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ON ENTIRE FUNCTIONS OF IMPROVED REGULAR GROWTH OF INTEGER ORDER WITH ZEROS ON A FINITE SYSTEM OF RAYS

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In the paper, a criterion of improved regular growth of entire functions of integer order with zeros on a finite system of rays is established.

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Установлен критерий улучшенного регулярного роста целых функций целого порядка с нулями на конечной системе лучей.

1^0. Introduction. It is well known [1, p. 38] that every entire function $f \not\equiv 0$ of order $\rho \in \mathbb{N}$ can be represented in the form

$$f(z) = z^{\lambda} e^{Q(z)} L(z), \tag{1}$$

where $Q(z)$ is a polynomial of degree $\le \rho$, λ_n are zeros of the function f, $\lambda \in \mathbb{Z}_+$, p is the smallest integer nonnegative number such that $\sum_{|\lambda_n|>0} 1/|\lambda_n|^{p+1} < +\infty$, $L(z) = \prod_{|\lambda_n|>0} E\left(\frac{z}{\lambda_n}, p\right)$ is the Weierstrass canonical product of genus p, $p = \rho$ or $p = \rho - 1$, and $E(w,p) = (1-w)\exp(w + w^2/2 + \cdots + w^p/p)$ is the Weierstrass primary factor.

In the theory of entire functions of completely regular growth in the sense of Levin-Pfluger [1, 2] in particular, the following is established. In order that for an entire function f of order $\rho \in \mathbb{N}$ with the indicator h the relation

$$\ln|f(z)| = |z|^{\rho} h(\varphi) + o(|z|^{\rho}) \quad (z \to \infty, \quad \varphi = \arg z \in [0, 2\pi))$$

hold outside some exceptional set $C_0 \subset \mathbb{C}$ of disks with linear density 0, it is necessary and sufficient that for almost all $\alpha \in \mathbb{R}$ and almost all $\beta \in \mathbb{R}$, $\alpha < \beta$, there exists [3, p. 43] an angle ρ-density

$$\Delta(\alpha,\beta) \overset{\text{def}}{=} \lim_{r\to+\infty} \frac{n(r,\alpha,\beta)}{r^{\rho}}, \tag{2}$$

where $n(r,\alpha,\beta) := n(\{z : 0 < |z| \le r,\ \alpha < \arg z \le \beta\})$, and, in addition, there is [3, p. 44] a finite limit

$$\delta_f = \lim_{r\to+\infty} \sum_{|\lambda_n|\le r} 1/\lambda_n^{\rho}. \tag{3}$$

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We remark that condition (3) is fulfilled for $\rho = p+1$, since in this case a series $\sum_n 1/|\lambda_n|^\rho$ converges. In the case when zeros of the function f are situated on a finite system of rays $\{z : \arg z = \psi_j,\ j \in \{1,\dots,m\}\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, condition (2) is equivalent [1, p. 129] to the condition

$$n_j(r) = \Delta_j r^\rho + o(r^\rho) \quad (\Delta_j \in [0,+\infty), \quad j \in \{1,\dots,m\})$$

as $r \to +\infty$, where $n_j(r)$ is the number of zeros of the function f from the disk $\{z : |z| \le r\}$, which are concentrated on a ray $\{z : \arg z = \psi_j\}$. Subtler asymptotics for entire functions are considered in [4–8].

An entire function f will be called of improved regular growth, if for some ρ and ρ_3, $0 < \rho_3 < \rho < +\infty$, and some 2π-periodic ρ-trigonometric convex function $h(\varphi) \not\equiv -\infty$ there exists an exceptional set $U \subset \mathbb{C}$ of disks with finite sum of radii such that

$$\ln|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_3}) \quad (U \not\ni z = re^{i\varphi} \to \infty). \tag{4}$$

If relation (4) holds, then [4] an entire function f has the order ρ and indicator h.

In the present paper we obtain a criterion of improved regular growth of an entire function of integer order with zeros on a finite system of rays and, in particular, prove the following propositions, analogue of which for the case of non-integer order is in [4]. Let Q_ρ be a coefficient at z^ρ in a polynomial $Q(z)$ of representation (1).

Theorem 1. *Let f be an entire function of order $\rho = p \in \mathbb{N}$ with zeros on a finite system of rays $\{z : \arg z = \psi_j,\ 1 \le j \le m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, and for some $\rho_1 \in (0,\rho)$ and every $j \in \{1,2,\dots,m\}$*

$$n_j(t) = \Delta_j t^\rho + o(t^{\rho_1}) \quad (t \to +\infty, \quad \Delta_j \in [0,+\infty)), \tag{5}$$

and let for some $\delta_f \in \mathbb{C}$ and $\rho_2 \in (0,\rho)$

$$\sum_{|\lambda_n| \le r} 1/\lambda_n^\rho = \delta_f + o(r^{\rho_2 - \rho}) \quad (r \to +\infty). \tag{6}$$

Then for some $\rho_3 \in (0,\rho)$ there exists an exceptional set $U \subset \mathbb{C}$ of disks with finite sum of radii such that relation (4) holds, where

$$h(\varphi) = \tau_f \cos(\rho\varphi + \theta_f) + \sum_{i=1}^m h_i(\varphi), \tag{7}$$

moreover, $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$ and $h_i(\varphi)$ is a 2π-periodic function such that on $[\psi_i, \psi_i + 2\pi)$

$$h_i(\varphi) = \Delta_i(\pi - \varphi + \psi_i)\sin\rho(\varphi - \psi_i) - \frac{\Delta_i}{\rho}\cos\rho(\varphi - \psi_i). \tag{8}$$

Conversely, if f, as above, is an entire function of order $\rho = p \in \mathbb{N}$ with zeros on a finite system of rays and for some $\rho_3 \in (0,\rho)$ there exists an exceptional set $U \subset \mathbb{C}$ of disks with finite sum of radii such that relation (4) holds with $h(\varphi)$ of form (7), then for some $\rho_1 \in (0,\rho)$ and every $j \in \{1,2,\dots,m\}$ (5) holds, and, in addition, for some $\rho_2 \in (0,\rho)$ (6) holds where $\delta_f = \rho(\tau_f e^{i\theta_f} - Q_\rho)$.

Theorem 2. *Let f be an entire function of order $\rho = p + 1 \in \mathbb{N}$ with zeros on a finite system of rays $\{z : \arg z = \psi_j,\ j \in \{1, 2, \dots, m\}\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, and for some $\rho_1 \in (0, \rho)$ and every $j \in \{1, 2, \dots, m\}$ condition (5) holds with $\Delta_j = 0$. Then for some $\rho_3 \in (0, \rho)$ there exists an exceptional set $U \subset \mathbb{C}$ of disks with finite sum of radii such that relation (4) holds, where*

$$h(\varphi) = Q_\rho \cos \rho\varphi. \tag{9}$$

Conversely, if f, as above, is an entire function of order $\rho = p + 1 \in \mathbb{N}$ with zeros on a finite system of rays and for some $\rho_3 \in (0, \rho)$ there exists an exceptional set $U \subset \mathbb{C}$ of disks with finite sum of radii such that relation (4) holds with $h(\varphi)$ of form (9), then for some $\rho_1 \in (0, \rho)$ and every $j \in \{1, 2, \dots, m\}$ (5) holds with $\Delta_j = 0$.

2^0. Auxiliary Lemmas. Everywhere further by c_1, c_2, c_3, … we denote arbitrary positive constants.

Lemma 1. *Let $\rho \in (0, +\infty)$, $\rho_3 \in (0, \rho)$ and f be an entire function of improved regular growth. Then there exists a sequence (r_k) such that $0 < r_k \uparrow +\infty$, $r_{k+1}^\rho - r_k^\rho = o(r_k^{\rho_3})$ $(k \to +\infty)$, and $\ln |f(r_k e^{i\varphi})| = r_k^\rho h(\varphi) + o(r_k^{\rho_3})$ $(k \to +\infty)$ uniformly in $\varphi \in [0, 2\pi]$.*

Lemma 1 is proved in [4].

Lemma 2. *Let $\rho \in (0, +\infty)$, $\rho_3 \in (0, \rho)$ and f be an entire function of improved regular growth with zeros on a finite system of rays $\{z : \arg z = \psi_j,\ 1 \le j \le m\}$. Then in every angle $\mathbb{C}[\varphi_j, \widetilde{\varphi_j}] = \{z : \varphi_j \le \arg z \le \widetilde{\varphi_j}\}$, $0 \le \psi_j < \varphi_j < \widetilde{\varphi_j} < \psi_{j+1} < 2\pi$, $j \in \{1, 2, \dots, m\}$, $\psi_{m+1} := \psi_1 + 2\pi$, the inequality*

$$\ln |f(z)| \ge |z|^\rho h(\varphi) - c_1 |z|^{\rho_3}$$

holds.

Proof. Let $\psi_{j+1} - \psi_j \le \min\{2\pi, \pi/\rho\}$. Since in every angle $\mathbb{C}[\varphi_j, \widetilde{\varphi_j}]$, $j \in \{1, 2, \dots, m\}$, the function f does not have zeros, then [2, p. 110] its indicator h in these angles is ρ-trigonometric, that is

$$h(\varphi) = A \cos \rho\varphi + B \sin \rho\varphi \quad (\varphi \in [\varphi_j, \widetilde{\varphi_j}], \quad j \in \{1, 2, \dots, m\}),$$

where A and B are some constants. We consider the function

$$F(z) = V(z)/f(z), \quad V(z) = \exp((A - iB) z_o^\rho - c_2 (z e^{-i\psi})_o^{\rho_3}),$$

where c_2 is a sufficiently large constant, $z_o^\rho = |z|^\rho (\cos \rho\varphi + i \sin \rho\varphi)$ is a holomorphic branch of the function z^ρ in angle $\mathbb{C}[\varphi_j, \widetilde{\varphi_j}]$, $\left(z e^{-i\psi}\right)_o^{\rho_3} = |z|^{\rho_3} (\cos \rho_3(\varphi - \psi) + i \sin \rho_3(\varphi - \psi))$ is a holomorphic branch of the function $(z e^{-i\psi})^{\rho_3}$ in this angle too, where $\psi = (\psi_{j+1} + \psi_j)/2$. Thereupon, in the same way as in the proof of Lemma 2 from [4], we obtain the required proposition. Lemma 2 is proved. □

Lemma 3. *Let $\rho \in (0, +\infty)$. If f is an entire function of improved regular growth, then the inequality*

$$\ln |f(z)| \le |z|^\rho h(\varphi) + c_3 |z|^{\rho_4}$$

holds for some $\rho_4 \in (0, \rho)$ and all $z \in \mathbb{C}$.

Lemma 3 is proved in [4]. The following lemma is an immediate corollary of lemmas 2 and 3.

Lemma 4. [4] *Let $\rho \in (0, +\infty)$ and f be an entire function of improved regular growth. Then there exists $\rho_5 \in (0, \rho)$ such that for every $j \in \{1, 2, \dots, m\}$*

$$J_f^t(\varphi) := \int_0^t \frac{\ln|f(ue^{i\varphi})|}{u}\,du = \frac{t^\rho}{\rho}h(\varphi) + o(t^{\rho_5}) \quad (t \to +\infty)$$

uniformly in $\varphi \in [\varphi_j, \widetilde{\varphi_j}]$, $\psi_j < \varphi_j < \widetilde{\varphi_j} < \psi_{j+1}$.

Lemma 5. *If an entire function f satisfies the hypothesis of the second part of Theorem 1 and $f(0) \neq 0$, then equality (5) holds for some $\rho_1 \in (0, \rho)$ and every $j \in \{1, 2, \dots, m\}$.*

Proof. Let

$$N_j(r) = \int_0^r \frac{n_j(t)}{t}\,dt, \quad n_j(t) := n(\{z : 0 < |z| \leq t, \arg z = \psi_j\}), \quad j \in \{1, 2, \dots, m\}.$$

For every $j \in \{1, 2, \dots, m\}$ we choose the numbers α and β so that $\psi_{j-1} < \alpha < \psi_j < \beta < \psi_{j+1}$, where $\psi_0 = \psi_m - 2\pi$, $\psi_{m+1} = \psi_1 + 2\pi$. Then, using Jensen general equality [1, p. 188], likewise as in the proof of Lemma 5 from [4], for some $\rho_6 \in (0, \rho)$ and every $j \in \{1, 2, \dots, m\}$ we obtain

$$N_j(r) = \frac{r^\rho}{2\pi\rho^2}s_f(\alpha, \beta) + o(r^{\rho_6}) \quad (r \to +\infty), \tag{10}$$

where

$$s_f(\alpha, \beta) = h'(\beta) - h'(\alpha) + \rho^2 \int_\alpha^\beta h(\varphi)\,d\varphi.$$

Since the functions $N_j(r)$, $j \in \{1, 2, \dots, m\}$, are independent of α and β from the gaps under consideration by (10), the function $s_f(\alpha, \beta)$ is the same. Hence, $s_f(\alpha, \beta) = s_j$ and

$$s_j = \lim_{\substack{\alpha \to \psi_j - \\ \beta \to \psi_j +}} \sum_{i=1}^m \left(h_i'(\beta) - h_i'(\alpha) + \rho^2 \int_\alpha^\beta h_i(\varphi)\,d\varphi \right), \tag{11}$$

where the function $h_i(\varphi)$ is of form (8). Let $i > j$. If $\varphi \in [\alpha, \beta]$ then, similarly, as in [4], we have that $\varphi + 2\pi \in [\psi_i, \psi_i + 2\pi)$, and, according to (8),

$$h_i(\varphi) = h_i(\varphi + 2\pi) = \Delta_i(\psi_i - \pi - \varphi)\sin\rho(\varphi - \psi_i) - \frac{\Delta_i}{\rho}\cos\rho(\varphi - \psi_i). \tag{12}$$

Therefore,

$$h_i'(\beta) - h_i'(\alpha) + \rho^2 \int_\alpha^\beta h_i(\varphi)\,d\varphi = 2\Delta_i(\sin\rho(\alpha - \psi_i) - \sin\rho(\beta - \psi_i)). \tag{13}$$

When $i < j$ and $\varphi \in [\alpha, \beta]$, then $\varphi \in [\psi_i, \psi_i + 2\pi)$ and the function $h_i(\varphi)$ is of form (8). Hence, in this case, we have (13). Let $i = j$. If $\varphi \in [\alpha, \psi_j]$, then $\varphi + 2\pi \in [\psi_j, \psi_j + 2\pi)$ and the function $h_j(\varphi) = h_j(\varphi + 2\pi)$ can be chosen in form (12). If $\varphi \in [\psi_j, \beta]$, then $\varphi \in [\psi_j, \psi_j + 2\pi)$ and $h_j(\varphi)$ is defined by (8). Therefore, for $i = j$, we obtain

$$h'_j(\beta) - h'_j(\alpha) + \rho^2 \int\limits_{\alpha}^{\beta} h_j(\varphi)\, d\varphi = h'_j(\beta) - h'_j(\alpha) + \rho^2 \left(\int\limits_{\alpha}^{\psi_j} + \int\limits_{\psi_j}^{\beta} \right) h_j(\varphi)\, d\varphi =$$

$$= 2\pi\rho\Delta_j + 2\Delta_j(\sin\rho(\alpha - \psi_j) - \sin\rho(\beta - \psi_j)). \tag{14}$$

Thus, the union of formulas (10), (11), (13) and (14) implies that for some $\rho_6 \in (0, \rho)$ and every $j \in \{1, 2, \ldots, m\}$

$$N_j(r) = \frac{\Delta_j}{\rho} r^\rho + o(r^{\rho_6}) \quad (r \to +\infty, \quad \Delta_j \in [0, +\infty)).$$

Hence, by Lemma 6 from [4] (see also the proof of Lemma 3 in [5, p. 143]), for some $\rho_1 \in (0, \rho)$ and every $j \in \{1, 2, \ldots, m\}$ equality (5) is valid. The proof of Lemma 5 is completed. □

Remark 1. Since in the case $\rho = p + 1$, the function $h(\varphi)$ is of form (9), then it follows from (10) that the relation $N_j(r) = o(r^{\rho_6})$ $(r \to +\infty)$ holds for some $\rho_6 \in (0, \rho)$ and every $j \in \{1, 2, \ldots, m\}$. Therefore, as above, relation (5) holds with $\Delta_j = 0$.

Lemma 6. *If an entire function f satisfies the hypothesis of the second part of Theorem 1, then for some $\rho_2 \in (0, \rho)$ condition (6) holds with $\delta_f = \rho(\tau_f e^{i\theta_f} - Q_\rho)$.*

Proof. Let ([2, p. 10])

$$\Phi_\nu(r, f) = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-i\nu\varphi} \ln |f(re^{i\varphi})|\, d\varphi \quad (\nu \in \mathbb{Z}, \quad r > 0)$$

be a Fourier coefficients of $\ln |f(re^{i\varphi})|$. Since for $\nu = \rho = p$ we have ([6, p. 101])

$$\Phi_\rho(r, f) = \frac{1}{2} Q_\rho r^\rho + \frac{1}{2\rho} \sum_{|\lambda_n| \le r} \left(\left(\frac{r}{\lambda_n} \right)^\rho - \left(\frac{\overline{\lambda}_n}{r} \right)^\rho \right),$$

we see that

$$\Phi_\rho(r, f) = \frac{1}{2} Q_\rho r^\rho + \frac{1}{2\rho} \sum_{|\lambda_n| \le r} \left(\frac{r}{\lambda_n} \right)^\rho - \frac{1}{2\rho} \sum_{j=1}^{m} I(j), \tag{15}$$

where

$$I(j) = \sum_{\substack{|\lambda_n| \le r, \\ \arg \lambda_n = \psi_j}} \left(\frac{|\lambda_n|}{r} \right)^\rho e^{-i\rho\psi_j}.$$

In view of Lemma 5, using the integration by parts, we get

$$I(j) = \frac{e^{-i\rho\psi_j}}{r^\rho} \int\limits_0^r t^\rho\, dn_j(t) = \frac{\Delta_j}{2} r^\rho e^{-i\rho\psi_j} + o(r^{\rho_1}) \quad (r \to +\infty, \quad \rho_1 < \rho).$$

For this reason, from (15) we obtain

$$\Phi_\rho(r,f)=\frac{1}{2}Q_\rho r^\rho+\frac{1}{2\rho}\sum_{|\lambda_n|\le r}\left(\frac{r}{\lambda_n}\right)^\rho-\frac{r^\rho}{4\rho}\sum_{j=1}^m\Delta_j\,e^{-i\rho\psi_j}+o(r^{\rho_1})\quad(r\to+\infty).\qquad(16)$$

Further, taking into account Lemma 1, by Theorem 2 from [6, p. 100], we have

$$\Phi_\rho(r,f)=\frac{r^\rho}{2\pi}\int_0^{2\pi}h(\varphi)\,e^{-i\rho\varphi}\,d\varphi+o(r^{\rho_7})\quad(r\to+\infty,\quad\rho_7<\rho).\qquad(17)$$

Thus, (17) together with (16), (8) and (7) gives that there exists $\rho_2\in(0,\rho)$, for which

$$\sum_{|\lambda_n|\le r}1/\lambda_n^\rho=\rho\left(\frac{2}{r^\rho}\Phi_\rho(r,f)+\frac{1}{2\rho}\sum_{j=1}^m\Delta_j\,e^{-i\rho\psi_j}-Q_\rho\right)+o(r^{\rho_1-\rho})=$$

$$=\frac{\rho}{\pi}\int_0^{2\pi}e^{-i\rho\varphi}(\tau_f\cos(\rho\varphi+\theta_f))\,d\varphi+o(r^{\rho_7-\rho})+\frac{1}{2}\sum_{j=1}^m\Delta_j\,e^{-i\rho\psi_j}-\rho Q_\rho+o(r^{\rho_1-\rho})+$$

$$+\frac{\rho}{\pi}\sum_{j=1}^m\Delta_j\left\{\int_{\psi_j}^{\psi_j+2\pi}e^{-i\rho\varphi}\left((\pi-\varphi+\psi_j)\sin\rho(\varphi-\psi_j)-\frac{1}{\rho}\cos\rho(\varphi-\psi_j)\right)d\varphi\right\}=$$

$$=\rho(\tau_f e^{i\theta_f}-Q_\rho)+o(r^{\rho_2-\rho})\quad(r\to+\infty),$$

whence the required proposition follows. Lemma 6 is proved. □

3^0. Proof of Theorem 1. Without loss of generality we can assume that in (1) $\lambda=0$. The second part of Theorem 1 follows directly from Lemmas 5 and 6. Now we prove the first part. Let conditions (5) and (6) hold. We have

$$f(z)=e^{Q(z)}\prod_{j=1}^m L_j(z),\qquad(18)$$

where a canonical product L_j is constructed in zeros of the function f which lie on the ray $\{z:\arg z=\psi_j\}$. By Theorem 1, from [7, p. 106] we get that for some $\rho_8\in(\rho-1,\rho)$ and $z=re^{i\varphi}\to\infty$ the relation

$$\ln|L_j(z)|=\frac{1}{\rho}\mathrm{Re}\left\{e^{i\rho\varphi}\sum_{\substack{|\lambda_n|\le r,\\ \arg\lambda_n=\psi_j}}\left(\frac{r}{|\lambda_n|\,e^{i\psi_j}}\right)^\rho\right\}+|z|^\rho h_j(\varphi)+o(|z|^{\rho_8})\qquad(19)$$

holds outside each exceptional set $U_j\subset\mathbb{C}$, $j\in\{1,2,\dots,m\}$, of disks with finite sum of radii, where $h_j(\varphi)$ is defined by (8). Next, the union of equalities (6), (18) and (19) implies that there exist $\rho_3\in(0,\rho)$ and an exceptional set $U=\bigcup_{j=1}^m U_j$ of disks in $\mathbb{C}$ with finite sum of radii such that for $U\not\ni z=re^{i\varphi}\to\infty$ it is easy to obtain the following relation

$$\ln|f(z)|=\mathrm{Re}\,\{Q(z)\}+\sum_{j=1}^m\ln|L_j(z)|=$$

$$= \operatorname{Re}\left\{Q_\rho z^\rho + \frac{e^{i\rho\varphi}}{\rho}\sum_{j=1}^{m}\sum_{\substack{|\lambda_n|\le r,\\ \arg\lambda_n=\psi_j}}\left(\frac{r}{|\lambda_n|e^{i\psi_j}}\right)^\rho\right\} + |z|^\rho\sum_{j=1}^{m} h_j(\varphi) + o(|z|^{\rho_8}) =$$

$$= |z|^\rho \operatorname{Re}\left\{e^{i\rho\varphi}\left(Q_\rho + \frac{1}{\rho}\sum_{|\lambda_n|\le r} 1/\lambda_n^\rho\right)\right\} + |z|^\rho\sum_{j=1}^{m} h_j(\varphi) + o(|z|^{\rho_8}) =$$

$$= |z|^\rho \operatorname{Re}\{(\delta_f/\rho + Q_\rho)e^{i\rho\varphi}\} + |z|^\rho\sum_{j=1}^{m} h_j(\varphi) + o(|z|^{\rho_2}) + o(|z|^{\rho_8}) =$$

$$= |z|^\rho\left(\tau_f\cos(\rho\varphi+\theta_f) + \sum_{j=1}^{m} h_j(\varphi)\right) + o(|z|^{\rho_3}) = |z|^\rho h(\varphi) + o(|z|^{\rho_3}),$$

moreover $\tau_f = |\delta_f/\rho + Q_\rho|$ and $\theta_f = \arg(\delta_f/\rho + Q_\rho)$. The proof of Theorem 1 is thus completed.

4^0. Proof of Theorem 2. The second part of Theorem 2 follows from Lemma 5 and Remark 1. Now we prove the first part. Since (5) holds with $\Delta_j = 0$, by Theorem 1 from [7, p. 106], we have

$$\ln|L_j(z)| = o(|z|^{\rho_9}) \quad (z\to\infty), \tag{20}$$

for some $\rho_9 \in (\rho-1,\rho)$ outside each exceptional set $U_j \subset \mathbb{C}$, $j\in\{1,2,\ldots,m\}$, of disks with finite sum of radii. Therefore, from (18) and (20) we obtain that there exist $\rho_3\in(0,\rho)$ and an exceptional set $U = \bigcup_{j=1}^m U_j$ of disks in $\mathbb{C}$ with finite sum of radii such that, for $U \not\ni z = re^{i\varphi}\to\infty$,

$$\ln|f(z)| = \operatorname{Re}\{Q(z)\} + \sum_{j=1}^{m}\ln|L_j(z)| = |z|^\rho Q_\rho\cos\rho\varphi + o(|z|^{\rho_9}) = |z|^\rho h(\varphi) + o(|z|^{\rho_3}).$$

Theorem 2 is proved.

Remark 2. In the case $\rho = p+1$, condition (6) follows from (5), because $\delta_f = \sum_{n=1}^\infty 1/\lambda_n^\rho$ and it is easy to show that, for $\rho_1\in(0,\rho)$,

$$\left|\sum_{|\lambda_n|>r} 1/\lambda_n^\rho\right| \le \sum_{j=1}^{m}\left(\sum_{\substack{|\lambda_n|>r,\\ \arg\lambda_n=\psi_j}} |1/\lambda_n^\rho|\right) = \sum_{j=1}^{m}\left(\int_r^{+\infty}\frac{dn_j(t)}{t^\rho}\right) = o(r^{\rho_1-\rho}) \quad (r\to+\infty).$$

In view of this, Theorems 1 and 2 together imply the following statement.

Theorem 3. *In order that an entire function f of order $\rho\in\mathbb{N}$ with zeros on a finite system of rays $\{z : \arg z = \psi_j,\ j\in\{1,2,\ldots,m\}\}$, $0\le\psi_1<\psi_2<\ldots<\psi_m<2\pi$, be of improved regular growth, it is necessary and sufficient that for some $\rho_1\in(0,\rho)$ and every $j\in\{1,2,\ldots,m\}$ (5) holds, and, for some $\delta_f\in\mathbb{C}$ and $\rho_2\in(0,\rho)$, (6) holds.*

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