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ON ZEROS OF DERIVATIVES OF AN ENTIRE FUNCTION

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For a finite system $S(f) = \{f^{(n_1)}, f^{(n_2)}, \dots, f^{(n_k)}\}$ of derivatives of an entire transcedental function f let $d_{S(f)}(z)$ be the radius of the largest disk with the center at z in which any derivative of $S(f)$ does not vanish. Conditions under which $\sup\{d_{S(f)}(z) : z \in \mathbb{C}\} = +\infty$ are investigated.

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Для конечной системы $S(f) = \{f^{(n_1)}, f^{(n_2)}, \dots, f^{(n_k)}\}$ производных целой трансцендентной функции f пусть $d_{S(f)}(z)$ — радиус наибольшего круга с центром в z, в котором ни одна из производных из $S(f)$ не превращается в ноль. Исследуются условия, при которых $\sup\{d_{S(f)}(z) : z \in \mathbb{C}\} = +\infty$.

1. Introduction. Let $f \not\equiv 0$ be an entire function and $C_f(z) = \max\{|f^{(n)}(z)|/n! : n \in \mathbb{Z}_+\}$. Then there exist non-negative integers $n_1, n_2, \dots, n_k$, which depend on z, such that

$$C_f(z) = \frac{|f^{(n_1)}(z)|}{n_1!} = \frac{|f^{(n_2)}(z)|}{n_2!} = \dots = \frac{|f^{(n_k)}(z)|}{n_k!}. \tag{1}$$

As in [1], we shall call $f^{(n_1)}, \dots, f^{(n_k)}$ the maximal derivatives of f at z and by $\rho_{n_j}(z)$ we denote the radius of the largest disk with the center at z, in which the derivative $f^{(n_j)}$ does not vanish. Finally, let $\rho_f = \sup\{\rho_{n_1}(z) : z \in \mathbb{C}\}$.

In [1] the following theorem is formulated.

Theorem 0. *For an entire function f, $\rho_f = +\infty$ iff $f \not\equiv 0$.*

The proof of Theorem 0 is based on the following lemma.

Lemma 0. *Let f be an entire function and equality (1) is true at z with $k \ge 1$. Then there exists a positive integer $s \in [1, k]$ and a domain W such that $z \in W$ and $|f^{(j)}(w)|/j! < |f^{(n_s)}(w)|/n_s!$ for every $w \in W \setminus \{z\}$ and all $j \ne n_s$.*

The entire function $f(z) = e^{(k+1)z} + P_{k-1}(z)$, where $k \in \mathbb{N}$ and P_{k-1} is a polynomial of the degree $\le k-1$, shows that Lemma 0 is false, because in this case $f^{(k)}(z)/k! = f^{(k+1)}(z)/(k+1)!$ for all $z \in \mathbb{C}$.

It seems that Theorem 0 is also false and the following conjecture is plausible.

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Conjecture. *There exists $R > 0$ such that each derivative $\sigma^{(k)}$ of the Weierstrass sigma-function*

$$\sigma(z) = z \prod_{m,n=-\infty;(m,n)\neq(0,0)}^{+\infty} \left(1 - \frac{z}{m+in}\right) \exp\left\{\frac{z}{m+in} + \frac{z^2}{2(m+in)^2}\right\} \tag{2}$$

has at least one zero in every disk of radius R.

Our investigations arise by the influence of the paper [1].

2. Entire functions with supermaximal derivatives. If for the fixed point $z \in \mathbb{C}$ there exists a $N = N(z) \in \mathbb{Z}_+$ such that $|f^{(j)}(z)|/j! < |f^{(N)}(z)|/N!$ for all $j \neq N$ then we shall call $f^{(N)}$ the supermaximal derivative of f at z. In fact, Lemma 0 asserts that (1) implies the existence of a supermaximal derivative in some pointed neighborhood of z. As we showed above, this statement is false. However there exists a class B of entire non-constant functions f which have a supermaximal derivative at every point $z \in \mathbb{C}$. In fact, the following theorem is true.

Theorem 1. *The class B coincides with the class of entire functions of the form $f(z) = e^{az+b}$, where $a \in \mathbb{C}$, $b \in \mathbb{C}$ and $|a| \notin \mathbb{Z}_+$.*

Proof. It is known [2] that the function $C_f(z)$ is continuous in the whole complex plane. Therefore for each $k \in \mathbb{Z}_+$, where k is the order of maximal derivative, a set

$$G_k = \left\{z: \quad C_f(z) = \frac{|f^{(k)}(z)|}{k!}\right\}$$

is closed.

Since $f \in B$ at every point $z \in \mathbb{C}$ has a supermaximal derivative, hence it follows that there exists a unique number $N_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}$ the inequality $|f^{(j)}(z)|/j! < < |f^{N_0}(z)|/N_0!$ holds, i.e. $N(z) \equiv N_0$.

Indeed, we assume that $N(z) \not\equiv \text{const}$. Let $N_1,\ N_2, \dots$ be the orders of supermaximal derivatives. Remark that every $G_{N_j} \neq \mathbb{C},\ \varnothing$ and G_{N_j} is closed. Since $f \in B$, we have $G_{N_j} \cap G_{N_k} = \varnothing\ (j \neq k)$ and $\mathbb{C} = \bigcup_j G_{N_j}$. Let $I = [A, B]$ be an arbitrary segment with $A \in G_{N_1}$ and $B \in G_{N_2}$. Then $I = \bigcup_{j=1}^{m} I_j$, where $I_j = G_{N_j} \cap I$, I_j are closed, $2 \leq m \leq \infty$ and $I_j \cap I_k = \varnothing\ (j \neq k)$ which is impossible (if m is infinite, this representation is impossible in view of Serpinski Theorem ([6], p.182)).

Thus, there exists $N_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}$

$$|f^{(N_0)}(z)|/N_0! > \max\{|f^{(j)}(z)|/j!:\ j \neq N_0\}, \tag{3}$$

i.e. f is of bounded index. We remind that an entire function f is said to be of bounded index [3] if there exists $N \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}$ and $j \in \mathbb{Z}_+$

$$|f^{(j)}(z)|/j! \leq \max\{|f^{(k)}(z)|/k!:\ 0 \leq k \leq N\}, \tag{4}$$

and the least such integer N is called the index of f and is denoted by $N(f)$.

Since an entire function f of bounded index is [4] of exponential type, the derivative $f^{(N_0)}$ is of exponential type too. Relation (3) implies also that $f^{(N_0)}$ does not vanish in $\mathbb{C}$.

Therefore, the derivative is of the form $f^{(N_0)}(z) = \exp\{az + c\}$, where a and c are complex numbers and $a \neq 0$. Hence

$$f(z) = \frac{e^{az+c}}{a^{N_0}} + a_{N_0-1}z^{N_0-1} + a_{N_0-2}z^{N_0-2} + \cdots + a_0. \tag{5}$$

If $N_0 = 0$ then from (5) we obtain $f(z) = e^{az+b}$ with $b = c$. If $N_0 \geq 1$ then from (3) we have $|f^{N_0}(z)|/N_0! > |f^{(N_0-1)}(z)|/(N_0-1)!$ and, therefore, from (5) we obtain $|e^{az+c}|/N_0 > |e^{az+c}/a + a_{N_0-1}|$, that is $|a_{N_0-1}e^{-az-c} + 1/a| < 1/N_0$ for all $z \in \mathbb{C}$. By Liouville theorem the last inequality is possible only if $a_{N_0-1} = 0$.

Now if $N_0 = 1$ then we have $f(z) = e^{az+b}$ with $b = c + \ln a$. If $N_0 \geq 1$ then by analogy we obtain the inequality $|a_{N_0-2}e^{-az-c} + 1/a^2| < 1/(N_0(N_0-1))$ for all $z \in \mathbb{C}$, whence $a_{N_0-2} = 0$. Continuing the process we obtain the equalities $a_{N_0-2} = \cdots = a_0 = 0$, thus $f(z) = e^{az+b}$ with $b = c + N_0 \ln a$.

In order to complete the proof of Theorem 1 we consider the power series $e^r = \sum_{n=0}^{\infty} \frac{r^n}{n!}$, $r \geq 0$. Let $\mu(r) = \max\{r^n/n! : n \geq 0\}$ be the maximal term of this series. It is known that if $n < r < n+1$, $n \in \mathbb{Z}_+$, then $\mu(r) = r^n/n! > r^k/k!$ for all $k \neq n$ and if $r = n$ then $\mu(r) = r^n/n! = r^{n-1}/(n-1)!$ Therefore, in order that there exist N_0 such that $|a|^{N_0}/N_0! > |a|^j/j!$ for all $j \neq N_0$, it is necessary and sufficient that $|a| \notin \mathbb{Z}_+$. Since for $f(z) = e^{az+b}$ the last relation coinsides with (3), the proof of Theorem 1 is complete. □

We remark that $\rho_f = +\infty$ for $f \in B$.

3. Main theorem. Let f be an entire transcendental function, $S(f) = \{f^{(n_1)}, f^{(n_2)}, \ldots, f^{(n_k)}\}$ be a finite system of its derivatives and $z \in \mathbb{C}$. If $f^{(j)}(z) = 0$ at least for one $j = n_1, n_2, \ldots, n_k$, we put $d_{S(f)}(z) = 0$. If $f^{(j)}(z) \neq 0$ for all $j = n_1, n_2, \ldots, n_k$, then let $d_{S(f)}(z)$ be the radius of the largest disk centered at z, in which any derivative of $S(f)$ does not vanish. Here we investigate conditions on f in order that $\sup\{d_{S(f)}(z) : z \in \mathbb{C}\} = +\infty$. We need the following simple lemma.

Lemma. *If $0 < r < R < +\infty$ then in the ring $\{z : R - r < |z| < R + r\}$ one can inscribe at least $[2R/r]$ non-intersecting disks with centers on the circle $\{z : |z| = R\}$ and with the radius r.*

Proof. Let z_1 and z_2 be points on the circle $\{z : |z| = R\}$ such that the boundaries of the disks $\{z : |z - z_1| < r\}$ and $\{z : |z - z_2| < r\}$ are tangential. Let α be the angle with the apex at the origin and with the sides $[0, z_1]$ and $[0, z_2]$. Since $|z_1 - z_2| = 2r$, we have $\alpha = 2\arcsin(r/R)$. Clearly, one can inscribe $[2\pi/\alpha]$ such tangential disks. Since $\arcsin x \leq \pi x/2$, we have $\alpha \leq \pi r/R$ that is $[2\pi/\alpha] \geq [2R/r]$ and Lemma is proved. □

Theorem 2. *For every entire transcendental function f, the growth of which does not exceed of minimal type of the order 2, and every finite system $S(f)$ of its derivatives* $\sup\{d_{S(f)}(z) : z \in \mathbb{C}\} = +\infty$. *On the other hand, there exists an entire function f of the normal type of the order 2 and a finite system $S(f)$ of its derivatives such that* $\sup\{d_{S(f)}(z) : z \in \mathbb{C}\} < +\infty$.

Proof. Let an entire function f have the growth which does not exceed the minimal type of the order 2 and the system $S(f)$ contains m different derivatives. We remark that the growth of each derivative $f^{(s)}$ does not exceed the minimal type of the order 2 too and, thus, the counting function $n(r, f^{(s)})$ of its zeros has the same bounds on growth.

We assume, on the contrary, that there exists $d > 0$ such that $d_{S(f)}(z) < d$ for all $z \in \mathbb{C}$, that is in every disk $\{z : |z - z_0| < d\}$ there exists at least one zero of at least one derivative of $S(f)$.

We put $R_k = kd$ and $r = d$. Then by Lemma in the the ring $C_k = \{z : R_{2k-1} \le |z| \le R_{2k+1}\}$ there are at least $[2R_{2k}/d] = 4k$ points, in which at least one derivative of $S(f)$ vanishes. Therefore, the disk $\{z : |z| < R_{2k+1}\}$ contains at least $2k^2$ such points. Hence it follows that there exists at least one derivative $f^{(s)} \in S(f)$, which has in the disk $\{z : |z| < R_{2k+1}\}$ at least $2k^2/m$ zeros for infinity numbers of k.

Now if $3d < R_{2k-1} \le t < R_{2k+1}$ then

$$n(t, f^{(s)}) \ge n(R_{2k-1}, f^{(s)}) \ge \frac{2(k-1)^2}{m} = \frac{2}{m}\left(\frac{1}{2}\left(\frac{R_{2k+1}}{d} - 1\right) - 1\right)^2 \ge \frac{1}{2m}\left(\frac{t}{d} - 3\right)^2,$$

i.e. $n(r, f^{(s)})$ has at least normal type of the order 2 and we obtain a contradiction. The first part of Theorem 2 is proved.

On the other hand, let $f(z) = \sigma(z)$, where σ is given by (2). It is known (see, for example, [5]) that the sigma-function has normal type of the order 2. If we put $S(f) = \{f^{(0)}\} = \{\sigma\}$ then $d_{S(f)} = \sqrt{2}/2$. The proof of Theorem 2 is complete. □

If we denote by E_ρ the class of entire transcedental functions of the order $\le \rho$ then from Theorem 2 the following result follows.

Corollary 1. *In order that* $\sup\{d_{S(f)}(z) : z \in \mathbb{C}\} = +\infty$ *for every entire function* $f \in E_\rho$ *and every finite system* $S(f)$ *of its derivatives, it is necessary and sufficient that* $\rho < 2$.

Remark 1. Since an entire function f of bounded index is of exponential type and by definition (4) at each point $z \in \mathbb{C}$ equality (1) holds with $n_1 \le N(f)$, then from Theorem 2 it follows that $\rho_f = +\infty$ for each entire function f of bounded index.

4. Other results. Let $N(r, f)$ and $M(r, f)$ be the usual Nevanlinna characteristics of an entire function f. Theorem 2 admits different generalizations and analogies. For example, the following results are true.

Theorem 3. *Let* f *be an entire transcedental function and let* l *be a positive continuous increasing to* $+\infty$ *function on* $[0, +\infty)$. *If*

$$\ln M(r, f) = o\left(\int_1^{r-1} \frac{dx}{x} \int_0^{x-1} t l(t) dt\right) \quad (r \to +\infty), \tag{6}$$

then $\sup\{d_{S(f)}(z) l(|z|) : z \in \mathbb{C}\} = +\infty$ *for each finite system* $S(f)$ *of derivatives of* f.

Proof. We assume, on the contrary, that there exists $d > 0$ such that $d_{S(f)}(z) < d/l(|z|)$ for all $z \in \mathbb{C}$, that is in every disk $\{z : |z - z_0| < d/l(|z_0|)\}$ there exists at least one zero of at least one derivative of $S(f)$.

Let $C_k = \{z : k - d/l(k) \le |z| < k + d/l(k)\}$. As in the proof of Theorem 2 we can show that in the ring C_k there are at least $[2kl(k)/d]$ points, in which at least one derivative of $S(f)$ has at least one zero.

It is clear that $C_k \cap C_{k+1} = \varnothing$ for all $k \geq k_0$. Let $k > k_0$ and $r \in [k, k+1)$. Then there exists $q > 0$ such that in $\{z : k_0 \leq |z| \leq r\}$ there is at least

$$\sum_{j=k_0}^{k} [2jl(j)/d] \geq q \int_{k_0+1}^{k} tl(t)dt \geq q \int_{k_0+1}^{r-1} tl(t)dt, \quad q = \text{const} > 0,$$

points, in which at least one derivative of $S(f)$ has at least one zero. Hence, as above, it follows that there exists at least one derivative $f^{(s)} \in S(f)$ such that

$$n(r, f^{(s)}) \geq p \int_{0}^{r-1} tl(t)dt, \; p = \text{const} > 0,$$

that is

$$N(r, f^{(s)}) \geq p \int_{1}^{r} \frac{dx}{x} \int_{0}^{x-1} tl(t)dt + O(\ln r) \quad (r \to +\infty),$$

and by the Jensen inequality

$$\ln M(r, f^{(s)}) \geq p \int_{1}^{r} \frac{dx}{x} \int_{0}^{x-1} tl(t)dt + O(\ln r) \quad (r \to +\infty).$$

Since $M(r, f^{(s)})/s! \leq M(r+1, f)$, it follows that

$$\ln M(r+1, f) \geq p \int_{1}^{r} \frac{dx}{x} \int_{0}^{x-1} tl(t)dt + O(\ln r) \quad (r \to +\infty),$$

that is we obtain the contradiction to (6). Theorem 3 is proved. □

Remark 2. If $\overline{\lim\limits_{r\to+\infty}} \frac{l(r+1)}{l(r)} = \eta < +\infty$ then by l'Hospital rule

$$\lim_{r\to+\infty} \frac{\int_1^{r-1} \frac{dx}{x} \int_0^{x-1} tl(t)dt}{\int_1^{r} \frac{dx}{x} \int_0^{x} tl(t)dt} \geq \lim_{r\to+\infty} \frac{\int_0^{r-2} tl(t)dt}{\int_0^{r} tl(t)dt} \geq \lim_{r\to+\infty} \frac{l(r-2)}{l(r)} = \frac{1}{\eta^2} > 0,$$

that is one can replace condition (6) by the condition

$$\ln M(r, f) = o\left(\int_{1}^{r} \frac{dx}{x} \int_{0}^{x} tl(t)dt\right) \quad (r \to +\infty).$$

In particular, for $l(x) = e^x$ the last condition has the form $\ln M(r, f) = o(e^r)$, $r \to +\infty$.

Remark 3. If $\frac{rl'(r)}{l(r)} \leq \beta < +\infty$ for $r \geq r_0$, then

$$\lim_{x\to+\infty} \frac{\int_0^{x} tl(t)dt}{x^2 l(x)} \geq \lim_{r\to+\infty} \frac{xl(x)}{2xl(x) + x^2 l'(x)} \geq \frac{1}{2+\beta}$$

and, therefore,

$$\int_1^r \frac{dx}{x}\int_0^x tl(t)dt \geq (1+o(1))\frac{r^2l(r)}{(2+\beta)^2} \quad (r\to+\infty),$$

that is, in view on Remark 2 condition (6) can be replaced by the condition $\ln M(r,f)=o(r^2l(r))$ $(r\to+\infty)$.

Theorem 4. *Let f be an entire transcedental function and let l be a positive differentiable increasing to $+\infty$ function on $[0,+\infty)$ such that $l'(r)\to 0$ $(r\to+\infty)$. If*

$$\ln M(r,f)=o\left(\int_1^{r-1}\frac{dx}{x}\int_0^x \frac{t}{l^2(t)}dt\right) \quad (r\to+\infty), \tag{7}$$

then $\sup\{d_{S(f)}(z)/l(|z|) : z\in\mathbb{C}\}=+\infty$ for each finite system $S(f)$ of derivatives of f.

Proof. At first note that $l'(r)\to 0$ implies $l(r)=o(r)$ and $l(r+O(l(r)))\sim l(r)$ as $r\to+\infty$.

We assume, on the contrary, that there exists $d>0$ such that $d_{S(f)}(z)<dl(|z|)$ for all $z\in\mathbb{C}$, that is in every disk $\{z: |z-z_0|<dl(|z|)\}$ there exists at least one zero of at least one derivative of $S(f)$.

It is easy to see that $r-dl(r)$ is increasing to $+\infty$, therefore there exists a sequence of positive numbers (r_k) such that $r_{k+1}-dl(r_{k+1})=r_k+dl(r_k)$ and $r_k\uparrow+\infty$, $k\to\infty$.

Let $C_k=\{z: r_k-dl(r_k)<|z|<r_k+dl(r_k)\}$. As above, we can show that in the ring C_k there is at least $\left[\frac{2r_k}{dl(r_k)}\right]$ points, in which at least one derivative of $S(f)$ has at least one zero. On the other hand,

$$\int_{r_k-dl(r_k)}^{r_k+dl(r_k)}\frac{t}{l^2(t)}dt \leq \frac{r_k}{l^2(r_k)}\int_{r_k-dl(r_k)}^{r_k+dl(r_k)} dt = 2Kd\frac{r_k}{l(r_k)}, \quad K=\text{const}>0,$$

that is there exists $\eta>0$ such that in the ring C_k there are more than $\eta\int_{r_k-dl(r_k)}^{r_k+dl(r_k)}\frac{t}{l^2(t)}dt$ points, in which at least one derivative of $S(f)$ has at least one zero.

It is clear that $C_k\cap C_{k+1}=\varnothing$ for all $k>k_0$. Let $r\in[r_k-dl(r_k),\, r_k+dl(r_k))$. Then we have

$$\eta\sum_{j=k_0}^{k-1}\int_{r_j-dl(r_j)}^{r_j+dl(r_j)}\frac{t}{l^2(t)}dt \geq \eta_1\int_{r_0}^{r_k-dl(r_k)}\frac{t}{l^2(t)}dt =$$

$$=\eta_1\left(\int_{r_0}^{r_k+dl(r_k)}\frac{t}{l^2(t)}dt-\int_{r_k-dl(r_k)}^{r_k+dl(r_k)}\frac{t}{l^2(t)}dt\right)\geq \eta_2\int_{r_0}^{r}\frac{t}{l^2(t)}dt,$$

since

$$\int_{r_k-dl(r_k)}^{r_k+dl(r_k)}\frac{t}{l^2(t)}dt = o\left(\int_{r_0}^{r_k+dl(r_k)}\frac{t}{l^2(t)}dt\right)$$

as $k \to \infty$ by l'Hospital rule and in view of the remarks at the beginning of the proof.

Then in $\{z : r_0 \le |z| \le r\}$ there are more than $\eta_2 \int_{r_0}^{r} \frac{t}{l^2(t)} dt$ points, in which at least one derivative of $S(f)$ has at least one zero. Hence, as above, it follows that there exists at least one derivative $f^{(s)} \in S(f)$ such that

$$n(r, f^{(s)}) \ge p \int_0^x \frac{t}{l^2(t)} dt, \quad p = \text{const} > 0,$$

that is

$$N(r, f^{(s)}) \ge p \int_1^r \frac{dx}{x} \int_0^x \frac{t}{l^2(t)} dt + O(\ln r) \quad (r \to +\infty),$$

whence, as in the proof of Theorem 3, we obtain

$$\ln M(r+1, f) \ge p \int_1^r \frac{dx}{x} \int_0^x \frac{t}{l^2(t)} dt + O(\ln r) \quad (r \to +\infty),$$

that is we obtain the contradiction to (7). Theorem 4 is proved. □

Remark 4. Let λ be an arbitrary slowly increasing to $+\infty$ function and $l(r) = r/\lambda(r)$. Since

$$\varliminf_{r\to+\infty} \frac{1}{\ln^2 r} \int_1^{r-1} \frac{dx}{x} \int_0^x \frac{t}{l^2(t)} dt = \varliminf_{r\to+\infty} \frac{1}{\ln^2 r} \int_1^{r-1} \frac{dx}{x} \int_0^x \frac{\lambda^2(t)}{t} dt \ge$$

$$\ge \varliminf_{r\to+\infty} \frac{1}{2\ln r} \int_0^{r-1} \frac{\lambda^2(t)}{t} dt \ge \varliminf_{r\to+\infty} \frac{\lambda^2(r-1)}{2} = +\infty,$$

condition (7) holds provided $\ln M(r, f) = O(\ln^2 r)\,(r \to +\infty)$.

Thus, if $\ln M(r, f) = O(\ln^2 r)\,(r \to +\infty)$ then $\sup\left\{\frac{d_{S(f)}(z)}{|z|}\lambda(|z|) : z \in \mathbb{C}\right\} = +\infty$ for any slowly increasing to $+\infty$ function λ.

5. Possible generalizations. The obtained above results one can extend from the system of derivatives of an entire function on a system of arbitrary entire functions of given growth. For example, Corollary 1 has the following extension.

Let $f = (f_n)$ be a sequence of entire functions of the order $\le \rho$ and $S(f) = \{f_{n_1}, f_{n_2}, \dots, f_{n_k}\}$ be a finite system from f. As above, we define $d_{S(f)}(z)$ by the replacement of $f^{(j)}(z)$ on $f_j(z)$. If $\rho < 2$ then $\sup\{d_{S(f)}(z) : z \in \mathbb{C}\} = +\infty$ and the condition $\rho < 2$ is essential.

In particular, for entire curves we obtain the following proposition: *if the order of the growth of an entire curve $f = (f_1, f_2, \dots, f_p)$ is less than 2 then* $\sup\{d_f(z) : z \in \mathbb{C}\} = +\infty$.

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