,dy, \tag{5}$$

$M_0 = (x_0,y_0) \in \mathbb{R}^2$, $M = (x,y) \in \mathbb{R}^2$, *$C$ be some $(K\times K)$-constant matrix.*

Operator W transforms operator (1) into operator $\hat{L} = WLW^{-1}$ of the form

$$\hat{L} = \begin{pmatrix} \alpha\partial_y - \partial_x & \hat{u}_1 \\ \hat{u}_2 & \alpha\partial_y + \partial_x \end{pmatrix}, \tag{6}$$

where

$$\hat{u}_1 = u_1 - \varphi_1(C + \Omega[\psi,\varphi])^{-1}\psi_2^{\top},$$

$$\hat{u}_2 = u_2 + \varphi_2(C + \Omega[\psi, \varphi])^{-1}\psi_1^{\top}.$$

The function $\hat{Y} := WY$ is the general solution of system $\hat{L}\hat{Y} = 0$ with the coefficients (potentials) $\hat{u}_1$, $\hat{u}_2$ (6).

Let $Y_0 = \begin{pmatrix} Y_1(x, y) \\ Y_2(x, y) \end{pmatrix}$ be an arbitrary solution and

$$\varphi = \begin{pmatrix} \varphi_1(y + \alpha x) \\ \varphi_2(y - \alpha x) \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1(y + \alpha x) \\ \psi_2(y - \alpha x) \end{pmatrix} \tag{7}$$

be a fixed $(2 \times K)$-matrix solutions respectively of systems

$$L_0 Y_0 = 0, \;\; L_0\varphi = 0, \;\; L_0^{\tau}\psi = 0,$$

where

$$L_0 = \begin{pmatrix} \alpha\partial_y - \partial_x & 0 \\ 0 & \alpha\partial_y + \partial_x \end{pmatrix}, \; L_0^{\tau} = \begin{pmatrix} -\alpha\partial_y + \partial_x & 0 \\ 0 & -\alpha\partial_y - \partial_x \end{pmatrix}. \tag{8}$$

On the assumption (7), (8) from formulas (4), (6) we obtain

$$Y = WY_0 = Y_0 - \varphi(C + \Omega[\psi, \varphi])^{-1}\Omega[\psi, Y_0],$$

$$L = WL_0W^{-1},$$

$$\begin{aligned} u_1 &= -\varphi_1(C + \Omega[\psi, \varphi])^{-1}\psi_2^{\top}, \\ u_2 &= \varphi_2(C + \Omega[\psi, \varphi])^{-1}\psi_1^{\top}. \end{aligned} \tag{9}$$

In this paper we construct the scattering operator S (2) for Dirac operator L (1) with potentials u_1, u_2 (9).

From formulas (5), (7) we have the matrix potentials $\Omega_1[\psi, \varphi]$ (putting $x_0 = x$, $y_0 = -\infty$) and $\Omega_2[\psi, \varphi]$ (putting $x_0 = x$, $y_0 = +\infty$):

$$\begin{aligned} \Omega_1[\psi, \varphi] &= C_1 + \alpha_1^{-1}\int_{-\infty}^{y} \left[\psi_1^{\top}(s + \alpha x)\varphi_1(s + \alpha x) - \psi_2^{\top}(s - \alpha x)\varphi_2(s - \alpha x)\right] ds, \\ \Omega_2[\psi, \varphi] &= C_2 + \alpha_1^{-1}\int_{+\infty}^{y} \left[\psi_1^{\top}(s + \alpha x)\varphi_1(s + \alpha x) - \psi_2^{\top}(s - \alpha x)\varphi_2(s - \alpha x)\right] ds, \end{aligned} \tag{10}$$

where C_1, C_2 be some $(K \times K)$-constant matrices.

Let $Y_0 = \begin{pmatrix} a_1(y + \alpha x) \\ a_2(y - \alpha x) \end{pmatrix}$.

From formulas (9), (10) we obtain solution of system (1) of the form

$$Y = W_1 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = W_1 a, \tag{11}$$

where

$$W_1 = I - \alpha^{-1} \begin{pmatrix} \varphi_1(y + \alpha x)\Delta_1^{-1}\int_{-\infty}^{y} \psi_1^{\top}(s + \alpha x)\cdot ds & -\varphi_1(y + \alpha x)\Delta_1^{-1}\int_{-\infty}^{y} \psi_2^{\top}(s - \alpha x)\cdot ds \\ \varphi_2(y - \alpha x)\Delta_1^{-1}\int_{-\infty}^{y} \psi_1^{\top}(s + \alpha x)\cdot ds & -\varphi_2(y - \alpha x)\Delta_1^{-1}\int_{-\infty}^{y} \psi_2^{\top}(s - \alpha x)\cdot ds \end{pmatrix},$$

$$\Delta_1 = C_1 + \alpha^{-1} \int_{-\infty}^{y} \left[\psi_1^\top(s+\alpha x)\varphi_1(s+\alpha x) - \psi_2^\top(s-\alpha x)\varphi_2(s-\alpha x)\right] ds.$$

In a way analogous to that, let $Y_0 = \begin{pmatrix} b_1(y+\alpha x) \\ b_2(y-\alpha x) \end{pmatrix}$, we obtain

$$Y = W_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = W_2 b, \qquad (12)$$

where

$$W_2 = I - \alpha^{-1} \begin{pmatrix} \varphi_1(y+\alpha x)\Delta_2^{-1} \int_{+\infty}^{y} \psi_1^\top(s+\alpha x)\cdot ds & -\varphi_1(y+\alpha x)\Delta_2^{-1} \int_{+\infty}^{y} \psi_2^\top(s-\alpha x)\cdot ds \\ \varphi_2(y-\alpha x)\Delta_2^{-1} \int_{+\infty}^{y} \psi_1^\top(s+\alpha x)\cdot ds & -\varphi_2(y-\alpha x)\Delta_2^{-1} \int_{+\infty}^{y} \psi_2^\top(s-\alpha x)\cdot ds \end{pmatrix},$$

$$\Delta_2 = C_2 + \alpha^{-1} \int_{+\infty}^{y} \left[\psi_1^\top(s+\alpha x)\varphi_1(s+\alpha x) - \psi_2^\top(s-\alpha x)\varphi_2(s-\alpha x)\right] ds.$$

From (9), (10) we obtain: $\Delta_1 = \Delta_2$ and

$$C_2 = C_1 + \alpha^{-1} \int_{-\infty}^{+\infty} \left[\psi_1^\top(s+\alpha x)\varphi_1(s+\alpha x) - \psi_2^\top(s-\alpha x)\varphi_2(s-\alpha x)\right] ds.$$

Direct calculation shows that W_1^{-1}, W_2^{-1} are given by

$$W_1^{-1} = I + \alpha^{-1} \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix},$$

where

$$\omega_{11} = \varphi_1(y+\alpha x) \int_{-\infty}^{y} \Delta_1^{-1}(s,x)\psi_1^\top(s+\alpha x)\cdot ds, \quad \omega_{12} = -\varphi_1(y+\alpha x) \int_{-\infty}^{y} \Delta_1^{-1}(s,x)\psi_2^\top(s-\alpha x)\cdot ds,$$

$$\omega_{21} = \varphi_2(y-\alpha x) \int_{-\infty}^{y} \Delta_1^{-1}(s,x)\psi_1^\top(s+\alpha x)\cdot ds, \quad \omega_{22} = -\varphi_2(y-\alpha x) \int_{-\infty}^{y} \Delta_1^{-1}(s,x)\psi_2^\top(s-\alpha x)\cdot ds,$$

and

$$W_2^{-1} = I + \alpha^{-1} \begin{pmatrix} \upsilon_{11} & \upsilon_{12} \\ \upsilon_{21} & \upsilon_{22} \end{pmatrix},$$

where

$$\upsilon_{11} = \varphi_1(y+\alpha x) \int_{+\infty}^{y} \Delta_2^{-1}(s,x)\psi_1^\top(s+\alpha x)\cdot ds, \quad \upsilon_{12} = -\varphi_1(y+\alpha x) \int_{+\infty}^{y} \Delta_2^{-1}(s,x)\psi_2^\top(s-\alpha x)\cdot ds,$$

$$\upsilon_{21} = \varphi_2(y-\alpha x)\int\limits_{+\infty}^{y} \Delta_2^{-1}(s,x)\psi_1^{\top}(s+\alpha x)\cdot ds,\quad \upsilon_{22} = -\varphi_2(y-\alpha x)\int\limits_{+\infty}^{y} \Delta_2^{-1}(s,x)\psi_2^{\top}(s-\alpha x)\cdot ds.$$

Taking into account (10), (11), (12) we have

$$Y = W_1 a = W_2 b. \tag{13}$$

The scattering operator given by $b = Sa$ (2) using (13) is then a composition of the binary Darboux-type transformation operators of the form

$$S = W_2^{-1}W_1,$$

$$S = I - \alpha^{-1}\begin{pmatrix} \varphi_1(y+\alpha x)\int\limits_{-\infty}^{+\infty} C_2^{-1}\psi_1^{\top}(s+\alpha x)\cdot ds & -\varphi_1(y+\alpha x)\int\limits_{-\infty}^{+\infty} C_2^{-1}\psi_2^{\top}(s-\alpha x)\cdot ds \\ \varphi_2(y-\alpha x)\int\limits_{-\infty}^{+\infty} C_2^{-1}\psi_1^{\top}(s+\alpha x)\cdot ds & -\varphi_2(y-\alpha x)\int\limits_{-\infty}^{+\infty} C_2^{-1}\psi_2^{\top}(s-\alpha x)\cdot ds \end{pmatrix}.$$

In a way analogous to that we obtain

$$S^{-1} = W_1^{-1}W_2,$$

$$S^{-1} = I + \alpha^{-1}\begin{pmatrix} \varphi_1(y+\alpha x)\int\limits_{-\infty}^{+\infty} C_1^{-1}\psi_1^{\top}(s+\alpha x)\cdot ds & -\varphi_1(y+\alpha x)\int\limits_{-\infty}^{+\infty} C_1^{-1}\psi_2^{\top}(s-\alpha x)\cdot ds \\ \varphi_2(y-\alpha x)\int\limits_{-\infty}^{+\infty} C_1^{-1}\psi_1^{\top}(s+\alpha x)\cdot ds & -\varphi_2(y-\alpha x)\int\limits_{-\infty}^{+\infty} C_1^{-1}\psi_2^{\top}(s-\alpha x)\cdot ds \end{pmatrix}.$$

Similar results for Dirac operator in cone variables are obtained in papers [3,4].

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