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LIE ALGEBRAS ASSOCIATED WITH WREATH PRODUCTS OF ELEMENTARY ABELIAN GROUPS

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We study Lie algebras associated with the lower central series of the wreath product $P_{m,n}$ of m copies of elementary abelian p-groups of degree n. It is shown that these Lie algebras have special "tableau" representation. We define a wreath product of a Lie algebra L with an abelian finite-dimensional Lie algebra over the field $\mathbb{F}_p$. We prove that the Lie algebra associated with the lower central series of the group $P_{m,n}$ is isomorphic to the wreath product of m copies of the abelian Lie algebra of dimension n over the field $\mathbb{F}_p$.

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Изучаются алгебры Ли, ассоциированные с нижним центральным рядом m-кратного сплетения $P_{m,n}$ элементарных абелевых групп ранга n. Показано, что рассматриваемые алгебры Ли имеют некоторое специальное "табличное" представление. Определяется сплетение произвольной алгебры Ли L с абелевой алгеброй Ли конечной размерности над полем $\mathbb{F}_p$. Показано, что алгебра Ли, ассоциированная с нижним центральным рядом группы $P_{m,n}$, изоморфна m-кратному сплетению абелевых алгебр Ли размерности n над полем $\mathbb{F}_p$.

Introduction. The idea of constructing Lie rings from abstract groups was raised by Magnus [9]. Since then Lie algebras associated with nilpotent or residually-nilpotent groups are one of the important objects of research in the group theory and the theory of pro-finite groups [4], [16]. Thanks to Lie algebra methods it was possible to solve different group-theoretical problems, one of the well-known of them is the restricted Burnside problem [17]. The Lie methods are applied in the study of growth of groups, of some group identities, of Hausdorff dimension and spectrum of pro-p groups. Association Lie algebra with a group enables one to use linear methods in calculations and proofs. But Lie algebras associated with groups are also interesting by themselves. For example the structures of Lie algebras associated with the Grigorchuk group, Gupta-Sidki group, Fabrykowski-Gupta group were studied in [2], [3]. Using these descriptions the width of these groups were estimated. In [11], [15], [14] Lie algebras associated with the Sylow p-subgroups of finite symmetric groups were investigated.

In the present paper we investigate Lie algebras associated with the lower central series of wreath products of elementary abelian p-groups. In [13] V.I. Sushchansky has shown that the elements of the wreath product $P_{m,n}$ of m copies of elementary abelian p-groups of degree n, have special "tableau" representation. In particular due to this representation the lower

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central series of the group $P_{m,n}$ was described. We construct the Lie algebra associated with the lower central series of the group $P_{m,n}$ and show that this Lie algebra also has the tableau representation.

In [15] was defined the notion of a wreath product of a Lie algebra with the one-dimensional Lie algebra over the field $\mathbb{F}_p$ residues modulo p. Now we extend this definition and define the wreath product of a Lie algebra with an abelian finite-dimensional Lie algebra over the field $\mathbb{F}_p$. We prove that the wreath product of a nilpotent (solvable) Lie algebra with an abelian Lie algebra is nilpotent (solvable). Our definition of the wreath product allows us to prove that the Lie algebra associated with the lower central series of the group $P_{m,n}$ is isomorphic to the wreath product of abelian Lie algebras of dimension n over the field $\mathbb{F}_p$,

$$L\big((C_p)^n \wr \ldots \wr (C_p)^n\big) = L\big((C_p)^n\big) \wr \ldots \wr L\big((C_p)^n\big).$$

1. Wreath product of Lie algebras. We first recall the definition of the semidirect product of Lie algebras (see [1]).

Let M and N be Lie algebras over the field K and $a \mapsto \varphi_a$ be a homomorphism of the Lie algebra M to the Lie algebra $\mathcal{D}(N)$ of differentiations of the algebra N. Define a Lie bracket on the direct sum L of K-modules M and N by the equality:

$$([a_1, b_1], [a_2, b_2]) = [(a_1, a_2), \varphi_{a_1}(b_2) - \varphi_{a_2}(b_1) + (b_1, b_2)],$$

where $a_1, a_2 \in M$ and $b_1, b_2 \in N$.

Definition 1. The Lie algebra L is called *the semidirect product* of algebra M and algebra N which corresponds to the homomorphism $\varphi : M \to \mathcal{D}(N)$. We denote this algebra as $L = M \leftthreetimes_{\varphi} N$.

Let L be a Lie algebra over the field $\mathbb{F}_p$ and A_n be the n-dimensional abelian Lie algebra over the field $\mathbb{F}_p$.

Let $L[x_1, \ldots, x_n]_{(p-1)}$ be the Lie algebra of polynomials of n variables over L of degree at most $p-1$ for each variable. The addition and the multiplication on the elements of the field $\mathbb{F}_p$ for the polynomials are defined in the natural way. The Lie bracket of the monomials $lx_1^{k_1} \ldots x_n^{k_n}$, $l'x_1^{m_1} \ldots x_n^{m_n}$ in this algebra is defined in the following way:

$$(lx_1^{k_1} \ldots x_n^{k_n}, l'x_1^{m_1} \ldots x_n^{m_n}) = \\ = \begin{cases} (l, l')x_1^{k_1+m_1} \ldots x_n^{k_n+m_n}, & \text{if } k_i + m_i < p \text{ for all } i \in \{1, 2, \ldots, n\}; \\ 0, & \text{if } k_i + m_i \geq p \text{ for some } i, 1 \leq i \leq n. \end{cases} \tag{1}$$

By linearity the Lie bracket is determined for all polynomials.

Farther we fix some basis of the abelian Lie algebra A_n.

Proposition 1. *Every map $f : A_n \to L$ corresponds to the unique polynomial $q(x_1, \ldots, x_n)$ over L of degree at most $p-1$ for each variable such that $f(\alpha) = q(\alpha_1, \ldots, \alpha_n)$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in A_n$.*

Proof. The proof goes by induction on dimension n of the Lie algebra A_n. For the case $n = 1$, a map $f : A_1 \to L$ corresponds to the unique polynomial $q(x)$ over L of degree at most $p-1$ such that $f(\alpha) = q(\alpha)$, $\alpha \in \mathbb{F}_p$, is constructed in [15].

Suppose that the statement is correct for some $n > 1$. Let us prove it for $n+1$. Consider A_{n+1} as $A_n \oplus A_1$. Then for any fixed $\beta_i \in A_1$, $i \in \{0, 1, \dots, p-1\}$, the map $f : A_n \oplus A_1 \to L$ induces the map $f_i : A_n \to L$ such that

$$f_i\big((\alpha_1, \dots, \alpha_n)\big) = f\big((\alpha_1, \dots, \alpha_n, \beta_i)\big)$$

for any $(\alpha_1, \dots, \alpha_n) \in A_n$. By induction maps f_i, $i \in \{0, \dots, p-1\}$, correspond to the unique polynomials $q_i(x_1, \dots, x_n)$ over L of degree at most $p-1$ for each variable such that $f_i(\alpha) = q_i(\alpha_1, \dots, \alpha_n)$, $\alpha = (\alpha_1, \dots, \alpha_n) \in A_n$. Thus the map f induces the map

$$g : A_1 \to L[x_1, \dots, x_n]_{p-1}$$

such that $g(\beta_i) = q_i(x_1, \dots, x_n)$, $\beta_i \in A_1$. This map corresponds to the unique polynomial $q(x_{n+1})$ over $L[x_1, \dots, x_n]_{(p-1)}$ of degree at most $p-1$ such that $g(\beta) = q(\beta)$ for any $\beta \in A_1$ ([15]). Hence $q(x_{n+1}) = q(x_1, \dots, x_n, x_{n+1}) \in L[x_1, \dots, x_{n+1}]_{(p-1)}$. Moreover, $f\big((\alpha_1, \dots, \alpha_n, \beta)\big) = q(\alpha_1, \dots, \alpha_n, \beta)$ for any $(\alpha_1, \dots, \alpha_n, \beta) \in A_{n+1}$. □

Consequently there exists the bijection between the set of all maps $f : A_n \to L$ and the set of all polynomials over L of n variables of degree at most $p-1$ for each variable. The structure of the Lie algebra $L[x_1, \dots, x_n]_{(p-1)}$ defines the structure of Lie algebra on the set of all maps $f : A_n \to L$. The addition, the multiplication on the elements of the field $\mathbb{F}_p$ and the Lie bracket $(\, , \,)$ for maps $f_1, f_2 : A_n \to L$ are defined as the respective operations for corresponding polynomials q_1, q_2 of the Lie algebra $L[x_1, \dots, x_n]_{(p-1)}$. We will denote this Lie algebra as $Fun(A_n, L)$.

Let $f \in Fun(A_n, L)$. Denote by f' the derivative of the polynomial f defined by

$$f(x_1, \dots, x_n)' = \left(\frac{\partial f(x_1, \dots, x_n)}{\partial x_1}, \dots, \frac{\partial f(x_1, \dots, x_n)}{\partial x_n} \right).$$

Proposition 2. *For any $\alpha = (\alpha_1, \dots, \alpha_n) \in A_n$ the map $D_\alpha : Fun(A_n, L) \to Fun(A_n, L)$ which is defined by the rule*

$$D_\alpha(f) = \alpha f' = \sum_{i=1}^{n} \alpha_i \frac{\partial f(x_1, \dots, x_n)}{\partial x_i}$$

is a differentiation.

Proof. The linearity of the map D_α follows from the linearity of the derivative. So the fact that D_α is differentiation is enough to verify for monomials.

$$D_\alpha(l x_1^{k_1} \dots x_n^{k_n}, l' x_1^{m_1} \dots x_n^{m_n}) =$$

$$= \begin{cases} (l, l') \sum\limits_{i=1}^{n} \alpha_i (k_i + m_i) x_1^{k_1+m_1} \dots x_i^{k_i+m_i-1} \dots x_n^{k_n+m_n}, \\ \qquad\qquad \text{if } \ k_i + m_i < p \text{ for all } i \in \{1, 2, \dots, n\}; \\ 0, \quad \text{if } \ k_i + m_i \geq p \text{ for some } i, 1 \leq i \leq n. \end{cases}$$

$$(D_\alpha(lx_1^{k_1}\dots x_n^{k_n}),l'x_1^{m_1}\dots x_n^{m_n})+(lx_1^{k_1}\dots x_n^{k_n},D(l'x_1^{m_1}\dots x_n^{m_n}))=$$
$$=(l\sum_{i=1}^n\alpha_i k_i x_1^{k_1}\dots x_i^{k_i-1}\dots x_n^{k_n},l'x_1^{m_1}\dots x_n^{m_n})+$$
$$+(lx_1^{k_1}\dots x_n^{k_n},l'\sum_{i=1}^n\alpha_i m_i x_1^{m_1}\dots x_i^{m_i-1}\dots x_n^{m_n})=$$

$$=\begin{cases}(l,l')\sum_{i=1}^n\alpha_i(k_i+m_i)x_1^{k_1+m_1}\dots x_i^{k_i+m_i-1}\dots x_n^{k_n+m_n}, \\ \qquad\qquad \text{if } \ k_i+m_i<p \text{ for all } i\in\{1,2,\dots,n\}; \\ 0, \text{ if } \ k_i+m_i\ge p \text{ for some } i, 1\le i\le n.\end{cases}$$

Thus the upper equality coincides with the lower one and D_α is a differentiation. □

Define a map φ from the Lie algebra A_n to the Lie algebra of differentiations $\mathcal{D}(Fun(A_n,L))$ given by the rule $\alpha\mapsto D_\alpha$. The map φ is a homomorphism. Really, $\varphi((\alpha,\beta))=0$ and

$$D_\alpha D_\beta(f)-D_\beta D_\alpha(f)=D_\alpha\left(\sum_{j=1}^n\beta_j\frac{\partial f}{\partial x_j}\right)-D_\beta\left(\sum_{i=1}^n\alpha_i\frac{\partial f}{\partial x_i}\right)=$$
$$=\sum_{i=1}^n\alpha_i\frac{\partial}{\partial x_i}\left(\sum_{j=1}^n\beta_j\frac{\partial f}{\partial x_j}\right)-\sum_{j=1}^n\beta_j\frac{\partial}{\partial x_j}\left(\sum_{i=1}^n\alpha_i\frac{\partial f}{\partial x_i}\right)=$$
$$=\sum_{i=1}^n\sum_{j=1}^n\alpha_i\beta_j\frac{\partial^2 f}{\partial x_i\partial x_j}-\sum_{i=1}^n\sum_{j=1}^n\alpha_i\beta_j\frac{\partial^2 f}{\partial x_i\partial x_j}=0.$$

Definition 2. The semidirect product of the Lie algebra A_n with the Lie algebra $Fun(A_n,L)$, which corresponds to the homomorphism φ, we call the *wreath product of Lie algebra L with* A_n and denote by $L\wr A_n$.

Thus, $L\wr A_n:=A_n\leftthreetimes_\varphi Fun(A_n,L)=\{[a,f]|\ a\in A_n, f\in Fun(A_n,L)\}$ with the Lie bracket

$$([a_1,f_1],[a_2,f_2])=[0,a_1f_2'-a_2f_1'+(f_1,f_2)]. \tag{2}$$

Remark 1. The definition of the wreath product of a Lie algebra L with the one-dimensional Lie algebra introduced in [15] coincides with Definition 2 for $n=1$.

Remark 2. Definition 2 allows us to consider the wreath product $L\wr A_{n_1}\wr A_{n_2}\wr\dots\wr A_{n_k}$ for arbitrary Lie algebra L.

The subset of the Lie algebra $L\wr A_n$ of elements of the form $[a,e]$, where $e(\alpha)=0_L$ for any $\alpha\in A_n$ and 0_L is the null element of L, forms the subalgebra $\hat{A}$ which is isomorphic to A_n. The subset H of elements of the form $[0,f]$, where 0 is the null element of the Lie algebra A_n, is the subalgebra of $L\wr A_n$ which is isomorphic to $Fun(A_n,L)$.

Let H be a subset of the Lie algebra $Fun(A_n,L)$. We denote by $[0,H]$ the subset $\{[0,f]\mid f\in H\}$ of the Lie algebra $L\wr A_n$.

Proposition 3. *Let $Z(L)$ be the center of the Lie algebra L then*

$$Z(L\wr A_n)=[0,Z(L)],$$

where the elements of $Z(L)$ are considered as the constant polynomials of $Fun(A_n,L)$.

Proof. By definition of the center we have

$$Z(L \wr A_n) = \{[\chi, g] \in L \wr A_n \mid ([\eta, g], [\alpha, f]) = [0, \chi f' - \alpha g' + (f, g)] = \\ = [0, e] \text{ for all } [\alpha, f] \in L \wr A_n\},$$

where $e(\alpha) = 0_L$ for any $\alpha \in A_n$.

The following equality

$$\chi f' - \alpha g' + (f, g) = e \tag{3}$$

take place for any $\alpha \in A_n$ and any $f \in Fun(A_n, L)$ if and only if the following conditions hold simultaneously

$$\chi = 0, \ g' = e, \ (f, g) = e. \tag{4}$$

Really, if (4) is correct then (3) holds.

Suppose that. Then $\chi \neq 0$ then for $\alpha = 0$ and $f = lx_1$, $l \in L$, $l \neq 0$

$$\chi f' - \alpha g' + (f, g) = \chi l + (lx_1, g) \neq e,$$

for any $g \in A_n$, since the degree of the polynomial (lx_1, g) is greater than the degree of lx_1 or $(lx_1, g) = 0$.

Suppose that $g \in Fun(A_n, L)$ such that $g' \neq e$ then for $f = l$ and some $\alpha \in A_n$

$$\chi f' - \alpha g' + (f, g) = 0 - \alpha g' + (l, g) \neq e,$$

for any $\chi \in A_n$, since the degree of the polynomial (l, g) is equal to the degree of g and is greater than the degree of the polynomial g'.

Thus if (3) is correct then (4) holds.

Since $g' = e$ then g is a constant polynomial of $Fun(A_n, L)$, i.e. $g = l$, where $l \in L$. Since $(f, l) = e$ for any $f \in Fun(A_n, L)$ then from the definition of the Lie bracket (1) in the Lie algebra $Fun(A_n, L)$ we have $l \in Z(L)$.

Thus we obtain

$$Z(L \wr A_n) = \{[0, l] \mid l \in Z(L)\}.$$

□

Proposition 4. *Let L be a solvable Lie algebra of the derived length l. Then the Lie algebra $L \wr A_n$ is solvable of the derived length $l + 1$.*

Proof. Since the calculation of a term of the commutant series of the Lie algebra $L \wr A_n$ does not depend on the dimension of the abelian Lie algebra A_n, the proof is carried without changes from the proof of Proposition 3 for the case $n = 1$ in [15]. □

Proposition 5. *Let L be a nilpotent Lie algebra of the nilpotent class l. Then the Lie algebra $L \wr A_n$ is nilpotent of the nilpotent class ld, where $d = n(p - 1) + 1$.*

Proof. Consider the lower central series of the Lie algebra $L \wr A_n$. Let $\gamma_{k+1}(L \wr A_n) = (\gamma_k(L \wr A_n), L \wr A_n)$, $k \geq 1$, be the $k + 1$-th term of the lower central series.

Denote $F_k = \{f \mid [0, f] \in \gamma_k(L \wr A_n)\} \subseteq Fun(A_n, L)$, $k \geq 2$. Then $\gamma_k(L \wr A_n) = [0, F_k]$. It follows from (2) that if we take any polynomial $f \in F_k$, $k \in \{2, \ldots, n(p - 1) + 1\}$, then the degrees of its monomials (i.d. the sum of degrees of its variables) whose coefficients belong to $L \setminus \gamma_2(L)$ are $\leq n(p-1)-k+1$ and f has also monomials of degrees $\leq n(p-1)$ whose coefficients belong to $\gamma_2(L)$.

It is easy to see that $F_{n(p-1)+1+1} \subseteq Fun(A_n, \gamma_2(L))$. In a similar way we obtain $F_{d+d+1} \subseteq Fun(A_n, \gamma_3(L))$ and so on. Thus,

$$\gamma_{(d\cdot l+1)}(L \wr A_n) = [0, F_{(d\cdot l+1)}] \subseteq [0, Fun(A_n, \gamma_{l+1}(L))] = [0,0].$$

Therefore, if L is nilpotent Lie algebra of the nilpotent class l then the Lie algebra $L \wr A_n$ is nilpotent of the nilpotent class at most ld.

Notice that $[0, L] \subseteq \gamma_k(L \wr A_n)$, $2 \leq k \leq d$, where we consider the elements of L as constant polynomials. In a similar way we obtain $[0, \gamma_2(L)] \subseteq \gamma_s(L \wr A_n), d+1 \leq s \leq 2d$. Hence we have

$$[0, \gamma_{(l)}(L)] \subseteq \gamma_s(L \wr A_n), (l-1)d+1 \leq s \leq ld.$$

Consequently, the Lie algebra $L \wr A_n$ is nilpotent of the nilpotent class at least ld.

Thus the Lie algebra $L \wr A_n$ is nilpotent of the nilpotent class ld. □

2. Wreath product of elementary abelian groups. Consider the group $P_{m,n} = \underbrace{(C_p)^n \wr \ldots \wr (C_p)^n}_{m}$, where $(C_p)^n$ is the direct product of n copies of the cyclic group C_p. The elementary abelian p-group of rank n can be considered as additive group of the vector space V_n of dimension n over the field $\mathbb{F}_p$. The elements of this group in fixed basis of the space V_n have the tableau representation of the form ([13])

$$\begin{bmatrix} a_{11} & a_{12}(x_{11},\ldots,x_{1n}) & \ldots & a_{1m}(x_{11},\ldots,x_{1n},\ldots,x_{m-1,1},\ldots,x_{m-1,n}) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2}(x_{11},\ldots,x_{1n}) & \ldots & a_{nm}(x_{11},\ldots,x_{1n},\ldots,x_{m-1,1},\ldots,x_{m-1,n}) \end{bmatrix} \tag{5}$$

where $a_{i1} \in \mathbb{F}_p$ and on the intersection of k-th column and i-th row stands the polynomial $a_{ik}(x_{11},\ldots,x_{1n},\ldots,x_{k-1,1},\ldots,x_{k-1,n})$ over $\mathbb{F}_p$ reduced by modulo of ideal $\langle x_{11}^p - x_{11},\ldots,x_{1n}^p - x_{1n},\ldots,x_{k-1,1}^p - x_{k-1,1},\ldots,x_{k-1,n}^p - x_{k-1,n}\rangle$ $(i \in \{1,\ldots,n\}; k \in \{2,\ldots,m\})$.

We denote the k-th column of the tableau (5) by $\mathbf{a}(X_1,\ldots,X_{k-1})$ or $\mathbf{a}(\overline{X}_{k-1})$ and the (i,k)-th coordinate of the tableau (5) as $a_{ik}(X_1,\ldots,X_{k-1})$ or a_{ik} for short. We say that the tableau (5) has depth k if its k first columns are zeros and at least one coordinate of the $k+1$-th column is nonzero.

Then the identity of the group $P_{m,n}$ is the tableau with zeros coordinates. The product of two tableaux with coordinates $a_{ik}(x_{11},\ldots,x_{1n},\ldots,x_{k-1,1},\ldots,x_{k-1,n})$ and $b_{ik}(x_{11},\ldots,x_{1n},\ldots,x_{k-1,1},\ldots,x_{k-1,n})$ is the tableau with coordinates

$$\begin{gathered} a_{ik}(x_{11},\ldots,x_{1n},\ldots,x_{k-1,1},\ldots,x_{k-1,n}) + b_{ik}(x_{11}-a_{11},\ldots,x_{1n} - \\ -a_{n1},\ldots,x_{k-1,1}-a_{1,k-1}(x_{11},\ldots,x_{k-2,n}),\ldots,x_{k-1,n} - \\ -a_{n,k-1}(x_{11},\ldots,x_{k-2,n})),\ i \in \{1,\ldots,n\}; k \in \{1,\ldots,m\}. \end{gathered}$$

The inverse for the tableau (5) is the tableau with coordinates

$$\begin{gathered} -a_{ik}(x_{11}+a_{11},\ldots,x_{1n}+a_{n1},\ldots,x_{k-1,1}+a_{1,k-1}(x_{11}+ \\ +a_{11},\ldots),\ldots,x_{k-1,n}+a_{n,k-1}(x_{11}+a_{11},\ldots)), \\ i \in \{1,\ldots,n\}; k \in \{1,\ldots,m\}. \end{gathered}$$

The height of a monomial $x_{11}^{i_{11}}\dots x_{1n}^{i_{1n}}\dots x_{k1}^{i_{k1}}\dots x_{kn}^{i_{kn}}$ is defined as the positive integer number

$$h=\sum_{l=1}^{k} d^{l-1}\sum_{j=1}^{n} i_{lj}+1,$$

where $d=n(p-1)+1$. The height of zero monomial is equal to zero. The height of a reduced polynomial is defined as the largest height of its monomials.

Denote by $\{u\}_{ik}$ the (i,k)-th coordinate of the tableau $u\in P_{m,n}$. The matrix $\|h(\{u\}_{ik})\|$ is called the characteristic of the tableau u and is denoted by $h(u)$. Introduce the partial order of coordinate-wise comparison on the set of all characteristics of tableaux: $h(u)<h(v)$ if and only if $h(\{u\}_{ik})\le h(\{v\}_{ik})$ for all $i\in\{1,\dots,n\}; k=\{1,\dots,m\}$.

Definition 3. A subgroup $U\subset P_{m,n}$ is called *parallelotopic* if for every $u\in U$ and $v\in P_{m,n}$ the inequality $h(v)\le h(u)$ implies $v\in U$. Such a subgroup is completely determined by the matrix with elements

$$|U|_{ij}=\max_{u\in U}\{h(\{u\}_{ij})\},\quad i\in\{1,\dots,n\};\ j\in\{1,\dots,m\}.$$

This matrix is called the characteristic of the parallelotopic subgroup U and is denoted by $h(U)$.

Definition 4. A normal parallelotopic subgroup U of $P_{m,n}$ is called *homogeneous* if the elements of each column of the matrix $h(U)$ coincide.

Hence each homogeneous parallelotopic subgroup is uniquely determined by the sequence $\langle k_1,k_2,\dots,k_m\rangle$, where

$$k_j=\max_{u\in U}\{h(\{u\}_{i,j}),\quad i\in\{1,\dots,n\}\}.$$

The following theorem is proved in [13].

Theorem 6. *The k-th term $\gamma_k(P_{m,n})$ of the lower central series of the group $P_{m,n}$ is homogeneous parallelotopic subgroup with characteristic*

$$<(1-k+1)^+,(d-k+1)^+,\dots,(d^{m-1}-k+1)^+>$$

where $d=n(p-1)+1$, $(d^{i-1}-k+1)^+=max\{0,d^{i-1}-k+1\}$ ($i\in\{1,\dots,m\}; k\ge 1$).

Corollary 6.1. *The group $P_{m,n}$ is nilpotent of the nilpotent class d^{m-1}.*

The subgroup $\gamma_k(P_{m,n})$ has depth l if and only if the inequality $d^{l-1}+1\le k<d^l+1$ holds.

Corollary 6.2. *The subgroup $\gamma_k(P_{m,n})$, $d^{l-1}+1\le k<d^l+1$, is generated by modulo of $\gamma_{k+1}(P_{m,n})$ by tableaux of the form*

$$u_{ijt}=[\underbrace{0,\dots,0}_{i},a_{jt}(X_1,\dots,X_i),0,\dots,0],$$

where $a_{jt}(X_1,\dots,X_i)$ is the $(i+1)$-th column with the unique nonzero coordinate which contains the monomial of the height d^i-k+1, $i\in\{l,\dots,m-1\}$, $j\in\{1,\dots,n\}$, $t\in\{1,\dots,r_{ji}\}$, where r_{ji} is the number of monomials on the (j,i)-th coordinate of the height d^i-k+1.

Corollary 6.3. *The quotients $\gamma_k(P_{m,n})/\gamma_{k+1}(P_{m,n})$ of the lower central series of the group $P_{m,n}$ are elementary abelian p-groups.*

3. Lie algebra $L_{m,n}$. Consider the set $L_{m,n}$ of tableaux of the form

$$\begin{bmatrix} c_{11} & c_{12}(x_{11},\dots,x_{1n}) & \dots & c_{1m}(x_{11},\dots,x_{1n},\dots,x_{m-1,1},\dots,x_{m-1,n}) \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2}(x_{11},\dots,x_{1n}) & \dots & c_{nm}(x_{11},\dots,x_{1n},\dots,x_{m-1,1},\dots,x_{m-1,n}) \end{bmatrix} \tag{6}$$

where $c_{i1} \in \mathbb{F}_p$ and $c_{ik}(x_{11},\dots,x_{1n},\dots,x_{k-1,1},\dots,x_{k-1,n})$ is a polynomial over $\mathbb{F}_p$ reduced by modulo of ideal $\langle x_{11}^p,\dots,x_{1n}^p,\dots,x_{k-1,1}^p,\dots,x_{k-1,n}^p\rangle$ $(i \in \{1,\dots,n\}; k \in \{2,\dots,m\})$.

We will use the same notations for tableau (6) as in §3.

On the set $L_{m,n}$ we introduce the structure of Lie algebra in the following way. Define the addition, the multiplication on the elements of the field $\mathbb{F}_p$ and the Lie bracket $(\,,\,)$ for tableaux u, v by the following equalities $(1 \le i \le n, 1 \le k \le m)$:

$$(i) \quad \{u+v\}_{ik} = \{u\}_{ik} + \{v\}_{ik};$$

$$(ii) \quad \{\alpha u\}_{ik} = \alpha\{u\}_{ik}, \alpha \in \mathbb{F}_p;$$

$$(iii) \quad \{(u,v)\}_{ik} = \sum_{s=1}^{k-1}\sum_{j=1}^{n}\left(\frac{\partial\{v\}_{ik}}{\partial x_{sj}}\cdot\{u\}_{js} - \frac{\partial\{u\}_{ik}}{\partial x_{sj}}\cdot\{v\}_{js}\right).$$

Theorem 7. *The set $L_{m,n}$ with operations (i) – (iii) forms a Lie algebra over the field $\mathbb{F}_p$.*

Proof. The set $L_{m,n}$ with the addition (i) is obviously an elementary abelian p-group. It is also easy to check that the Lie bracket defined by the equality (iii) is linear and satisfies the identity $(u,u) = 0$. Hence it is sufficient to check the Jakobi identity. Let $u, v, w \in L_{m,n}$ be any tableaux. Then

$$\{(u,(v,w))\}_{ik} = \sum_{s=1}^{k-1}\sum_{j=1}^{n}\left(\frac{\partial\{(v,w)\}_{ik}}{\partial x_{sj}}\cdot\{u\}_{js} - \{(v,w)\}_{js}\cdot\frac{\partial\{u\}_{ik}}{\partial x_{sj}}\right) =$$

$$= \sum_{s=1}^{k-1}\sum_{j=1}^{n}\Bigg(\frac{\partial}{\partial x_{sj}}\left(\sum_{t=1}^{k-1}\sum_{r=1}^{n}\left(\frac{\partial\{w\}_{ik}}{\partial x_{tr}}\cdot\{v\}_{rt} - \{w\}_{rt}\cdot\frac{\partial\{v\}_{ik}}{\partial x_{tr}}\right)\right)\cdot\{u\}_{js} -$$
$$- \left(\sum_{t=1}^{s-1}\sum_{r=1}^{n}\left(\frac{\partial\{w\}_{js}}{\partial x_{tr}}\cdot\{v\}_{rt} - \{w\}_{rt}\cdot\frac{\partial\{v\}_{js}}{\partial x_{tr}}\right)\right)\cdot\frac{\partial\{u\}_{ik}}{\partial x_{sj}}\Bigg) =$$

$$= \sum_{s=1}^{k-1}\sum_{j=1}^{n}\sum_{t=1}^{k-1}\sum_{r=1}^{n}\Bigg(\frac{\partial\{w\}_{ik}}{\partial x_{sj}\partial x_{tr}}\cdot\{v\}_{rt}\{u\}_{js} + \frac{\partial\{w\}_{ik}}{\partial x_{tr}}\frac{\partial\{v\}_{rt}}{\partial x_{sj}}\cdot\{u\}_{js} -$$
$$- \frac{\partial\{v\}_{ik}}{\partial x_{sj}\partial x_{tr}}\cdot\{w\}_{rt}\{u\}_{js} - \frac{\partial\{v\}_{ik}}{\partial x_{tr}}\frac{\partial\{w\}_{rt}}{\partial x_{sj}}\cdot\{u\}_{js}\Bigg) -$$
$$- \sum_{s=1}^{k-1}\sum_{j=1}^{n}\sum_{t=1}^{s-1}\sum_{r=1}^{n}\left(\frac{\partial\{u\}_{ik}}{\partial x_{sj}}\frac{\partial\{w\}_{js}}{\partial x_{tr}}\cdot\{v\}_{rt} - \frac{\partial\{u\}_{ik}}{\partial x_{sj}}\frac{\partial\{v\}_{js}}{\partial x_{tr}}\cdot\{w\}_{rt}\right).$$

Notice that the index t in the last sum changes from 1 to $s-1$, because its components with indices $t \geq s$ are equal to zero. Similarly fill expressions $\{(v,(w,u))\}_{ik}$, $\{(w,(u,v))\}_{ik}$. It remains to write down in expanded form the sum

$$\{(u,(v,w))\}_{ik} + \{(v,(w,u))\}_{ik} + \{(w,(u,v))\}_{ik}.$$

It is easy to see that this sum is equal to zero for any $i \in \{1,\dots,n\}; k \in \{1,2,\dots,m\}$. The proof is complete. □

4. Lie algebra associated with group $P_{m,n}$. Recall the general construction of a Lie ring associated with the lower central series of a nilpotent group, described for instance in [8], [10].

Let G be a nilpotent group and

$$G = \gamma_1(G) > \gamma_2(G) > \cdots > \gamma_l(G) = \{e\}$$

be its lower central series.

The Lie ring associated with this series is the direct sum of its quotients

$$L(G) = \bigoplus_{i=1}^{l-1} \gamma_i(G)/\gamma_{i+1}(G)$$

with Lie bracket defined for homogeneous elements $\overline{u} = u \cdot \gamma_{i+1}(G)$, $\overline{v} = v \cdot \gamma_{j+1}(G) \in L(G)$ by the equality

$$(\overline{u},\overline{v}) = (u \cdot \gamma_{i+1}(G), v \cdot \gamma_{j+1}(G)) = [u,v] \cdot \gamma_{i+j+1}(G),$$

For all elements of $L(G)$ the Lie bracket extends by bi-additivity.

If each quotient $\gamma_i(G)/\gamma_{i+1}(G)$ $(i \in \{1,2,\dots\})$ is an elementary abelian p-group for fixed prime p then each of these quotients, and thus the whole ring $L(G)$, can be considered as an additive group of a vector space over the field $\mathbb{F}_p$ of appropriate dimension. In this case it is possible to introduce the operation of multiplication of the elements of $L(G)$ on the elements of $\mathbb{F}_p$. This operation transforms the Lie ring $L(G)$ to the Lie algebra over the field $\mathbb{F}_p$.

Theorem 8. *The Lie algebra $L(P_{m,n})$ associated with the lower central series of the group $P_{m,n}$ is isomorphic to the Lie algebra $L_{m,n}$.*

Proof. Let $\overline{u} = u \cdot \gamma_{k+1}(P_{m,n})$ be a homogeneous element of the Lie algebra $L(P_{m,n})$, i.d. $\overline{u} \in \gamma_k(P_{m,n})/\gamma_{k+1}(P_{m,n})$ for some $k \in \{1,2,\dots,d^{m-1}\}$. By Theorem 6 we have

$$h(\gamma_k(P_{m,n})) =< \underbrace{0,\dots,0}_{l}, (d^l - k + 1), \dots, (d^{m-1} - k + 1) >$$

where $d = n(p-1)+1$ and $l = [\log_d(k-1)] + 1$.

>From the corollaries of Theorem 6 it follows that $\gamma_k(P_{m,n})/\gamma_{k+1}(P_{m,n})$ is an elementary abelian p-group generated by modulo of $\gamma_{k+1}(P_{m,n})$ by tableaux of the form

$$u_{ijt} = [\underbrace{\mathbf{0},\dots,\mathbf{0}}_{i}, \mathbf{a}_{jt}(X_1,\dots,X_i), 0,\dots,0], \tag{7}$$

where $\mathbf{a}_{jt}(X_1, \ldots, X_i)$ is the $(i+1)$-th column with the unique nonzero coordinate which contains the monomial of the height $d^i - k + 1$, $i \in \{l, \ldots, m-1\}$, $j \in \{1, \ldots, n\}$, $t \in \{1, \ldots, r_{ji}\}$, where r_{ji} is the number of monomials on the (j, i)-th coordinate of the height $d^i - k + 1$. Then the tableau u can be represented in the form

$$u = \sum_{i=l}^{m-1} \sum_{j=1}^{n} \sum_{t=1}^{r_{ji}} \alpha_{ijt} u_{ijt}, \quad \alpha_{ijt} \in \mathbb{F}_p. \tag{8}$$

Consider the map $\varphi : L(P_{m,n}) \to L_{m,n}$ which maps the homogeneous element $\overline{u} = u \cdot \gamma_{k+1}(P_{m,n}) \in L(P_{m,n})$ to the tableau $u \in L_{m,n}$. Using (7) and (8), we have

$$\varphi(\overline{u}) = \left[\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{l}, \sum_{j=1}^{n} \sum_{t=1}^{r_{ji}} \alpha_{ljt} \mathbf{a}_{jt}(\overline{X}_l), \ldots, \sum_{j=1}^{n} \sum_{t=1}^{r_{ji}} \alpha_{m-1,j,t} \mathbf{a}_{jt}(\overline{X}_{m-1}) \right]$$

Arbitrary element $\overline{v} \in L(P_{m,n})$ is the sum of homogeneous elements $\overline{u}_i$. Then the map φ for any $\overline{v} \in L(P_{m,n})$ is defined by the following equality

$$\varphi(\overline{v}) = \sum_{i=1}^{d^m} \varphi(\overline{u}_i)$$

The map φ is well-defined, because $\varphi(\overline{v})$ is a tableau. Moreover each tableau $w \in L_{m,n}$ can be uniquely presented as a sum of tableaux of the form (7). Thus the constructed map is a bijection.

It is easy to check that $\varphi(\overline{u}+\overline{v}) = \varphi(\overline{u}) + \varphi(\overline{v})$ and $\varphi(\alpha\overline{u}) = \alpha\varphi(\overline{u})$ for any $\overline{u}, \overline{v} \in L(P_{m,n})$, $\alpha \in \mathbb{F}_p$. It remains to check that the map φ is a homomorphism, i.d.

$$\varphi((\overline{u}, \overline{v})) = (\varphi(\overline{u}), \varphi(\overline{v})).$$

Let $\overline{u} = u \cdot \gamma_{i+1}(P_{m,n})$, $\overline{v} = v \cdot \gamma_{j+1}(P_{m,n})$ be some homogeneous elements of the Lie algebra $L(P_{m,n})$. Moreover such that elements $u \in \gamma_i(P_{m,n})$, $v \in \gamma_j(P_{m,n})$ have unique nonzero (r, k)-th coordinate with the monomial $a_r x_{11}^{i_{11}^r} \ldots x_{1n}^{i_{1n}^r} \ldots x_{k1}^{i_{k1}^r} \ldots x_{kn}^{i_{kn}^r}$ of the tableau u and (t, l)-th coordinate with the monomial $b_r x_{11}^{j_{11}^t} \ldots x_{1n}^{j_{1n}^t} \ldots x_{l1}^{j_{l1}^t} \ldots x_{ln}^{j_{ln}^t}$ of the tableau v. Notice, that the hight of the monomial $a_r x_{11}^{i_{11}^r} \ldots x_{1n}^{i_{1n}^r} \ldots x_{k1}^{i_{k1}^r} \ldots x_{kn}^{i_{kn}^r}$ is equal to $d^k - i + 1$ and the hight of the monomial $b_r x_{11}^{j_{11}^t} \ldots x_{1n}^{j_{1n}^t} \ldots x_{l1}^{j_{l1}^t} \ldots x_{ln}^{j_{ln}^t}$ is equal to $d^l - j + 1$.

If $k = l$ then $[u, v] = 0$. Let $k \neq l$ and without loss of generality suppose that $k < l$. Then

$$(\overline{u}, \overline{v}) = [u, v] \cdot \gamma_{i+j+1}(P_{m,n})$$

$$\text{and} \quad (i) \quad \{[u, v]\}_{ts} = 0, \quad \text{for} \quad s \neq l+1, (t \in \{1, \ldots, n\});$$

$$(ii) \qquad \{[u, v]\}_{t,l+1} = \{u^{-1} v^{-1} u v\}_{t,l+1} =$$

$$= b_t x_{11}^{j_{11}^t} \ldots x_{1n}^{j_{1n}^t} \ldots x_{l1}^{j_{l1}^t} \ldots x_{ln}^{j_{ln}^t} - b_t x_{11}^{j_{11}^t} \ldots x_{1n}^{j_{1n}^t} \ldots x_{k1}^{j_{k1}^t} \ldots x_{k+1,r-1}^{j_{k+1,r-1}^t} \times$$

$$\times \left(x_{k+1,r} - a_r x_{11}^{i_{11}^r} \ldots x_{kn}^{i_{kn}^r} \right)^{j_{k+1,r}^t} \cdot x_{k+1,r+1}^{j_{k+1,j-1}^t} \ldots x_{k+1,n}^{j_{k+1,n}^t} \ldots x_{ln}^{j_{ln}^t} =$$

$$= b_t x_{11}^{j_{11}^t} \ldots x_{ln}^{j_{ln}^t} - b_t x_{11}^{j_{11}^t} \ldots x_{k+1,r-1}^{j_{k+1,r-1}^t} \times$$

$$\times \left(\sum_{s=0}^{j^t_{k+1,r}} (-1)^s C^s_{j^t_{k+1,r}} x_{k+1,r}^{j^t_{k+1,r}-s} \left(a_r x_{11}^{i^r_{11}} \dots x_{kn}^{i^r_{kn}}\right)^s \right) \cdot x_{k+1,r+1}^{j^t_{k+1,j-1}} \dots x_{ln}^{j^t_{ln}} =$$

$$= b_t x_{11}^{j^t_{11}} \dots x_{ln}^{j^t_{ln}} - b_t x_{11}^{j^t_{11}} \dots x_{kn}^{j^t_{kn}} \dots x_{k+1,r-1}^{j^t_{k+1,r-1}} \times$$

$$\times \left(x_{k+1,r}^{j^t_{k+1,r}} - j^t_{k+1,r} x_{k+1,r}^{j^t_{k+1,r}-1} a_r x_{11}^{i^r_{11}} \dots x_{kn}^{i^r_{kn}} \right) x_{k+1,r+1}^{j^t_{k+1,r+1}} \dots x_{ln}^{j^t_{ln}} =$$

$$= j^t_{k+1,r} b_t x_{11}^{j^t_{11}} \dots x_{kn}^{j^t_{kn}} x_{k+1,r-1}^{j^t_{k+1,r-1}} x_{k+1,r}^{j^t_{k+1,r}-1} x_{k+1,r+1}^{j^t_{k+1,r+1}} \dots \times$$

$$\times \dots x_{k+1,n}^{j^t_{k+1,n}} \dots x_{ln}^{j^t_{ln}} \cdot a_r x_{11}^{i^r_{11}} \dots x_{kn}^{i^r_{kn}} =$$

$$= \frac{\partial}{\partial x_{k+1,r}} \left(b_t x_{11}^{j^t_{11}} \dots x_{1n}^{j^t_{1n}} \dots x_{l1}^{j^t_{l1}} \dots x_{ln}^{j^t_{ln}} \right) \cdot a_r x_{11}^{i^r_{11}} \dots x_{kn}^{i^r_{kn}} =$$

$$= \frac{\partial \{v\}_{t,l+1}}{\partial x_{k+1,r}} \cdot \{u\}_{r,k+1} - \frac{\partial \{u\}_{t,l+1}}{\partial x_{k+1,r}} \cdot \{v\}_{r,k+1}, \quad \text{since } \{u\}_{t,l+1} = 0.$$

Since each element of the Lie algebra $L(P_{m,n})$ can be represented as the sum of elements of the form

$$\begin{bmatrix} 0 & \dots & 0 & & 0 & & 0 & \dots & 0 \\ \dots & \dots & \dots & x_{11}^{i^t_{11}} \dots x_{1n}^{i^t_{1n}} \dots & x_{k-1,1}^{i^t_{k-1,1}} & \dots x_{k-1,n}^{i^t_{k-1,n}} & \dots & \dots & \dots \\ 0 & \dots & 0 & & 0 & & 0 & \dots & 0 \end{bmatrix} \cdot \gamma_{i+1}(P_{m,n}),$$

($i \in \{1, \dots, d^{m-1}\}; k \in \{1, \dots, m\}$), using linearity of the Lie bracket we have that the constructed bijection agrees with the Lie bracket.

Remark that under taking the commutator of two tableaux of the group $P_{m,n}$ in case (ii), the elements of the form x^s_{rq}, where $s \geq p$ in $P_{m,n}$ reduced using the equality $x^p_{rq} = x_{rq}$ are equal to x^l_{rq}, where $l < p$. In this case the calculation of the hight of the polynomial $\{[u,v]\}_{t,l+1}$ gives that $\{[u,v]\}_{t,l+1} \in \gamma_{i+j+1}(P_{m,n})$. Hence $[u,v]$ is equal to zero by modulo $\gamma_{i+j+1}(P_{m,n})$. Thus in the definition of the Lie algebra $L(P_{m,n})$ we should use equations $x^p_{rq} = 0$. □

Theorem 9. $L_{m,n} \simeq \underbrace{A_n \wr A_n \wr \dots A_n}_{m}$, *where A_n is the abelian Lie algebra of dimension n over the field $\mathbb{F}_p$.*

Proof. We note that since $P_{m,n} = (C_p)^n \wr (C_p)^n \wr \dots \wr (C_p)^n$ and by Theorem 8 $L_{m,n} \simeq L(P_{m,n})$ then we can replace the assertion of the theorem by

$$L((C_p)^n \wr (C_p)^n \wr \dots \wr (C_p)^n) \simeq A_n \wr A_n \wr \dots A_n.$$

We will prove the theorem by induction on the number of the components of the wreath product. Define $\mathcal{L}_m = \underbrace{A_n \wr \dots \wr A_n}_{m}$.

If $m = 1$ then $L((C_p)^n) \simeq A_n$ and the assertion is correct. Suppose that the assertion is true for m, that is $L(P_{m,n}) \simeq \mathcal{L}_m$. We will show that

$$\mathcal{L}_m \wr A_n \simeq L(P_{m,n} \wr (C_p)^n).$$

Every function $f : A_n \to \mathcal{L}_m$ can be uniquely represented by the tableau

$$[\mathbf{a}(X_1), \mathbf{a}(X_1, X_2), \dots, \mathbf{a}(X_1, \dots, X_m)], \tag{9}$$

Really, $f(X_1)$ is a polynomial of n variables over the Lie algebra $\mathcal{L}_m$ of degree at most $p-1$ for each variable. According to the assumption of induction and Theorem 8 the coefficients of $f(X_1)$ are represented by tableaux of the form $[\mathbf{b}, \mathbf{b}(X_2), \dots, \mathbf{b}(X_2, \dots, X_m)] \in \mathcal{L}_m$. Then $f(X)$ is uniquely represented in the form $[\mathbf{a}(X_1), \mathbf{a}(X_1, X_2), \dots, \mathbf{a}(X_1, \dots, X_m)]$, where $\mathbf{a}(X_1, \dots, X_i)$ is the i-th column with polynomials of ni variables.

Then $D_\alpha(f)$, where $\alpha = (\alpha_1 \dots, \alpha_n) \in A_n$, is represented in the form $[\mathbf{q}(X_1), \mathbf{q}(X_1, X_2), \dots, \mathbf{q}(X_1, \dots, X_m)]$, where

$$q_{i,k}(X_1, \dots, X_{k-1}) = \sum_{j=1}^{n} \alpha_j \frac{\partial a_{ik}(X_1, \dots, X_{k-1})}{\partial x_{1j}} \tag{10}$$

Moreover, for any functions

$$f = [\mathbf{a}(X_1), \mathbf{a}(X_1, X_2), \dots, \mathbf{a}(X_1, \dots, X_m)], \quad g = [\mathbf{b}(X_1), \mathbf{b}(X_1, X_2), \dots, \mathbf{b}(X_1, \dots, X_m)]$$

the function (f, g) is of the form $[0, \mathbf{c}(X_1, X_2), \dots, \mathbf{c}(X_1, \dots, X_m)]$, where

$$c_{ik}(X_1, \dots, X_{k-1}) = \sum_{s=2}^{k-1} \sum_{j=1}^{n} \left(\frac{\partial b_{ik}}{\partial x_{s,j}} \cdot a_{js} - \frac{\partial a_{ik}}{\partial x_{s,j}} \cdot b_{js} \right) \tag{11}$$

Indeed, from the linearity of the representation (9) follows that it is enough to verify (11) only for monomials. Let $f = lx_{11}^{m_1} \dots x_{1n}^{m_n}$ and $g = hx_{11}^{k_1} \dots x_{1n}^{k_n}$, where $l = [\mathbf{l}, \mathbf{l}(X_2), \dots, \mathbf{l}(X_2, \dots, X_m)]$, $h = [\mathbf{h}, \mathbf{h}(X_2), \dots, \mathbf{h}(X_2, \dots, X_m)] \in \mathcal{L}_m$. Then

$$\begin{aligned} f &= [\mathbf{l}x_{11}^{m_1} \dots x_{1n}^{m_n}, \mathbf{l}(X_2)x_{11}^{m_1} \dots x_{1n}^{m_n}, \dots, \mathbf{l}(X_2, \dots, X_m)x_{11}^{m_1} \dots x_{1n}^{m_n}], \\ g &= [\mathbf{h}x_{11}^{k_1} \dots x_{1n}^{k_n}, \mathbf{h}(X_2)x_{11}^{k_1} \dots x_{1n}^{k_n}, \dots, \mathbf{h}(X_2, \dots, X_m)x_{11}^{k_1} \dots x_{1n}^{k_n}] \end{aligned}$$

The coefficients from (11) look like:

$$c_{ik}(X_1, \dots, X_{k-1}) = \sum_{s=2}^{k-1} \sum_{j=1}^{n} \Big(l_{js} x_{11}^{m_1} \dots x_{1n}^{m_n} \frac{\partial h_{ik}}{\partial x_{s,j}} x_{11}^{k_1} \dots x_{1n}^{k_n} - \\ - h_{js} x_{11}^{k_1} \dots x_{1n}^{k_n} \frac{\partial l_{ik}}{\partial x_{s,j}} x_{11}^{m_1} \dots x_{1n}^{m_n} \Big) =$$

$$= \begin{cases} \sum\limits_{s=2}^{k-1} \sum\limits_{j=1}^{n} \left(\frac{\partial h_{ik}}{\partial x_{sj}} l_{js} - \frac{\partial l_{ik}}{\partial x_{sj}} h_{js} \right) x_{11}^{m_1+k_1} \dots x_{1n}^{m_n+k_n}, \\ \qquad \text{if } \ i, m_i + k_i < p \text{ for all } i \in \{1, 2, \dots, n\}; \\ 0, \quad \text{if } \ m_i + k_i \geq p \text{ for some } i, 1 \leq i \leq n. \end{cases}$$

Then the Lie bracket (f,g) is represented by the tableau (9) of the form:

$$(f,g)=$$
$$=\begin{cases}(l,h)x_{11}^{m_1+k_1}\dots x_{1n}^{m_n+k_n}, & \text{if } m_i+k_i<p \text{ for all } i\in\{1,2,\dots,n\};\\ 0, & \text{if } m_i+k_i\ge p \text{ for some } i, 1\le i\le n.\end{cases}=$$

$$=\begin{cases}[0,\mathbf{d}(X_2),\dots,\mathbf{d}(X_2,\dots,X_m)]x_{11}^{m_1+k_1}\dots x_{1n}^{m_n+k_n},\\ \qquad\qquad \text{if } \ m_i+k_i<p \text{ for all } i\in\{1,2,\dots,n\}; \\ 0, \ \text{ if } \ m_i+k_i\ge p \text{ for some } i, 1\le i\le n.\end{cases}=$$

$$=\begin{cases}[0,\mathbf{d}(X_2)x_{11}^{m_1+k_1}\dots x_{1n}^{m_n+k_n},...,\mathbf{d}(X_2,...,X_m)x_{11}^{m_1+k_1}...x_{1n}^{m_n+k_n}],\\ \qquad\qquad \text{if } \ m_i+k_i<p \text{ for all } i\in\{1,2,\dots,n\}; \\ 0, \ \text{ if } \ m_i+k_i\ge p \text{ for some } i, 1\le i\le n,\end{cases}$$

$$\text{where } \ d_{ik}\left(X_2,\dots,X_{k-1}\right)=\sum_{s=2}^{k-1}\sum_{j=1}^{n}\left(\frac{\partial h_{ik}}{\partial x_{s,j}}l_{js}-\frac{\partial l_{ik}}{\partial x_{s,j}}h_{js}\right).$$

Thus the function (f,g) is of the form $[0,\mathbf{c}(X_1,X_2),\dots,\mathbf{c}(X_1,\dots,X_m)]$.

Let us construct the map $\varphi:\mathcal{L}_m\wr A_n\to L(P_{m,n}\wr C_p^n)$ by the rule

$$\varphi([\mathbf{a},f])=[\mathbf{a},\mathbf{a}(X_1),\dots,\mathbf{a}(X_1,\dots,X_m)].$$

Since to every tableau $[\mathbf{a}(X_1),\mathbf{a}(X_1,X_2),\dots,\mathbf{a}(X_1,\dots,X_m)]$ uniquely corresponds some function $f:\ A_n\to\mathcal{L}_m$, the map φ is a bijection.

Let us show that ψ is linear. Really:

$$\psi(\alpha[a,f]+\beta[b,g])=\psi([\alpha a+\beta b,\alpha f+\beta g])=$$
$$=[\alpha\mathbf{a}+\beta\mathbf{b},\alpha\mathbf{a}(X_1)+\beta\mathbf{b}(X_1),...,\alpha\mathbf{a}(X_1,...,X_m)+\beta\mathbf{b}(X_1,...,X_n)]=$$
$$=\alpha[\mathbf{a},\dots,\mathbf{a}(X_1,\dots,X_m)]+\beta[\mathbf{b},\dots,\mathbf{b}(X_1,\dots,X_m]=\alpha\varphi([a,f])+\beta\varphi([b,g]).$$

It remains to prove that $\varphi(([a,f],[b,g]))=(\varphi([a,f]),\varphi([b,g]))$. From (10) and (11) follow:

$$\varphi(([a,f],[b,g]))=\varphi([0,ag'-bf'+(f,g)])=[0,\mathbf{d}(X_1),\dots,\mathbf{d}(X_1,\dots,X_m)],$$

where

$$d_{ik}(\overline{X}_{k-1})=\sum_{j=1}^{n}\left(a_{j1}\frac{\partial b_{ik}}{\partial x_{1,j}}-b_{j1}\frac{\partial a_{ik}}{\partial x_{1,j}}\right)+\sum_{s=2}^{k-1}\sum_{j=1}^{n}\left(\frac{\partial b_{ik}}{\partial x_{s,j}}a_{js}-\frac{\partial a_{ik}}{\partial x_{s,j}}b_{js}\right)$$

Thus we obtain

$$\varphi(([a,f],[b,g]))=([\mathbf{a},\mathbf{a}(X_1),\dots,\mathbf{a}(\overline{X}_m)],[\mathbf{b},\mathbf{b}(X_1),\dots,\mathbf{b}(\overline{X}_m)])=(\varphi([a,f]),\varphi([b,g])).$$

□

Due to Propositions 5 and 4 we have the following

Corollary 9.1. *The Lie algebra $L_{m,n}$ is nilpotent of the nilpotent class d^{m-1} and is solvable of the derived length $m+1$.*

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