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ON THE HYPERSPACE OF ROTORS IN CONVEX POLYGONS

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A rotor in a polygon is a closed convex curve that can be completely rotated inside this polygon so that, in all its positions, it touches all the sides of the polygon. We prove that the hyperspace of all rotors (respectively, of all smooth rotors) in a regular polygon is homeomorphic to the Hilbert cube (respectively, the separable Hilbert space). In the case when the polygon is a square (i.e. for convex curves of constant width) we show that the hyperspace of spherical rotors is homeomorphic to the set of finite sequences in the Hilbert cube.

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Ротором в многоугольнике называется замкнутая выпуклая кривая, которую можно полностью повернуть в этом многоугольнике так, что во всех своих положениях она касается всех сторон многоугольника. Доказано, что гиперпространство всех роторов (соответственно всех гладких роторов) в правильном многоугольнике гомеоморфно гильбертову кубу (соответственно сепарабельному гильбертову пространству). В случае, когда многоугольник является квадратом (т.е. для выпуклых кривых постоянной ширины) мы показываем, что соответствующее гиперпространство сферических роторов гомеоморфно пространству финитных последовательностей в гильбертовом кубе.

1. Introduction. The hyperspaces of compact convex sets were first considered in [1]. The main result of [1] stated that the hyperspace of compact convex sets in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to a punctured Hilbert cube. Some other results on topological properties of the hyperspace of compact convex bodies and its subspaces are considered in [2]-[4].

A *rotor* in a polygon is a closed convex curve that can be completely rotated inside this polygon so that, in all its positions, it touches all the sides of the polygon. See [5]–[7] for various constructions of rotors. It is known ([8]) that the polygon under consideration must be either a regular one or a rhombus. The latter case corresponds to the convex curves of constant width. In [9] (see also [10]) it is proved that the hyperspace of compact convex sets of constant width d lying in a square of size d is homeomorphic to the Hilbert cube $Q = \{(x_i)_{i=1}^{\infty} \mid x_i \in [0,1]\}$ (the space Q is endowed with the metric d, $d((x_i),(y_i)) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$. The aim of this note is an extension of this result to the case of rotors in arbitrary regular polygons. Our main result states that the hyperspace of rotors in regular polygons is also homeomorphic to the Hilbert cube.

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Note that rotors exist also in some polyhedra in higher dimensions (see [11]). However, the results of [11] demonstrate that the hyperspaces of rotors in the polyhedra of dimension ≥ 3 are finite-dimensional, unless these polyhedra are parallellotopes in which case the hyperspaces of rotors coincide with those of bodies of constant width.

In the sequel, we fix a polygon P as above.

Let $\mathcal{R}$ denote the set of all rotors in P. The set $\mathcal{R}$ is naturally identified with a subset $\mathrm{cc}(\mathbb{R}^2)$ of all nonempty compact convex subsets in $\mathbb{R}^2$ with the topology generated by the Hausdorff metric, d_H,

$$d_H(A,B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}.$$

Using the fact that the hyperspace of the rotors inscribed in P is homeomorphic to the Hilbert cube, we are able to prove that the hyperspace of smooth rotors is homeomorphic to the Hilbert space. One more result, which is proved, however, under the restriction that P is the square, describes the topology of the rotors that consist of finite number of circular arcs.

By $\langle\cdot,\cdot\rangle$ we denote the inner product in $\mathbb{R}^2$. It is sometimes convenient to identify $\mathbb{R}^2$ and $\mathbb{C}$; to any point $(a,b) \in \mathbb{R}^2$ there corresponds $a + bi \in \mathbb{C}$. By S^1 we denote the unit circle, $S^1 = \{z \in \mathbb{R}^2 \mid \|z\| = 1\}$. By "dimension" we mean the covering dimension (see, e.g. [12]). By ANR we denote the class of absolute neighborhood retracts, i.e. the metrizable spaces which are neighborhood retracts of every enveloping metrizable space.

2. Result. The main result of this note is the following one.

Theorem 2.1. *The hyperspace $\mathcal{R}$ of rotors in a regular polygon P is homeomorphic to the Hilbert cube.*

Proof. First, we show that the hyperspace $\mathcal{R}$ is a convex subset in $\mathrm{cc}(\mathbb{R}^2)$ with respect to the convex combination given by the formula

$$tA + (1-t)B = \{ta + (1-t)b \mid a \in A,\ b \in B\}, \quad A, B \in \mathrm{cc}(\mathbb{R}^2),\ t \in [0,1].$$

Indeed, $A \in \mathcal{R}$ if there exists a continuous map $f_A\colon S^1 \to \mathbb{R}^2$ such that, for every $z \in S^1$ the set $zA + f_A(z)$ is inscribed into P.

We reformulate the condition in terms of the support functions. By $h_A\colon S^1 \to \mathbb{R}$ we denote the support function of a convex symmetric curve A, $h_A(z) = \max\{\langle a, z\rangle \mid a \in A\}$, $z \in S^1$.

For $j \in \{0, 1 \ldots, n-1\}$, let $c_j = \left(\cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n}\right)$.

Without loss of generality, one may assume that P is a regular n-gon centered at the origin, $n \in \{3, 4, \ldots\}$, located so that $h_P(c_j) = c$, where c is the radius of the inscribed in P circumference.

Then A is a rotor in P if and only if there exists a continuous function $f_A\colon S^1 \to \mathbb{R}^2$ such that

$$h_{zA+f_A(z)}(c_j) = h_{zA}(c_j) + \langle c_j, f_A(z)\rangle = h_A(c_j z^{-1}) + \langle c_j, f_A(z)\rangle = c,$$

for every $z \in S^1$ and $j \in \{0, 1 \ldots, n-1\}$.

Given $A, B \in \mathcal{R}$ and $t \in [0,1]$, we have

$$\begin{aligned} h_{tA+(1-t)B}(c_j z^{-1}) + \langle t f_A(z) + (1-t) f_B(z), c_j\rangle &= \\ = t h_A(c_j z^{-1}) + (1-t) h_B(c_j z^{-1}) + t\langle f_A(z), c_j\rangle + (1-t)\langle f_B(z), c_j\rangle &= \\ = t(h_A(c_j z^{-1}) + \langle f_A(z), c_j\rangle) + (1-t)(h_B(c_j z^{-1}) + \langle f_B(z), c_j\rangle) = tc + (1-t)c = c&, \end{aligned}$$

whence $tA + (1-t)B \in \mathcal{R}$.

Like in [1], we consider $\mathcal{R} \subset \mathrm{cc}(\mathbb{R}^2)$ affinely (with respect to the convex combination) embedded in a normed space.

We are going to prove that the space $\mathcal{R}$ is infinite-dimensional. Suppose the contrary, then the space $\mathcal{R}' = \mathcal{R} + \mathbb{R}^2 = \{A + z \mid A \in \mathcal{R},\ z \in \mathbb{R}^2\}$ is finite-dimensional.

It is known from [5] that there exists a noncircular rotor, A, formed by circular arcs. We may suppose now that the origin is the center of such an arc, which is supposed to be a maximal (with respect to inclusion) circular arc in A.

We naturally identify every support function on S^1 with the corresponding 2π-periodic function defined on $\mathbb{R}$. Without loss of generality, we may assume that the function $h_A\colon \mathbb{R} \to \mathbb{R}$ possesses the properties:

$$h_A|[0, t_0] = \mathrm{const} = c > 0,\ \ h_A((t_0, t_1]) \subset (0, c).$$

Let u_s, $s \in \mathbb{R}$, denote the shift operator, $(u_s f)(t) = f(t+s)$, $t \in \mathbb{R}$. It suffices to show that the set $\mathcal{A} = \{u_s h_A \mid s \in \mathbb{R}\}$ is infinite-dimensional.

Given $n \in \mathbb{N}$, let $\tau > 0$ be such that $n\tau < t_0$ and $\tau < t_1 - t_0$. Let $h_j = u_{-j\tau} h_A$, $j \in \{0, 1, \ldots, n\}$. Suppose that $h = \sum_{j=0}^{n} \lambda_j h_j = 0$, for some $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$. Evaluating the function h at the points $t_0 + j\tau$, $j \in \{0, 1, \ldots, n\}$, we obtain

$$\begin{gathered} h(t_0) = c \sum_{j=0}^{n} \lambda_j = 0, \\ h(t_0 + \tau) = \lambda_0 h_A(t_0 + \tau) + c \sum_{j=1}^{n} \lambda_j = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ h(t_0 + n\tau) = \sum_{j=0}^{n-1} \lambda_j h_A(t_0 + \tau) + c\lambda_n = 0. \end{gathered} \tag{1}$$

Since $h(t_0 + \tau) < c$, we consequently obtain from (1) that $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$. Thus, the set $\{h_0, \ldots, h_n\}$ is linearly independent. Because of arbitrariness of n, the set $\mathcal{A}$ is infinite-dimensional.

It is easy to see that the space $\mathcal{R}$ is compact. Now we are going to apply the classical Keller theorem ([13]). The space $\mathcal{R}$, being a compact convex infinite-dimensional metrizable space, is homeomorphic to the Hilbert cube.

□

Remark 2.2. In the above proof, one can also demonstrate the convexity of the hyperspace $\mathcal{R}$ by means of the Fourier expansions for the support functions of rotor curves. Indeed, according to Meissner ([14]), every rotor in a regular n-gon can be described by an equation of the form

$$p(\theta) = a_0 + \sum_{k=1}^{\infty} a_k \sin k\theta + \sum_{k=1}^{\infty} b_k \cos k\theta,$$

where $a_k = b_k = 0$ for $k \neq \pm 1 \mod n$, and that the convex combination of two such equations is also an equation of a rotor.

Let $\mathcal{R}_c$ denote the subset of $\mathcal{R}$ formed by the rotors consisting of finite number of circular arcs. We denote by σ the subspace $\{(x_i) \mid x_i = 0$ for all but finitely many $i\}$ of the Hilbert cube Q. The pair (Q,σ) is characterized in [13] in terms of skeletoids. One can conjecture whether the hyperspace $\mathcal{R}_c$ is homeomorphic to σ.

At the moment, we are able to prove this fact only for the case $P =$ square (i.e. for the case of curves of constant width).

Theorem 2.3. *The pair $(\mathcal{R},\mathcal{R}_c)$ is homeomorphic to the pair (Q,σ).*

Proof. One can easily see that $\mathcal{R}_c$ is a convex subset in $\mathcal{R}$ (i.e. $tA+(1-t)B \in \mathcal{R}_c$ for every $A, B \in \mathcal{R}_c$ and $t \in [0,1]$). It is known (see, e.g. [15]; a detailed proof is given also in [16]) that the hyperspace $\mathcal{R}_c$ is dense in $\mathcal{R}$.

Using the fact that each $A \in \mathcal{R}_c$ consists of finite number of circular arcs one can show that the space $\mathcal{R}_c$ is σ-finite-dimensional, i.e. is the union of countably many closed finite-dimensional subsets. Given natural l, m, n, let K_{lmn} denote set of all curves in $\mathcal{R}_c$ that can be represented as the union of l circular arcs of length $\geq \frac{1}{m}$ and such that the consequent arcs form the angle $\geq \pi-\frac{1}{l}$. It is easy to see that the set K_{lmn} is closed in $\mathcal{R}_c$ for every $l,m,n \in \mathbb{N}$. Let $A \in K_{lmn}$. Assign to every (maximal) circular arc forming A its center and the endpoints (taken counterclockwise), we obtain a point in $\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2$ which uniquely determines the arc and continuously depends on it. To the whole curve, there corresponds therefore the set of l points in $\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2$, i.e. an element of the hyperspace $\exp_l(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2)$ of subsets of cardinality $\leq l$ in $\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2$ endowed with the Hausdorff metric. It is easy to demonstrate that the described map $\mathcal{R}_c \to \exp_l(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2)$ is, in fact, a topological embedding. Since the dimension of the $\exp_l(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2)$ is finite, we conclude that the space $\mathcal{R}_c$ is finite-dimensional.

Note that the above argument also shows that the set $\mathcal{R}_c$ is σ-compact.

Finally, it follows from [17, Theorem 2(i)] that the pair $(\mathcal{R},\mathcal{R}_c)$ homeomorphic to the pair (Q,σ). □

Let $s = \{(x_i) \mid x_j \in (0,1)$ for every $j\}$ be the pseudointerior of the Hilbert cube Q. By $\mathcal{R}^1$ we denote the class of smooth rotors.

Theorem 2.4. *The pair $(\mathcal{R},\mathcal{R}^1)$ is homeomorphic to the pair (Q,s).*

Proof. We first remark that $\mathcal{R}^1$ is closed with respect to the Minkowski combination. Next, if $A \in \mathcal{R}\setminus\mathcal{R}^1$, $B \in \mathcal{R}$, then, for every $t \in (0,1]$, we have $tA+(1-t)B \in \mathcal{R}\setminus\mathcal{R}^1$. Given arbitrary $B \in \mathcal{R}$, for t small enough, we see that $tA+(1-t)B \in \mathcal{R}\setminus\mathcal{R}^1$ is close to B. Thus, $\mathcal{R}\setminus\mathcal{R}^1$ is dense in $\mathcal{R}$.

Note that the set $\mathcal{R}\setminus\mathcal{R}^1$ is σ-compact. Denote by C_n the set of all $A \in \mathcal{R}\setminus\mathcal{R}^1$ satisfying the condition

$(*)_n$ there exists a point $a \in \partial A$ and two supporting lines at a forming the (smaller) angle $\geq 1/n$.

By applying elementary geometric considerations, we can easily demonstrate that every set C_n is closed in $\mathcal{R}$. Obviously, $\mathcal{R}\setminus\mathcal{R}^1 = \cup_{n=1}^{\infty} C_n$.

Note also that the set $\mathcal{R}^1$ is dense in $\mathcal{R}$. Indeed, for any $r>0$, the closed r-ball of $A \in \mathcal{R}$ (we denote it by $\overline{O}_r(A)$) is a rotor in a polygon P_r whose sides are parallel to the corresponding sides of P and are of distance r from them (we suppose that $P_r \supset P$). Now, the homothetic copy of $\overline{O}_r(A)$ with coefficient $h/(h+r)$, where h denotes the height of P, is in $\mathcal{R}^1$ and is close enough to A provided r is small enough.

Finally, the set $\mathcal{R}\setminus\mathcal{R}^1$ contains a Keller cube (i.e. a convex subset homeomorphic to the Hilbert cube), namely, the set $(1/2)K+(1/2)\mathcal{R}$, where K is an arbitrary element of $\mathcal{R}\setminus\mathcal{R}^1$.

The rest of the proof now follows from the results of [18].

□

Corollary 2.5. *The hyperspace of smooth rotors in a regular polyhedron is homeomorphic to the separable Hilbert space ℓ_2.*

Proof. This follows from Theorem 2.4 and the fact that s is homeomorphic to ℓ_2 (see, e.g., [13]). □

The rotors in regular mn-gon P_{mn} are also the rotors in the regular n-gon P_n, $n,m\in\mathbb{N}$, $n\geq 3$, $m\geq 2$ (we assume that P_{mn} is inscribed into P_n and both are inscribed in the same circle).

In the sequel, we use the notation $\mathcal{R}(P)$ instead of $\mathcal{R}$ in order to indicate explicitly the polygon under consideration.

Recall that a set A in an ANR-space X is said to be a *Z-set* in X if the identity map 1_X can be approximated by maps into $X\setminus A$ (see, e.g. [19]).

Proposition 2.6. *The pair $(\mathcal{R}(P_4),\mathcal{R}(P_8))$ is homeomorphic to the pair $(Q\times[0,1],Q\times\{0\})$.*

Proof. We are going to prove that $\mathcal{R}(P_8)$ is nowhere dense in $\mathcal{R}(P_4)$. Simple geometric arguments demonstrate that the Reuleaux triangle inscribed in P_4 is not contained in P_8. Denoting the Reuleaux triangle by K, consider arbitrary $A\in\mathcal{R}(P_8)$. Given $\varepsilon>0$, we see that $\varepsilon K+(1-\varepsilon)A\notin\mathcal{R}(P_8)$. If ε is sufficiently small, the set $\varepsilon K+(1-\varepsilon)A$ is close enough to A. This shows that $\mathcal{R}(P_8)$ is nowhere dense in $\mathcal{R}(P_4)$. The set $\mathcal{R}(P_8)$, being a convex subset of $\mathcal{R}(P_4)$, is a Z-set in the latter.

Since $Q\times\{0\}$ is obviously a Z-set in the space $Q\times[0,1]$, which is homeomorphic to the Hilbert cube, it follows from the Z-set unknotting theorem ([19]) that the pairs $(\mathcal{R}(P_4),\mathcal{R}(P_8))$ and $(Q\times[0,1],Q\times\{0\})$ are homeomorphic. □

One can similarly prove that the pairs $(\mathcal{R}(P_4),\mathcal{R}(P_{2n}))$, $n\geq 3$, and $(\mathcal{R}(P_3),\mathcal{R}(P_{3m}))$, $m\geq 2$, are homeomorphic to the pair $(Q\times[0,1],Q\times\{0\})$.

3. Remarks and open questions. The results of this note demonstrate that the hyperspace of rotors behaves similarly to the hyperspace of compact convex sets in a euclidean space as well as the hyperspace of compact convex sets of constant width.

In connection with Theorem 2.3 the following question arises.

Question 3.1. Let $\mathcal{R}_c$ denote the subset of all circular rotors, i.e. rotors formed by circular arcs in P. Is the pair $(\mathcal{R},\mathcal{R}_c)$ homeomorphic to the pair (Q,σ)?

Analyzing the proof of Theorem 2.3 one can reduce this question to the following one.

Question 3.2. Is the set of all trammel (i.e., formed by circular arcs) rotors dense in $\mathcal{R}$?

The following question is related to Proposition 2.6.

Question 3.3. Is the pair $(\mathcal{R}(P_n),\mathcal{R}(P_{mn}))$, $n\geq 3$, $m\geq 2$, homeomorphic to the pair $(Q\times[0,1],Q\times\{0\})$?

The corresponding questions concerning rotors in polyhedra can be also considered in the realm of spherical and hyperbolic geometry (see [20] and [7] for known results on these rotors).

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