УДК 517.51

O. Karlova, V. Mykhaylyuk

ON BAIRE ONE MAPPINGS AND LEBESGUE ONE MAPPINGS WITH VALUES IN INDUCTIVE LIMITS

O. Karlova, V. Mykhaylyuk. *On Baire one mappings and Lebesgue one mappings with values in inductive limits*, Matematychni Studii, **25** (2006) 103–107.

We show that the classes $B_1(X,Y)$ and $H_1(X,Y)$ of the Baire one mappings and Lebesgue one mappings coincide when X is a hereditarily Baire separable metrizable space and Y is the strict inductive limit of an increasing sequence of metrizable locally convex spaces.

Е. Карлова, В. Михайлюк. *Об отображениях первого класса Бэра и первого класса Лебега со значениями в индуктивных пределах* // Математичні Студії. – 2006. – Т.25, №1. – С.103–107.

Показано, что классы $B_1(X,Y)$ и $H_1(X,Y)$ отображений первого класса Бэра и первого класса Лебега совпадают, если X — наследственно бэровское метризуемое сепарабельное пространство, а Y — строгий индуктивный предел возрастающей последовательности метризуемых локально выпуклых пространств.

1. Let X and Y be topological spaces. Recall that a mapping $f\colon X\to Y$ is said to be *a Baire one mapping* if f is the pointwise limit of a sequence of continuous functions $f_n\colon X\to Y$. We denote by $B_1(X,Y)$ the collection of all such mappings. We say that $f\colon X\to Y$ *belongs to the first Lebesgue class* or f *is a Lebesgue one mapping* if the preimage $f^{-1}(G)$ of each open set $G\subseteq Y$ is an F_σ-set in X. Let $H_1(X,Y)$ denote the class of all Lebesgue one mappings.

The classical Lebesgue-Hausdorff Theorem [1, p. 402] asserts that $H_1(X,Y)=B_1(X,Y)$ when X is a metric space and $Y=[0,1]^m$, where $m\le\aleph_0$. It was shown in [2] that the inclusion $H_1(X,Y)\subseteq B_1(X,Y)$ holds in the case when X is a perfectly normal space and Y is a separable metrizable topological vector space. R. Hansell [3] proved that every Lebesgue one mapping $f\colon X\to Y$ belongs to the first Baire class if X is a normal Hausdorff space and Y is a metric, complete, separable absolute retract. In [4] the inclusion $H_1(X,Y)\subseteq B_1(X,Y)$ was established for a perfectly normal space X and a connected and locally retractive set $Y\subseteq\mathbb{R}^n$. M. Fosgerau [5] proved that every σ-discrete Lebesgue one mapping $f\colon X\to Y$ belongs to the first Baire class when X is a metrizable space and Y is an arcwise connected and locally arcwise connected metrizable space. In particular, if X is a metrizable space, Y is an arcwise connected and locally arcwise connected separable metrizable space then $B_1(X,Y)=H_1(X,Y)$.

Note that in all above-mentioned results the range space is metrizable. The following question naturally arises: does the inclusion $H_1(X,Y)\subseteq B_1(X,Y)$ hold for a non-metrizable

2000 *Mathematics Subject Classification:* 54C10, 54C35.

range space? In this paper we prove that the equality $H_1(X,Y)=B_1(X,Y)$ takes place if X is a hereditarily Baire separable metrizable space, Y is the strict inductive limit of an increasing sequence of metrizable locally convex spaces.

2. In this section we will establish the inclusion $H_1(X,Y)\subseteq B_1(X,Y)$.

We first establish some auxiliary facts.

Lemma 2.1. *Let X be a normal space, Y be a topological vector space, $F_1,\dots,F_n$ be disjoint closed in X sets and $g_i\colon X\to Y$ be a continuous mapping for every $i\in\{1,\dots,n\}$. Then there exists a mapping $g\colon X\to Y$ such that $g(x)=g_i(x)$ on F_i for every $i\in\{1,\dots,n\}$.*

Proof. Since X is normal space, there exist open in X disjoint sets G_i such that $F_i\subseteq G_i$ for every $i\in\{1,\dots,n\}$. Let $A_i=X\setminus G_i$. Then for every $i\in\{1,\dots,n\}$ there exists a continuous function $\varphi_i\colon X\to[0,1]$ such that $F_i\subseteq\varphi_i^{-1}(1)$ and $A_i\subseteq\varphi_i^{-1}(0)$. For every $x\in X$ define $g(x)=\sum_{i=1}^{n}\varphi_i(x)g_i(x)$. It is evident that the mapping $g\colon X\to Y$ is continuous. If $x\in F_i$ for some $1\le i\le n$ then $\varphi_i(x)=1$ and $\varphi_j(x)=0$ for $j\ne i$. It follows immediately that $g(x)=g_i(x)$ on F_i. □

Lemma 2.2. *Let X be a perfectly normal space, Y be a topological vector space, $f\in H_1(X,Y)$, $Y=\bigcup_{n=1}^{\infty}Y_n$, where Y_n satisfies $H_1(X,Y_n)\subseteq B_1(X,Y_n)$ for each $n\in\mathbb{N}$, $(F_n)_{n=1}^{\infty}$ be an increasing sequence of F_σ-sets in X such that $X=\bigcup_{n=1}^{\infty}F_n$ and $f(F_n)\subseteq Y_n$. Then $f\in B_1(X,Y)$.*

Proof. According to the Reduction Theorem [1, p. 228] there exists a sequence $(B_n)_{n=1}^{\infty}$ of an ambiguous disjoint sets in X such that $B_n\subseteq F_n$ and $X=\bigcup_{n=1}^{\infty}B_n$. Let $f_n^*=f|_{B_n}\colon B_n\to Y_n$. Then $f_n^*\in H_1(B_n,Y_n)$. Fix a point $y_n\in Y_n$ for each n and let $f_n(x)=f_n^*(x)$ if $x\in B_n$ and $f_n(x)=y_n$ if $x\notin B_n$. Since $f_n|_{B_n}\in H_1(B_n,Y_n)$ and sets B_n are ambiguous, from [1, p. 385] it follows that $f_n\in H_1(X,Y_n)$ for every $n\in\mathbb{N}$. Since $H_1(X,Y_n)\subseteq B_1(X,Y_n)$, there exists a sequence $(g_{n,m})_{m=1}^{\infty}$ of continuous mappings $g_{n,m}\colon X\to Y_n$ such that $g_{n,m}(x)\to f_n(x)$. In particular, $\lim_{m\to\infty}g_{n,m}(x)=f(x)$ on B_n. The sets B_n are F_σ that is $B_n=\bigcup_{m=1}^{\infty}B_{n,m}$, where $(B_{n,m})_{m=1}^{\infty}$ is an increasing sequence of closed sets in X. Let $F_{n,m}=\varnothing$, when $n>m$, and $F_{n,m}=B_{n,m}$, when $n\le m$. From Lemma 2.1 it follows that for every $m\in\mathbb{N}$ there exists a continuous mapping $g_m\colon X\to Y$ such that $g_m|_{F_{n,m}}=g_{n,m}$ since the system $\{F_{n,m}:n\in\mathbb{N}\}$ is finite for every $m\in\mathbb{N}$. It remains to prove that $g_m(x)\to f(x)$ on X. Fix $x\in X$. There exists an integer $n\in\mathbb{N}$ such that $x\in B_n$. The sequence $(F_{n,m})_{m=1}^{\infty}$ is increasing therefore there exists an integer $m_0\ge n$ such that $x\in F_{n,m}$ for all $m\ge m_0$. Then $g_m(x)=g_{n,m}(x)$ for all $m\ge m_0$. Hence $\lim_{m\to\infty}g_m(x)=\lim_{m\to\infty}g_{n,m}(x)=f(x)$. □

Recall that a topological space X is *a Baire space* if every open nonempty set G in X is a set of the second category. A topological space X is called *a hereditarily Baire space* if every closed subset of X is a Baire space.

Proposition 2.3. *Let X be a hereditarily Baire metrizable separable space, Y be a locally convex space which is the strict inductive limit of an increasing sequence $(Y_n)_{n=1}^{\infty}$ of its closed subspaces Y_n and $f\in H_1(X,Y)$. Then there exists a sequence $(F_n)_{n=1}^{\infty}$ of F_σ-sets in X such that $X=\bigcup_{n=1}^{\infty}F_n$ and $f(F_n)\subseteq Y_n$.*

Proof. Let $E_{n,0}=\operatorname{int}f^{-1}(Y_n)$ for each $n\in\mathbb{N}$. We proceed to show that there exists an integer n such that $E_{n,0}\ne\varnothing$. Conversely, suppose that the set $E_{n,0}=\varnothing$ for all $n\in\mathbb{N}$. Then

$X \setminus E_{n,0} = \overline{X \setminus f^{-1}(Y_n)} = X$ that is the set $A_n = f^{-1}(Y \setminus Y_n)$ is dense in X for every $n \in \mathbb{N}$. Let $(U_k)_{k=1}^\infty$ denote a base in X. Then the set $U_k \cap A_n$ is nonempty for each n and k.

Fix $n_0 \in \mathbb{N}$. Prove that the preimage $f^{-1}(Y \setminus Y_{n_0})$ contains a dense in X G_δ-set. There exist an increasing sequence $n_0 < n_1 < \cdots < n_k < \ldots$ and a sequence of points x_k such that

$$x_k \in U_k \quad \text{and} \quad f(x_k) \in Y_{n_k} \setminus Y_{n_{k-1}}.$$

Since $W_0 = Y_{n_0}$ is a closed subspace of Y_{n_1} and $f(x_1) \notin Y_{n_0}$, there exists an open absolutely convex neighborhood W_1 of zero in Y_{n_1} such that $W_1 \cap Y_{n_0} = W_0$ and $f(x_1) \notin W_1$. This follows by the same method as in [6, p. 52]. Next, the set Y_{n_1} is a closed subspace of Y_{n_2}, $f(x_2) \notin Y_{n_1}$ and W_1 is an absolutely convex open neighborhood of zero in Y_{n_1}. Therefore there exists an absolutely convex open neighborhood W_2 of zero in Y_{n_2} with $W_2 \cap Y_{n_1} = W_1$ and $f(x_2) \notin W_2$. Continuing this process we obtain a sequence $(W_k)_{k=1}^\infty$ of the absolutely convex open neighborhoods of zero in Y_{n_k} such that $W_k \cap Y_{n_{k-1}} = W_{k-1}$ and $f(x_k) \notin W_k$. Let $W = \bigcup_{k=0}^\infty W_k$. Then W is an open neighborhood of zero in Y and $f(x_k) \notin W$ for every $k \in \mathbb{N}$. The set $F = Y \setminus W$ is closed in Y hence $f^{-1}(F)$ is G_δ-set in X. The inclusion $x_k \in f^{-1}(F) \cap U_k$ for every k implies $\overline{f^{-1}(F)} = X$. It follows from $F \subseteq Y \setminus W_0$ that $f^{-1}(F) \subseteq f^{-1}(Y \setminus W_0)$, i.e. the set $f^{-1}(Y \setminus Y_{n_0})$ contains a dense in X G_δ-set.

Since X is a Baire space, the intersection $\bigcap_{n=1}^\infty f^{-1}(Y \setminus Y_n)$ contains an everywhere dense G_δ-set. But $\bigcap_{n=1}^\infty f^{-1}(Y \setminus Y_n) = f^{-1}(Y \setminus \bigcup_{n=1}^\infty Y_n) = \varnothing$, a contradiction. Consequently, there exists an integer $n_0 \in \mathbb{N}$ such that $E_{n,0} \neq \varnothing$ for all $n \geq n_0$ provided the sequence $E_{n,0}$ is increasing.

Let $E_0 = \bigcup_{n=1}^\infty E_{n,0}$ and $X_0 = X$. Then $X_1 = X_0 \setminus E_0$ is a closed subset of X_0. Since X is a hereditarily Baire space, X_1 is a Baire space. Consider the mapping $f_1 = f|_{X_1}$. Clearly, $f_1 \in H_1(X_1, Y)$. Let $E_{n,1} = \operatorname{int}_{X_1} f_1^{-1}(Y_n)$ for every $n \in \mathbb{N}$. In the same manner we can see that there exists an integer $n_0 \in \mathbb{N}$ such that $E_{n,1} \neq \varnothing$ for all $n \geq n_0$. Let $E_1 = \bigcup_{n=1}^\infty E_{n,1}$. Since X_1 is closed in X_0, E_1 is an F_σ-set in X_0.

Assume the sets X_ξ, E_ξ have been constructed for all $\xi < \alpha < \omega_1$ and

(i) X_ξ is closed in X_0 for all $\xi < \alpha$;

(ii) $X_\xi \subseteq X_\beta$ for all $\beta < \xi < \alpha$;

(iii) $E_\xi = \bigcup_{n=1}^\infty E_{n,\xi}$, $E_{n,\xi} = \operatorname{int}_{X_\xi} f_\xi^{-1}(Y_n)$, where $f_\xi = f|_{X_\xi}$;

(iv) if $X_\xi \neq \varnothing$ then $E_\xi \neq \varnothing$;

(v) if $\xi = \eta + 1$ then $X_\xi = X_\eta \setminus E_\eta$ and if ξ is a limit ordinal then $X_\xi = \bigcap_{\eta<\xi} X_\eta$.

Let $X_\alpha = X_\xi \setminus E_\xi$ if $\alpha = \xi + 1$ and $X_\alpha = \bigcap_{\xi<\alpha} X_\xi$ if α is a limit ordinal. Let $E_\alpha = \bigcup_{n=1}^\infty E_{n,\alpha}$, where $E_{n,\alpha} = \operatorname{int}_{X_\alpha} f_\alpha^{-1}(Y_n)$ and $f_\alpha = f|_{X_\alpha}$.

Show that there exists $\alpha < \omega_1$ such that $X_\alpha = \varnothing$. Assume that $X_\alpha \neq \varnothing$ for all $\alpha < \omega_1$. By the Baire-Hausdorff Theorem [7, p. 162] there exists $\alpha < \omega_1$ such that $X_\alpha = X_{\alpha+1} = \ldots$. Then $E_\alpha = X_{\alpha+1} \setminus X_\alpha = \varnothing$, which contradicts property (iv). Denote $\beta = \inf\{\alpha < \omega_1 : X_\alpha = \varnothing\}$.

Thus $X_0 = \bigcup_{\xi<\beta}(X_\xi \setminus X_{\xi+1}) = \bigcup_{\xi<\beta} E_\xi = \bigcup_{n=1}^\infty \bigcup_{\xi<\beta} E_{n,\xi}$. Let $F_n = \bigcup_{\xi<\beta} E_{n,\xi}$. Then F_n is an F_σ-set in X for all $n \in \mathbb{N}$. Clearly, $X = \bigcup_{n=1}^\infty F_n$ and $f(F_n) \subseteq Y_n$. □

We say that $f\colon X \to Y$ is *pointwise discontinuous* if for every nonempty closed set $F \subseteq X$ the restriction $f|_F$ has a continuity point. It is easy to check that f is pointwise discontinuous on any closed subset of X if and only if the continuity point set $C(f)$ of the mapping f is dense in X. Note that for a mapping $f\colon X \to Y$ the set $C(f)$ is G_δ when X is a topological space and Y is a metrizable space.

Lemma 2.4. *Let X be a hereditarily Baire separable metrizable space, Y a metrizable space and $f\colon X\to Y$ be a Lebesgue one mapping. Then $f(Y)$ is separable.*

Proof. According to [8, Theorem 4.12], the function f is pointwise discontinuous. Consider an arbitrary G_δ-set A in X. Let $B=\overline{A}$. Since the continuity point set of $f|_B$ is a dense G_δ-set in B, the discontinuity point set of $f|_B$ is a set of the first category in B. Thus the discontinuity point set of $f|_A$ is of the first category in A and the continuity point set of $f|_A$ is dense in A, provided A is a Baire space. Consequently, f is pointwise discontinuous on any G_δ-set in X. By [1, p. 407], $f(X)$ is separable. □

Theorem 2.5. *Let X be a hereditarily Baire separable metrizable space, Y be the strict inductive limit of an increasing sequence of metrizable locally convex spaces Y_n. Then $H_1(X,Y)\subseteq B_1(X,Y)$.*

Proof. Without loss of generality we assume that all Y_n are closed in Y (see [9, Theorem 4.1.2]). By Proposition 2.3, there exists a sequence $(F_n)_{n=1}^\infty$ of F_σ-sets in X such that $X=\bigcup_{n=1}^\infty F_n$ and $f(F_n)\subseteq Y_n$. According to Lemma 2.4, the space $f(X)$ is separable. Then [5, Theorem 2] implies the inclusion $H_1(X,Y_n)\subseteq B_1(X,Y_n)$. Applying Lemma 2.2, we obtain $H_1(X,Y)\subseteq B_1(X,Y)$. □

3. We now prove the inclusion $B_1(X,Y)\subseteq H_1(X,Y)$.

Let Y be a topological space and $(Y_n)_{n=1}^\infty$ be a sequence of sets $Y_n\subseteq Y$ such that $Y=\bigcup_{n=1}^\infty Y_n$. We say that $(Y,(Y_n)_{n=1}^\infty)$ has *$(*)$-property* if for every convergent sequence $(y_k)_{k=1}^\infty$ in Y there exists an integer n such that $\{y_k : k\in\mathbb{N}\}\subseteq Y_n$.

Proposition 3.1. *Let X be a topological space, Y be a topological space, $(Y_n)_{n=1}^\infty$ be an increasing sequence of closed subspaces of Y such that $(Y,(Y_n)_{n=1}^\infty)$ has $(*)$-property and $f\in B_1(X,Y)$. Then there exists a sequence $(F_n)_{n=1}^\infty$ of closed sets in X such that $X=\bigcup_{n=1}^\infty F_n$ and $f(F_n)\subseteq Y_n$.*

Proof. The mapping f belongs to the first Baire class, therefore there exists a sequence of continuous mappings $f_n\colon X\to Y$ which pointwise converges to f on X. Let $G_n=\bigcup_{k\ge n} f_k^{-1}(Y\setminus Y_k)$ for each $n\in\mathbb{N}$ and prove the equality $\bigcap_{n=1}^\infty G_n=\varnothing$. Assume there exists $x\in\bigcap_{n=1}^\infty G_n$. Then there exists an increasing sequence $k_1<k_2<\dots<k_n<\dots$ such that $f_{k_n}(x)\notin Y_{k_n}$. From this we conclude that $f_{k_n}(x)$ is not convergent in Y. Thus $\bigcap_{n=1}^\infty G_n=\varnothing$. Let $F_n=X\setminus G_n$ for every $n\in\mathbb{N}$. Then all F_n are closed sets in X and $\bigcup_{n=1}^\infty F_n=X\setminus\bigcap_{n=1}^\infty G_n=X$. If $x\in F_n$ then $f_k(x)\in Y_n$ for every $k\ge n$. The space Y_n is a closed subspace of Y and $\lim_{k\to\infty} f_k(x)=f(x)$, hence $f(x)\in Y_n$, which completes the proof. □

The following fact was established in [9].

Theorem 3.2. *Let X be a topological space and Y be a perfectly normal space. Then $B_1(X,Y)\subseteq H_1(X,Y)$.*

Theorem 3.3. *Let X be a topological space, Y be a topological space, $(Y_n)_{n=1}^\infty$ be an increasing sequence of closed perfectly normal subspaces of Y such that $(Y,(Y_n)_{n=1}^\infty)$ has $(*)$-property. Then $B_1(X,Y)\subseteq H_1(X,Y)$.*

Proof. By Proposition 3.1, there exists a sequence $(F_n)_{n=1}^{\infty}$ of closed subsets of X such that $X = \bigcup_{n=1}^{\infty} F_n$ and $f(F_n) \subseteq Y_n$. Then $f|_{F_n} \in B_1(F_n, Y_n)$. According to Theorem 3.2, $f|_{F_n} \in H_1(F_n, Y_n) \subseteq H_1(F_n, Y)$. Then by [1, p. 385], $f \in H_1(X, Y)$. □

Remark that for the strict inductive limit Y of a sequence of metrizable locally convex spaces Y_n the pair $(Y, (Y_n)_{n=1}^{\infty})$ has $(*)$-property [9, Theorem 4.1.2]. Combining Theorem 2.5 and Theorem 3.3, we obtain the following result.

Theorem 3.4. *Let X be a hereditarily Baire separable metrizable space, Y the strict inductive limit of an increasing sequence of metrizable locally convex spaces. Then $H_1(X,Y) = B_1(X,Y)$.*

Finally, we give an example which shows that the conditions on the range space in Theorem 3.2 are essential.

It is well known [10] that the Nemytski plane $\mathbb{P}$ is a perfect space but is not a normal space.

Example. Define the function $f\colon \mathbb{R} \to \mathbb{P}$ by $f(x) = (x, 0)$. Then $f \in B_1(\mathbb{R}, \mathbb{P}) \setminus H_1(\mathbb{R}, \mathbb{P})$.

Let $f_n(x) = (x, \frac{1}{n})$, if $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Clearly, the functions $f_n\colon \mathbb{R} \to \mathbb{P}$ are continuous and $f_n(x) \to f(x)$ on $\mathbb{R}$. Hence, $f \in B_1(\mathbb{R}, \mathbb{P})$.

Consider the set $F = \{(r, 0) : r \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of all rational reals. The set $\mathbb{P}_1 = \{(x, 0) : x \in \mathbb{R}\}$ is a discrete closed subspace of $\mathbb{P}$, and so F is closed in $\mathbb{P}$. But $f^{-1}(F) = \mathbb{Q}$ is not a G_δ-set in $\mathbb{R}$. Thus, $f \notin H_1(X, Y)$.

REFERENCES

1. Куратовский К. Топология. Т.1. – М.: Мир, 1966. – 594 с.
2. Карлова О. *Перший клас Лебега і берівська класифікація нарізно неперервних функцій* // Наук. вісн. Чернів. у-ту., Вип. 191-192. Математика. – Чернівці: Рута, 2004. – С. 52–60.
3. Hansell R.W. *Lebesgue's theorem on Baire class 1 functions* // Topology with applications, Szekszard (Hungary). – 1993. – P. 251–257.
4. Карлова О. *Берівська класифікація відображень із значеннями у підмножинах скінченновимірних просторів* // Наук. вісн. Чернів. у-ту., Вип. 239. Математика. – Чернівці: Рута, 2005. – С. 59–65.
5. Fosgerau M. *When are Borel functions Baire functions?* // Fund. Math. – 1993. – V.143. – P. 137–152.
6. Маслюченко В.К. Лінійні неперервні оператори. – Чернівці: Рута, 2002. – 72 с.
7. Александров П.С. Введение в теорию множеств и общую топологию. – М.: Наука, 1977. – 368 с.
8. Koumoulis G. *A generalization of functions of the first class* // Top. Appl. – 1993. – V.50. – P. 217–239.
9. Маслюченко В.К. Нарізно неперервні відображення і простори Кете: Дис...докт. фіз.-мат. наук: 01.01.01. – Чернівці, 1999. – 345 с.
10. Енгелькинг Р. Общая топология. – М.: Мир, 1986. – 752 с.

Chernivtsi National University

Received 16.09.2005