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ON FREE PARATOPOLOGICAL GROUPS

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We consider topological properties of free paratopological groups. We also investigate the relations between separation properties of the free paratopological groups and their bases. Some results on cardinal invariants of free paratopological groups are obtained.

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Рассматриваются топологические свойства свободных паратопологических групп. Мы также исследуем соотношение между свойствами отделимости свободных паратопологических групп и их оснований. Получены некоторые результаты о кардинальных инвариантах свободных паратопологических групп.

1. Preliminaries. By a *paratopological group* we understand a pair (G,τ) consisting of a group G and a topology τ on G making the group operation $\cdot\colon G\times G\to G$ of G continuous. If, in addition, the operation $(\cdot)^{-1}\colon G\to G$ of taking the inverse is continuous with respect to the topology τ, then (G,τ) is a *topological group*.

In the paper the word “space” means “topological space”.

Definition 1.1. Let X be a subspace of a paratopological group G with the identity e such that $e\in X$. Suppose that

1. The set X generates G algebraically, that is $\langle X\rangle = G$.
2. Every continuous mapping $f\colon X\to H$ of X to an arbitrary paratopological group H satisfying $f(e)=e_H$, where e_H is the unit of the group H, extends to a continuous homomorphism $f^*\colon G\to H$.

Then G is called *Graev free paratopological group* on (X,e) and is denoted as $FG_p(X,e)$.

If we change the word “group” to the words “abelian group” in the above definition we obtain the definition of *Graev free abelian paratopological group* on X which we denote by $AG_p(X)$.

Definition 1.2. Let X be a subspace of a paratopological group G. Suppose that

1. The set X generates G algebraically, that is $\langle X\rangle = G$

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2. Every continuous mapping $f\colon X \to H$ of X to an arbitrary paratopological group H extends to a continuous homomorphism $f^*\colon G \to H$.

Then G is called *Markov free paratopological group* on X and is denoted by $F_p(X)$.

If we change the word "group" to the word "abelian group" in the above definition we get the definition of *Markov free abelian paratopological group* on X which we denote as $A_p(X)$.

Proposition 1.3. *[13] Let X be a space.*

1. *Let e be an arbitrary point of the space X. Then the free paratopological groups $F_p(X,e)$ and $A_p(X,e)$ exist.*
2. *Let e_1 and e_2 be arbitrary points of the space X. Then the free paratopological groups $F_p(X,e_1)$ and $F_p(X,e_2)$ are topologically isomorphic. The free paratopological groups $A_p(X,e_1)$ and $A_p(X,e_2)$ are topologically isomorphic as well.* □

Similarly to the case of free topological groups we can prove that the group $F_p(X)$ is topologically isomorphic to the group $FG_p(X^+)$ and the group $A_p(X)$ is topologically isomorphic to the group $AG_p(X^+)$, where X^+ is the space obtained from X by adding one isolated point.

Proposition 1.4. *For each space X the following claims hold.*

1. *Free paratopological groups $F_p(X)$ and $A_p(X)$ exist.*
2. *Let G_1, G_2 be Markov free (abelian) paratopological groups on X. Then there exists a topological isomorphism $i\colon G_1 \to G_2$ such that $i(x) = x$ for each point $x \in X$.*

Similarly to the case of free topological groups [4] one can easily check that the paratopological group $FG_p(X^+)$ satisfies conditions of definition 1.2. The proof of part 2 and part 3 is the similar to the proof of [6, Proposition 7.2].

It follows from the proof of existence of free partopological groups considered in [13] that that groups $FG_p(X)$, $AG_p(X)$, $F_p(X)$ and $A_p(X)$ are algebraically free.

The second section contains basic topological properties of free paratopological groups. Many of them have their counterparts in the theory of free topological groups. The third section is devoted to separation properties and cardinal invariants of free paratopological groups. Some results of the work were announced in [10].

2. On topological properties of free paratopological groups.

Definition 2.1. [7] *Markov free topological group* on a space X is a pair consisting of a topological group $F(X)$ and a continuous function $\eta_X\colon X \to F(X)$ such that any continuous function f from X to a topological group G "lifts" to a unique continuous group homomorphism $\widetilde{f}\colon F(X) \to G$ such that $\widetilde{f} \circ \eta_X = f$.

If we change the word "group" to the words "abelian group" in the above definition we obtain the definition of *Markov free abelian topological group* on a space X which we denote by $A(X)$. It was proved in [7] that the groups $F(X)$ and $A(X)$ exist for each functionally Hausdorff space X, are unique up to isomorphism which leaves all points from X and are Tychonoff for each functionally Hausdorff space X.

By *the group reflexion* $G^\flat = (G, \tau^\flat)$ of a paratopological group (G,τ) we understand the group G endowed with the strongest topology $\tau^\flat \subset \tau$ turning G into a topological group. This topology admits a categorial description: $\tau^\flat$ is a unique topology on G such that

- $(G, \tau^\flat)$ is a topological group;
- the identity homomorphism $\mathrm{id}_\flat\colon (G,\tau) \to (G,\tau^\flat)$ is continuous;
- for each continuous group homomorphism $h\colon G \to H$ into a topological group H the homomorphism $h \circ \mathrm{id}_\flat^{-1}\colon G^\flat \to H$ is continuous.

There is also a dual notion of a group coreflexion. Given a paratopological group G let $\tau_\sharp$ be the weakest group topology on G, stronger than the topology of G. The topological group $G^\sharp = (G, \tau_\sharp)$ is called *the group coreflexion* of G. According to [11], a neighborhood base of the unit of the group coreflexion $G^\sharp$ of a paratopological group G consists of the sets $U \cap U^{-1}$ where U runs over neighborhoods of the unit in G. A paratopological group is called $\sharp$-*discrete* provided its group co-reflection is discrete. Denote by FX the abstract free group with the set of generators of cardinality $|X|$.

Proposition 2.2. *If X is a functionally Hausdorff space then the topological groups $F_p(X)^\flat$ and $F(X)$ are topologically isomorphic.*

Proof. Define continuous homomorphisms $i_1\colon F_p(X)^\flat \to F(X)$ and $i_2\colon F(X) \to F_p(X)^\flat$ as follows. Let $\eta\colon F_p(X) \to F(X)$ be the continuous extension of the canonical condensation $X \to \eta_X(X)$ to a continuous homomorphism $F_p(X) \to F(X)$ of the paratopological groups. Let $\mathrm{id}_\flat\colon F_p(X) \to F_p(X)^\flat$ be the identity homomorphism. The properties of the group reflection imply that the mapping $\eta \circ \mathrm{id}_\flat^{-1} = i_1\colon F_p(X)^\flat \to F(X)$ is continuous. Let $i_2\colon F(X) \to F_p(X)^\flat$ be the continuous extension of the map $\mathrm{id}_\flat|X\colon X \to F_p(X)^\flat$.

Algebraically i_1 and i_2 are identity homomorphisms on abstract group FX with $i_1(x) = x$ $\forall x \in X$ and $i_2(x) = x$ $\forall x \in X$ so $i_1 \circ i_2 = 1_{F(X)}$ and $i_2 \circ i_1 = 1_{F_p(X)^\flat}$ and i_1 is topological isomorphism.

□

Let $G = (\mathbb{R}, +)$ and τ be the topology on G with the base $\mathcal{B} = \{[x; +\infty) : x \in \mathbb{R}\}$. Then (G, τ) is a paratopological group. We shall denote this group by $\mathbb{R}^*$. Denote by $\mathbb{Z}^*$ the group of integers equipped with the topology of the subspace of $\mathbb{R}^*$.

Lemma 2.3. *Let $F_1, F_2, ...F_n$ be closed disjoint subsets of a space X, and $x_0, \dots, x_n$ be points of the space $\mathbb{R}^*$ such that $x_i \le x_0$ for $i \in \{1, \dots, n\}$. We define the mapping $f\colon X \to \mathbb{R}^*$ putting $f(F_i) = \{x_i\}$ and $f(X \setminus (F_1 \cup F_2 \cup \dots \cup F_n) = \{x_0\}$. Then the mapping f is continuous.* □

Theorem 2.4. (E. Reznichenko) *Let X be a space and F be a closed subset of X. Then F^{-1} is an open subset of the space $F_p(X) \cap X^{-1}$.*

Proof. Let U be an open subset of the space X. Then the mapping $f\colon X \to \mathbb{Z}^*$ such that $f(F) = \{0\}$ and $f(X \setminus F) = \{1\}$ is continuous. Let f^* be the extension of the mapping f to a continuous homomorphism $f^*\colon F_p(X) \to \mathbb{Z}^*$. Then $F^{-1} = f^{*-1}([0; \infty)) \cap X^{-1}$ is an open subset of X^{-1}. □

Corollary 2.5. *Let X be a T_1 space. Then X^{-1} is a discrete subspace of the group $F_p(X)$ and $-X$ is a discrete subspace of the group $A_p(X)$.*

For a subset A of a space X denote by $\langle A \rangle$ the subgroup of $A_p(X)$ generated by A.

Proposition 2.6. *Let A be a subset of a space X. Then the set $\langle A \rangle$ is closed in $A_p(X)$ if and only if A is closed in X and every one-point subset of $X \setminus A$ is closed in X.*

Proof. *Necessity.* Suppose that the set $\langle A\rangle$ is closed in $A_p(X)$. The set $A=\langle A\rangle\cap X$ is closed in X. Suppose that there exist points $y\in X\backslash A$ and $x\in X$, $x\neq y$, such that $x\in\overline{y}$. Then $x-y\in\overline{y-y}=\overline{0}\subset\overline{\langle A\rangle}$. Since $x-y\notin\langle A\rangle$ the set $\langle A\rangle$ is not closed, a contradiction.

Sufficiency. Suppose that the set A is closed in X and every one-point subset of $X\backslash A$ is closed in X. Also suppose that the set $\langle A\rangle$ is not closed in $A_p(X)$. Then there exists a point $w=\lambda_1x_1+\lambda_2x_1+\dots\lambda_nx_n+v\in\overline{\langle A\rangle}$, where $x_i\in X\backslash A$, $v\in\langle A\rangle$, $\lambda_i\in\mathbb{Z}\setminus\{0\}$. Let $f\colon X\to\mathbb{R}^*$ be a function such that $f(x_i)=-1$ for $\lambda_i<0$, $f(x_i)=1$ for $\lambda_i>0$, $f(A)=\{0\}$ and $f(X\backslash\{x_1,x_2\dots x_n\}\cup A)=\{2\}$. The mapping f is continuous by Lemma 2.3. Denote by $f^*\colon A_p(X)\to\mathbb{R}^*$ the extension of the mapping f. The construction implies that $f(w)>0$. Then the set $f^{*-1}((-\infty,0])$ is closed in $A_p(X)$, contains $\langle A\rangle$ and do not contains w. Thus $w\notin\overline{\langle A\rangle}$, a contradiction. □

Example 2.7. *Let X be the space of size ω_1 with the cofinite topology. Then $F_p(X)$ is not $\sharp$-discrete.*

Proof. Indeed let $U\subset F_p(X)$ be an arbitrary neighborhood of the unit. Then for every point $a\in X$ there is a finite set $F(a)$ such that $X\backslash F(a)\subset aU$. Let Y be an arbitrary countable infinite subset of X and $F(Y)=\bigcup_{a\in Y}F(a)$. Then $Y^{-1}(X\backslash F(Y))\subset U$. Now choose a point $x\in X\backslash F(Y)$. Then $x^{-1}(X\backslash F(x))\subset U$. Hence $(X\backslash F(x))^{-1}x\subset U^{-1}$. Choose a point $y\in Y\backslash F(x)$, $y\neq x$. Then $y^{-1}x\in(X\backslash F(x))^{-1}x\subset U^{-1}$ and $y^{-1}x\in Y^{-1}x\subset Y^{-1}(X\backslash F(Y))\subset U$. Therefore the group $F_p(X)$ is not $\sharp$-discrete. □

The propositions from here and to the end of the section describe the properties of free paratopological groups which are similar to these of free topological groups. In most cases the proofs also are similar to the case of free topological groups.

Proposition 2.8. *Let X be a space. The identity mapping $id_X\colon X\to X$ can be extended to a continuous open homomorphism $id_X^*\colon F_p(X)\to A_p(X)$ and hence $A_p(X)$ is quotient group of $F_p(X)$.*

Proof. The proof is similar to that from the paper [4]. □

Proposition 2.9. *[4] Let X be a space. Then the following conditions are equivalent:*

1. *The space X is connected.*
2. *The space $FG_p(X)$ is connected.*
3. *The space $AG_p(X)$ is connected.*

Proof. Since the component of the unit of a paratopological group is a closed normal subgroup of the group [11], then we may prove the proposition similarly to Proposition 6.A from [4]. □

Proposition 2.10. *Let X, Y be spaces, $p\colon X\to Y$ be a continuous surjection and $p^*\colon F_p(X)\to F_p(Y)$ be the homomorphic extension of p. Then the mapping p is quotient if and only if the mapping p^* is open.*

We need the following lemma for the proof of the theorem.

Lemma 2.11. *A continuous surjection $p\colon X\to Y$ is quotient if and only if the following condition holds: a mapping $\psi\colon Y\to\mathbb{R}^*$ is continuous if and only if the composition $\psi\circ p$ is continuous.*

Proof. The necessity is evident. We shall prove the sufficiency. Let U be a subset of Y such that $p^{-1}(U)$ is open in X. Consider the mapping $\psi\colon Y \to \mathbb{R}^*$ defined by $\psi(U) = \{1\}$ and $\psi(Y \setminus U) = \{0\}$. Then the composition $\psi \circ p$ is continuous. Hence the mapping ψ is continuous too. Thus $U = \psi^{-1}([0,\infty))$ is open in Y. Therefore p is a quotient mapping. □

Proof of Proposition 2.10. Necessity. Let p be a quotient mapping. The set $H = \ker p^*$ is a normal subgroup of the group $F_p(X)$. We denote by G the quotient group $F_p(X)/H$. Let $\pi\colon F_p(X) \to G$ be the natural homomorphism. Then the mapping π is open [11]. There exists a unique isomorphism $i\colon G \to F_p(Y)$ such that $p^* = i \circ \pi$. The mapping i is continuous because p^* is a continuous homomorphism and π is an open homomorphism. Now we show that inverse isomorphism $j = i^{-1}$ is also continuous. It suffice to prove that the restriction $j|Y$ is continuous. Since $j \circ p^* = \pi$ then $(j|Y) \circ p = \pi|X$. Since the mapping p is quotient, the continuity of the mapping π implies the continuity of the mapping $j|Y$. Thus i is a topological isomorphism, and the homomorphism $p^* = i \circ \pi$ is open.

Sufficiency. Suppose that p^* is an open homomorphism. Let $f\colon Y \to \mathbb{R}^*$ be a mapping such that the composition $f \circ p$ is continuous. Then the composition $f^* \circ p^*\colon F_p(X) \to \mathbb{R}^*$ is continuous too. Since the mapping p^* is open then the mapping f^* is continuous and hence the mapping $f = f^*|Y$ is continuous too. □

Proposition 2.12. *[6] Let X, Y be topological spaces and $r\colon X \to Y$ be a retraction. Then the subgroup of $F_p(X)$ generated by Y is naturally isomorphic to $F_p(Y)$.*

We denote by $X \oplus Y$ the disjoint union of topological spaces X and Y.

Proposition 2.13. *For every topological spaces X and Y paratopological groups $A_p(X \oplus Y)$ and $A_p(X) \times A_p(Y)$ are topologically isomorphic.*

Proof. It is similar to the proof which was given in [15] for free abelian topological groups. □

Let X be a topological space, $x \in X$. The quasicomponent of x in X is the intersection of all clopen subsets in X containing x.

Proposition 2.14. *The quasicomponent of the unit of a paratopological group is a closed normal subgroup.*

Proof. Let G be the paratopological group, $a, b \in G$ such that $Q_a = Q_b$. From the homogeneity of paratopological group G it follows that $Q_{ac} = Q_{bc}$ for arbitary $c \in G$. Let e the unit of G, $a, b \in Q_e$, then $Q_e = Q_a$, $Q_e = Q_b$. Then $Q_{ae} = Q_{ab}$. So $Q_e = Q_a = Q_{ab}$, i.e. $ab \in Q_e$. Let $a \in Q_e$, then $Q_e = Q_a$, so $Q_{ea^{-1}} = Q_{aa^{-1}}$, i.e. $Q_{a^{-1}} = Q_e$, thus $a^{-1} \in Q_e$. Let $a \in Q_e$, $x \in G$. From $Q_e = Q_a$ it follows $Q_{x^{-1}ex} = Q_{x^{-1}ax}$, i.e. $Q_e = Q_{x^{-1}ax}$, so $x^{-1}ax \in Q_e$. □

Proposition 2.15. *Let X be a topological space. Then the quasicomponent of the unit in $F_p(X)$ is the smallest normal subgroup containing $\bigcup_{x\in X}(Q_x x^{-1})$, where Q_x denotes the quasicomponent of x in X.*

Proof. Let N be the smallest closed normal subgroup generated by the set $\bigcup_{x\in X}(Q_x x^{-1})$ and $\widetilde{Q}$ be the quasicomponent of the identity of the $F_p(X)$. For each point $x \in X$ define the mapping $\varphi_x\colon F_p(X) \to F_p(X)$ such that $\varphi_x(y) = yx^{-1}$ for each $y \in F_p(X)$. If f is a continuous mapping then $f(Q_x) \subseteq Q_{f(x)}$ [8]. So for all $x \in X$ we have $Q_x x^{-1} = \varphi_x(Q_x) \subseteq Q_{\varphi_x(x)} = \widetilde{Q}$. Hence $N \subseteq \widetilde{Q}$ by Proposition 2.14.

The proof of the inverse inclusion is similar to that from [8] □

3. Separation properties. The set of all words in the group $F_p(X)(A_p(X))$ having in the irreducible form the length not greater than n is denoted by $F_n(X)(A_n(X))$. The set of all words in the group $F_p(X)(A_p(X))$ having the sum of the degrees equal to n is denoted by $Z_n(X)$. Extending the constant mapping equal 1 on X to the group homomorphism p from F_p to $\mathbb{Z}$ we obtain that the set $Z_n(X) = p^{-1}(n)$ is clopen in the group $F_p(X)$ for each $n \in \mathbb{Z}$.

We denote by $B_n(X)$ the set of all words in $A_p(X))$ whose irreducible form has the following properties:

1) the length of the form is not greater than n;

2) all coefficients is the form are positive.

Obviously $B_n(X) = A_n(X) \cap Z_n(X)$. According to [5], a paratopological group G has a *suitable set* X provided X is a discrete subset of G, $X \cup \{e\}$ is closed and generates a dense subgroup of G.

Lemma 3.1. *A topological space X is a T_0-space if and only if for arbitrary points $x, y \in X$, $x \neq y$ there exists a continuous mapping $f\colon X \to \mathbb{R}^*$ such that $f(x) \neq f(y)$.*

Proof. Suppose that X is a T_0-space and $x, y \in X$, $x \neq y$. Then there exists an open set U, containing exactly one of the points x, y. Then the mapping $f\colon X \to \mathbb{R}^*$ defined as $f(U) = \{0\}, f(X \setminus U) = \{1\}$ is continuous and $f(x) \neq f(y)$.

Suppose that for arbitrary points $x, y \in X$, $x \neq y$ there exists a continuous mapping $f\colon X \to \mathbb{R}^*$ such that $f(x) \neq f(y)$. Without loss of generality we may suppose that $f(x) < f(y)$. Then the set $U = f^{-1}(-(\infty, f(x)])$ is open, contains x and does not contain y. Thus X is a T_0-space. □

For mappings $f, g\colon X \to \mathbb{R}^*$ we define the mapping $f+g\colon X \to \mathbb{R}^*$ such that $(f+g)(x) = f(x) + g(x)$ for each point $x \in X$.

Since $\mathbb{R}^*$ is paratopological group, the following lemma holds.

Lemma 3.2. *Let $f, g\colon X \to \mathbb{R}^*$ be continuous mappings. Then the mapping $f + g\colon X \to \mathbb{R}^*$ is continuous.*

For mappings $f, g\colon X \to \mathbb{R}^*$ we define the mapping $f \times g\colon X \to \mathbb{R}^*$ such that $(f \times g)(x) = f(x) \cdot g(x)$ for each point $x \in X$.

Lemma 3.3. *Let $f, g\colon X \to \mathbb{R}^*$ be continuous mappings, such that $f(x) > 0$ and $g(x) > 0$ for each $x \in X$. Then the mapping $f \times g\colon X \to \mathbb{R}^*$ is continuous.*

Proof. Put $P^* = \{x \in \mathbb{R}^* | x > 0\}$. Then the logarithmic mapping ln from P^* to $\mathbb{R}^*$ is a homeomorphism. The mappings $f, g\colon X \to \mathbb{R}^*$ are continuous and thus the mappings $f_1(x) = \ln(f(x))$, $g_1(x) = \ln(g(x))$ are continuous as well. Lemma 3.2 implies that the mapping $f_1 + g_1\colon X \to \mathbb{R}^*$ is continuous. Then the composition $f \times g = e^{f_1+g_1}$ is continuous. □

Proposition 3.4. *Let X be a T_0-space. Then $A_p(X)$ is a T_0-space.*

Proof. It suffices to prove that for any point $a \in A_p(X)$ there exists a continuous mapping $f^*\colon A_p(X) \to \mathbb{R}^*$ such that $f^*(a) \neq f^*(0)$. Let $a = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n$ where $x_i \in X$ and $\lambda_i \in \mathbb{Z} \setminus \{0\}$. Let $2 \leq i \leq n$. Since $x_1 \neq x_i$ then there exists a set U_i open in X containing exactly one of the points x_1, x_i. Define the mapping $f_i\colon X \to \mathbb{R}^*$ as follows. If $x_1 \in U_i$ then we put $f_i(U_i) = \{\pi\}$, $f_i(X \setminus U_i) = \{4\}$. If $x_i \in U_i$ then we put $f_i(U_i) = \{1\}$, $f_i(X \setminus U_i) = \{\pi\}$.

By Lemma 3.3 the mapping $f = f_2 \times f_3 \times ... \times f_n$ is continuous. Then $f(x_1) = \pi^{n-1}$ and $f(x_i) = m_i\pi^{k_i}$, where $m_i \in \mathbb{Z}$ and $k_i < n-1$. Let $f^*\colon A_p(X) \to \mathbb{R}^*$ be the extension of the mapping f to the continuous homomorphism. Then $f^*(a) = \lambda_1\pi^{n-1} + \lambda_2 m_2\pi^{k_2} + ... + \lambda_n m_n \pi^{k_n}$. Since $k_i < n-1$ for $2 \le i \le n$, then $f^*(a) = g(\pi)$, where g is a polynomial of the degree $n-1$ with integer coefficients. Since π is not an algebraic number, $f^*(0) = f(0) = 0 \neq g(\pi) = f^*(a)$. □

Proposition 3.5. *The following conditions are equivalent for a topological space X :*

1. *The space X is a T_1 space.*
2. *The space $A_p(X)$ is a T_1 space.*
3. *The subspace X is closed in $A_p(X)$.*
4. *The subspace $-X \subset A_p(X)$ is discrete.*
5. *The subspace $-X \subset A_p(X)$ is T_1 space.*
6. *The subspace $-X \subset A_p(X)$ is closed in $A_p(X)$.*
7. *The group $A_p(X)$ contains a closed suitable set.*
8. *The group $A_p(X)$ contains a suitable set.*
9. *The subspace $A_n(X)$ is closed in $A_p(X)$ for every positive integer n.*
10. *The subspace $A_n(X)$ is closed in $A_p(X)$ for some positive integer n.*
11. *The commutant K of the group $F_p(X)$ is closed in $F_p(X)$.*

Proof. We prove the following implications $3 \Rightarrow 2$, $2 \Rightarrow 1$, $1 \Rightarrow 3$, $1 \Rightarrow 4$ (is proved in Corollary 2.5), $4 \Rightarrow 5$ (obvious), $5 \Rightarrow 1$, $1 \Rightarrow 6$, $6 \Rightarrow 1$, $1 \Rightarrow 7$, $7 \Rightarrow 8$ (obvious), $8 \Rightarrow 1$, $1 \Rightarrow 9$, $9 \Rightarrow 10$ (obvious), $10 \Rightarrow 1$, $1 \Rightarrow 11$, $11 \Rightarrow 1$.

($3 \Rightarrow 2$). Let $a, b \in X$, $a \neq b$. Consider the subsets $V_a = X - a$, $V_b = X - b$ of $A_p(X)$. By the homogeneity of $A_p(X)$ the sets V_a and V_b are closed hence the set $V_a \cap V_b = \{0\}$ is closed in $A_p(X)$. Therefore $A_p(X)$ is T_1-space.

($2 \Rightarrow 1$) This implication holds since $X \subset A_p(X)$.

($1 \Rightarrow 3$) Let $a \in X$. Consider the continuous mapping $f\colon X \to \mathbb{R}^*$ such that $f(a) = -1$, $f(X \backslash \{a\}) = \{0\}$. Denote by $f^*\colon A_p(X) \to \mathbb{R}^*$ its extension. Then the set $B_a = \{\lambda a + v \mid \lambda \geq 0, v \in \langle X \backslash \{a\} \rangle\} = f^{*-1}((-\infty, 0])$ is closed in $A_p(X)$. Hence the set $B = \bigcap_{x \in X} B_x$ is closed in $A_p(X)$. Thus the set $X = B \cap Z_1(X)$ is closed in $A_p(X)$.

($5 \Rightarrow 1$) Suppose that X is not a T_1-space. Then there exist distinct points $x, y \in X$ such that $x \in \overline{y}$. Then $x - x - y \in \overline{y - x - y}$. Thus $-y \in \overline{-x}$. So X^{-1} is not a T_1-space, a contradiction.

($1 \Rightarrow 6$) Suppose that there exists a point $w = \lambda_1 x_1 + \lambda_2 x_1 + ... \lambda_n x_n \in \overline{-X} \backslash (-X)$ with $\lambda_i \in \mathbb{Z} \setminus \{0\}$ and $x_i \in X$ for all $1 \le i \le n$. Then $w \in Z_{-1}(X)$. Consider the continuous mapping $f\colon X \to \mathbb{R}^*$ such that $f(\{x_1, \ldots, x_n\}) = \{-1\}$, $f(X \backslash \{x_1, x_2 ... x_n\}) = \{0\}$. Denote by $f^*\colon A_p(X) \to \mathbb{R}^*$ its extension. Then the set $f^{*-1}((-\infty, 0])$ is closed. Since X is a T_1-space then $A_p(X)$ is a T_1-space. Thus the set $V = f^{*-1}((-\infty, 0]) \cup \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$ is closed. The set V is closed, contains $-X$ and does not contain w. Therefore $w \notin \overline{-X}$, a contradiction.

($6 \Rightarrow 1$) Let $a, b \in X$ be distinct points. Consider the sets $V_a = -X + a$, $V_b = -X + b$. By the homogeneity of $A_p(X)$ the sets V_a and V_b are closed. Hence $V_a \cap V_b = \{0\}$ is closed in $A_p(X)$. Therefore $A_p(X)$ is a T_1-space.

(1⇒7) We may choose $-X$ as a closed suitable set.

(8⇒1) Suppose that X is not a T_1-space and M is a suitable set in $A_p(X)$. Then there exist distinct points $x, y \in X$ such that $x \in \overline{y}$. By the homogeneity $x - y \in \overline{y - y} = \overline{0}$. Since $M \cup \{0\}$ is closed and $x - y \neq 0$ then $x - y \in M$. Since $x - y \in \overline{0}$ then $2(x-y) \in \overline{x - y}$. Thus $2(x-y) \in M$. Therefore M is not T_1-space, a contradiction.

(1⇒9) We prove the statement by induction.

The set $A_1(X) = X \cup (-X) \cup \{0\}$ is closed in $A_p(X)$. Suppose that for all $k \leq n-1$ the set $A_k(X)$ is closed in $A_p(X)$. Assume that $A_n(X)$ is not closed in $A_p(X)$. Then there exists a point $w = \lambda_1 x_1 + \lambda_2 x_1 + \ldots \lambda_t x_t \in \overline{A_n(X)} \backslash A_n(X)$. Then $m = \sum_{i=1}^{t} \mid \lambda_i \mid > n$. Since $A_n(X) \subset \bigcup\{Z_s : -n \leq s \leq n\}$ and all the sets Z_s are clopen in $A_p(X)$ then $-n \leq \sum_{i=1}^{t} \lambda_i \leq n$. Thus there exists an index j with $\lambda_j < 0$. Consider the function $f \colon X \to \mathbb{R}^*$ defined as $f(x_j) = -m$, $f(x_i) = -1$ for $i \neq j$ and $f(X \backslash \{x_1, x_2 \ldots x_n\}) = \{0\}$. Denote by $f^* \colon A_p(X) \to \mathbb{R}^*$ its extension. By the construction, $f(w) > 0$. Put $Y = \{x_1, x_2 \ldots x_n\}$ and $A_i(Y) = A_i(X) \cap \langle Y \rangle$ for every i. The set $f^{*-1}((-\infty, 0])$ is closed. Consider the set $V = f^{*-1}((-\infty, 0]) \cup A_1(Y)A_{n-1}(X) \cup A_2(Y)A_{n-2}(X) \cup \ldots \cup A_n(Y) \subset A_p(X)$. Every $A_i(Y)$ is finite and $A_k(X)$ is closed in $A_p(X)$ for all $k \leq n-1$, hence the set V is closed, contains $A_n(X)$ and does not contain w. Thus $w \notin \overline{A_n(X)}$, a contradiction

(10⇒1) Let $A_n(X)$ be a closed subset of $A_p(X)$. Then the set $B_n(X) = A_n(X) \cap Z_n(X)$ is closed in $A_p(X)$. Let $a, b \in X$ be distinct points. Consider the sets $V_a = B_n(X) - na$, $V_b = B_n(X) - nb$. By the homogeneity of $A_p(X)$ the sets V_a and V_b are closed. Hence the set $V_a \cap V_b = \{0\}$ is closed in $A_p(X)$. Therefore $A_p(X)$ is a T_1-space.

Proposition 2.8 implies the equivalence of conditions 1 and 11. □

Let X be a space. The *extent* $e(X)$ of X is the smallest cardinal κ such that X contains no closed discrete subspace of size $> \kappa$.

Corollary 3.6. *Let X be a T_1-space. Then $e(A_p(X)) = e(F_p(X)) = d(F_p(X)) = d(A_p(X)) = |X|$.*

Proof. We can take $-X$ as a closed discrete subspace. Thus $e(A_p(X)) = |X|$. $A_p(X)$ is a continuous image of $F_p(X)$. Hence $e(A_p(X)) \leq e(F_p(X))$ and thus $e(F_p(X)) = |X|$.

Let $w = x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}$ be a word in the irreducible form. Put $\operatorname{supp}(w) = \{x_1, x_2 \ldots x_n\}$. For a subset A of A_p we define $\operatorname{supp}(A) = \underset{w \in A}{\cup} \operatorname{supp}(w)$. Let A be a dense subset of $F_p(X)$. Suppose that there exists a point $x \in X \backslash \operatorname{supp} A$. Define the continuous mapping $f \colon X \to \mathbb{R}^*$ as follows: $f(x) = -1$ and $f(X \backslash \{x\}) = \{0\}$. Let $f^* \colon F_p(X) \to \mathbb{R}^*$ be the homomorphic extension of the mapping f. Then the set $f^{*-1}((-\infty, 0])$ is closed, contains A and does not contain x^{-1}. Therefore $x^{-1} \notin \overline{A}$, which is impossible since the set A is dense in $F_p(X)$. Thus $\operatorname{supp} A = X$ and $|A| = |X|$. Hence $d(F_p(X)) = |X|$. Similarly we can prove that $d(A_p(X)) = |X|$. □

Example 3.7. *There is a Hausdorff space X such that $F_p(X)$ is not Hausdorff.*

Proof. A space X is called an Urysohn space if every two distinct points of X have the disjoint closed neighborhoods. The subspace of an Urysohn space is again Urysohn. Every Urysohn space is Hausdorff, but the opposite is not true (see [3]). Let X be a Hausdorff space which is not Urysohn. Suppose that $F_p(X)$ is a Hausdorff space then $F_p(X)$ is an Urysohn space [12] and hence X is an Urysohn space too. The obtained contradiction shows that $F_p(X)$ is not Hausdorff. □

A paratopological group G is called $\flat$-separated if G admits a continuous injective homomorphism into a Tychonoff topological group. It is clear that a paratopological group G is $\flat$-separated if and only if the group reflexion $G^\flat$ is Tychonoff. A paratopological group G is *maximally almost periodic (MAP)* if there exists a continuous monomorphism of G into a compact Hausdorff topological group. A space X is *functionally Hausdorff* if for every distinct points $x, y \in X$ there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(x) = 1$.

Proposition 3.8. *The following claims are equivalent for an arbitrary space X:*

1. *The space X is functionally Hausdorff,*
2. *Free Markov paratopological group over X is MAP-group*

2'. *Free Markov abelian paratopological group over X is MAP-group*

3. *Free Markov paratopological group over X $\flat$-separated,*

3'. *Free abelian Markov paratopological group over X $\flat$-separated,*

4. *Free Markov paratopological group over X is functionally Hausdorff,*

4'. *Free Markov abelian paratopological group over X is functionally Hausdorff.*

Proof. (1⇒2) This implication holds since the space X admits a continuous injection into a Tychonoff space Y. Thus the group $F_p(X)$ admits a continuous isomorphism into the group $F_p(Y)$. Moreover, the group $F_p(Y)$ admits a continuous isomorphism into the group $F(Y)$ and the group $F(Y)$ admits a continuous isomorphism into a compact Hausdorff topological group [9, Theorem 23]. The proof of the implication (1⇒2) is similar.

The implications 2⇒3, 3⇒4, 4⇒1, 2'⇒3', 3'⇒4' and 4'⇒1 are obvious. □

Proposition 3.9. *Let X be a Tychonoff space, n be a positive integer. Let $j\colon X^n \to F_p(X)$ be a mapping such that $j(x_1, x_2, \dots, x_n) = x_1 x_2 \cdots x_n$. Then j is a homeomorphism and $j(X^n)$ is a closed subset of $F_p(X)$.*

Proof. Consider the free topological semigroup over the space X. It is the disjoint sum of the spaces X^i, where $i \geq 1$ with the natural multiplication [2]. Let $j^*\colon S(X) \to F_p(X)$ be the continuous homomorphic extension of the natural embedding of the space X into the group $F_p(X)$. Since $j = j^*|X^n$ then the map j is continuous. Let $i^*\colon F_p(X) \to F(X)$ be the continuous homomorphic extension of the natural embedding $X \to F(X)$. In the paper [1] is proved that the set $i^*j(X^n)$ is a closed subset of the space $F(X)$ and the map $i^*j\colon X^n \to i^*j(X^n)$ is a homeomorphism. Thus the map j is a homeomorphism and $j(X^n) = i^{*-1}(i^*j(X^n))$ is a closed subset of the space $F_p(X)$. □

Example 3.10. *There is a normal space X such that the group $F_p(X)$ is not normal.*

Proof. Let X be the Sorgenfrey line. Since the group $F_p(X)$ contains a closed subset homeomorphic to X^2 then $F_p(X)$ is not normal. □

Definition 3.11. A space X is called *submetrizable* if it admits a continuous injection into a metrizable space. A space X is called *subquasimetrizable* if it admits a continuous injection into a quasimetrizable space. Recall that a *quasimetric* on a set X is a nonnegative function $d : X \times X \to \mathbb{R}$ such that $d(x, y) = 0$ iff $x = y$ and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$, see [13].

Proposition 3.12. *The following conditions are equivalent for an arbitrary space X.*

1. *The space X is submetrizable.*
2. *The space $F_p(X)$ is submetrizable.*
3. *The space $A_p(X)$ is submetrizable.*

Proof. The implications 2⇒1 and 3 ⇒1 are obvious.

(1⇒2) Let (Y, d) be a metric space and $f\colon X \to Y$ be a continuous injection. Let $F_d(Y)$ be the free group over the set Y equipped with the metric topology generated by Graev's extension of the metric d onto $F(Y)$ [4]. Let $f^*\colon F_p(X) \to F_d(Y)$ be the homomorphic extension of the map f. Since f^* is a continuous injection then $F_p(X)$ is submetrizable.

(1⇒2) Let (Y, d) be a metric space and $f\colon X \to Y$ be a continuous injection. Let $A_d(Y)$ be the free group over the set Y equipped with the metric topology generated by Graev's extension of the metric d onto $A(Y)$ [4]. Let $f^*\colon A_p(X) \to A_d(Y)$ be the homomorphic extension of the map f. Since f^* is a continuous injection then $A_p(X)$ is submetrizable. □

Proposition 3.13. *The following conditions are equivalent for an arbitrary space X.*

1. *The space X is subquasimetrizable.*
2. *The space $F_p(X)$ is subquasimetrizable.*
3. *The space $A_p(X)$ is subquasimetrizable.*

Proof. The proof is similar to the proof of the previous proposition. We should use the extension of the continuous quasimetric from the space X onto the spaces $F_p(X)$ and $A_p(X)$ which is described in [13] instead of Graev's extension of the metric.

□

A paratopological group G is called an NSS-group if there exist a neighborhood of the unit of G which does no contain nontrivial subgroups (recall that NSS is abbreviated from **N**o **S**mall **S**ubgroups).

Corollary 3.14. *Let X be a submetrizable space. Then $F_p(X)$ is an NSS-group.*

Proof. Let Y be a metrizable space and $f\colon X \to Y$ be a continuous injection. Since Y is a metrizable space then $F(Y)$ is an NSS-group [14]. Let $f^*\colon F_p(X) \to F(Y)$ be the homomorphic extension of the map f. Since f^* is a continuous injection then $F_p(X)$ is an NSS-group too. □

Example 3.15. *Let X be a space of size ω_1 with the cofinite topology. Then every neighborhood U of the zero of the group $A_p(X)$ contains a group of countable index.*

Proof. Consider a sequence $\{U_i\}$ of the zero neighborhoods such that $U_0 = U$ and $U_{i+1} + U_{i+1} \subset U_i$ for each i. We put $X_i(x) = \{y \in X : y - x \notin U_i\}$ for each point $x \in X$ and each index $i \in \omega$. Let $i \in \omega$ be an arbitrary index. Since $U_{i+1} + U_{i+1} \subset U_i$ the following condition holds

$(\forall x \in X)(\forall y \in X \backslash X_{i+1}(x))(z \in X_i(x) \implies z \in X_{i+1}(y))$.

Thus for each point $x \in X$ holds $X_i(x) \subset \bigcap\{X_{i+1}(y) : y \in X \backslash X_{i+1}(x)\}$. Suppose that $|\bigcup\{X_i(x) : x \in X\}| \geq \omega$. Then there exists a countable set $Y \subset X$ such that $|\bigcup\{X_i(x) : x \in Y\}| \geq \omega$. Let $y \in X \backslash \bigcup\{X_{i+1}(x) : x \in Y\}$ be an arbitrary point. Then $\bigcup\{X_i(x) : x \in Y\} \subset X_{i+1}(y)$, a contradiction, since the set $X_{i+1}(y)$ is finite. Therefore the set $\bigcup\{X_i(x) : x \in X\}$ is finite.

Put $X_\infty = \bigcup\{X_i(x) : x \in X, i \in \omega\}$. Let H be a group generated by the set $X \backslash X_\infty - X \backslash X_\infty$. Then $H \subset U$ and $|G : H| \leq \omega$. □

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