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P. I. Kalenyuk, Z. M. Nytrebych, P. Drygaś

METHOD OF SOLVING A CAUCHY PROBLEM FOR HOMOGENEOUS DIFFERENTIAL-OPERATOR EQUATION AND ITS APPLICATIONS

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We propose a scheme of solving a Cauchy problem for a homogeneous differential-operator equation of high order with operator coefficients in Banach space. For initial data from a special subspace of the Banach space in which the vectors are represented as Stieltjes integrals with respect to a certain measure, the solution of the problem is represented as a sum of certain Stieltjes integrals with respect to the same measure. We give some examples of the application of the scheme to solving a Cauchy problem for partial differential equations of infinite order in the spatial variable in infinite order Sobolev classes and in classes of entire analytical functions of certain orders.

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Предложен метод решения задачи Коши для однородного дифференциально-операторного уравнения высокого порядка с операторными коэффициентами бесконечного порядка в банаховом пространстве. Для начальных данных из специального подпространства банахового пространства, в котором векторы представляются как интегралы Стильтьеса по некоторой мере, решение задачи представлено в виде суммы некоторых интегралов Стильтьеса по этой же мере. Приведены примеры применения метода к решению задачи Коши для дифференциального уравнения в частных производных бесконечного порядка по пространственной переменной в классах Соболева бесконечного порядка и классах целых аналитических функций некоторых порядков.

1. Statement of the problem. Let $\mathfrak{H}$ be a Banach space, A be a given linear operator acting in it. Suppose that for this operator, all powers A^j, $j \in \{2, 3, \dots\}$, are defined in $\mathfrak{H}$, i.e. any vector $h \in \mathfrak{H}$ is a C^∞-vector of the operator A (cf. [1]). Let Λ be an arbitrary subset of $\mathbb{C}$. Denote by $x(\lambda)$ the solution of the equation

$$Ax(\lambda) = \lambda x(\lambda), \quad \lambda \in \Lambda,$$

considering $x(\lambda)$ to be an eigenvector of the operator A corresponding to the eigenvalue λ, and $x(\lambda) = 0$ when λ is not an eigenvalue of the operator A. Besides, we consider analytic on Λ functions $b_j(\lambda)$, $j \in \{1, 2, \dots, n\}$, which are the symbols of, in general, infinite

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order operators $b_j(A)$, $j \in \{1, 2, \ldots, n\}$. We also suppose that $b_j(A)x(\lambda) = b_j(\lambda)x(\lambda)$, $j \in \{1, 2, \ldots, n\}$.

We shall consider the following Cauchy problem:

$$L\left(\frac{d}{dt}, A\right)U(t) \equiv \frac{d^nU}{dt^n} + \sum_{j=1}^{n} b_j(A)\frac{d^{n-j}U}{dt^{n-j}} = 0, \quad t \in \mathbb{R}_+, \tag{1}$$

$$\left.\frac{d^kU}{dt^k}\right|_{t=0} = h_k, \quad k \in \{0, 1, \ldots, n-1\}, \tag{2}$$

where h_k, $k \in \{1, 2, \ldots, n-1\}$, are given vectors in $\mathfrak{H}$, $U\colon \mathbb{R}_+ \to \mathfrak{H}$ is a sought function.

Numerous investigations of scientists concerning problem (1), (2) originate from the Cauchy problem for evolutional equation, i.e. problem (1), (2) where $n = 1$ and $b_1(A) = -A$. The semigroup theory takes a great place in such investigations. Significant results of this theory can be found in the books [9–14].

The Cauchy problem for differential-operator equation has been studied by many authors (see in addition to the above papers also [1–3, 6, 7]). When solving such a problem we often use the technique of separation of variables whose generalization for partial differential equations has been proposed in [4, 5]. In recent years, there appear new approaches to investigating Cauchy problem both for differential-operator equations and for partial differential equations. In particular, in [2, 3], for investigating problem (1), (2), where $A = -i\frac{d}{dx}$, Yu. A. Dubinskiy has proposed the technique of using infinite order differential operators in specially defined infinite order Sobolev spaces and has represented the solution of the problem in integral form. In [4, 5], by means of the differential-symbol method proposed by P. I. Kalenyuk and Z.M. Nytrebych, the solution of problem (1), (2), where $A = \frac{d}{dx}$, has been represented in a differential form as an action of infinite order differential expressions, whose symbols are the initial functions, onto certain entire functions of parameters, with respect to which the differential expressions act.

In the present paper, we propose a method of solving Cauchy problem (1), (2), which embraces the approaches mentioned above.

2. Method of solving. We shall show the method of constructing a solution of the problem (1), (2) for the special right-hand sides h_k.

Definition 1. We shall say that the vector h from $\mathfrak{H}$ *belongs to* $\mathfrak{L}$ if on Λ there exist a depending on h linear operator $R_h(\lambda)\colon \mathfrak{H} \to \mathfrak{H}$ and a measure $\mu(\lambda)$ such that

$$h = \int_{\Lambda} R_h(\lambda)x(\lambda)d\mu(\lambda). \tag{3}$$

Thus, each vector h from $\mathfrak{L}$ can be represented in the form of Stieltjes integral (3) with respect to a certain measure.

In the operator $L\left(\frac{d}{dt}, A\right)$ which is in equation (1), we replace the operator A with parameter λ and write down the ordinary differential equation as follows:

$$L\left(\frac{d}{dt}, \lambda\right)T(t, \lambda) = 0. \tag{4}$$

Consider the normal fundamental system of solutions of equation (4), i.e. the system $T_0(t,\lambda), T_1(t,\lambda), \dots, T_{n-1}(t,\lambda)$ of solutions of equation (4) satisfying the conditions

$$\left.\frac{d^k T_j}{dt^k}\right|_{t=0} = \delta_{kj}, \quad k, j \in \{0, 1, \dots, n-1\}, \tag{5}$$

where δ_{kj} is the Kronecker delta.

Remark 1. The coefficients of ordinary differential equation (4) are analytic on Λ functions, thus the elements of a normal fundamental system of solutions of equation (4), by the Poincaré theorem on analytic dependence of the Cauchy problem solution on a parameter (cf. [8]), are also analytic on Λ functions.

Lemma 1. *For an arbitrary function $\chi(t,\lambda), n$ times differentiable with respect to the variable t and analytic on Λ in variable λ, the following equality holds on the set $\mathbb{R}_+ \times \Lambda$:*

$$L\left(\frac{d}{dt}, A\right)[\chi(t,\lambda)x(\lambda)] = L\left(\frac{d}{dt}, \lambda\right)[\chi(t,\lambda)x(\lambda)],$$

where $L\left(\frac{d}{dt}, A\right)$ is the operator of equation (1).

Proof. As supposed, for the operator A, all powers A^n, $n \in \mathbb{N}$, are defined in $\mathfrak{H}$. Then, for each $t \in \mathbb{R}_+$ and $\lambda \in \Lambda$, we have

$$L\left(\frac{d}{dt}, A\right)[\chi(t,\lambda)x(\lambda)] \equiv \left(\frac{d^n}{dt^n} + \sum_{j=1}^{n} b_j(A)\frac{d^{n-j}}{dt^{n-j}}\right)[\chi(t,\lambda)x(\lambda)] \equiv$$

$$\equiv \frac{d^n}{dt^n}[\chi(t,\lambda)x(\lambda)] + \sum_{j=1}^{n}\left(\frac{d^{n-j}}{dt^{n-j}}\chi(t,\lambda)\right)[b_j(A)x(\lambda)] \equiv$$

$$\equiv \frac{d^n}{dt^n}[\chi(t,\lambda)x(\lambda)] + \sum_{j=1}^{n}\left(\frac{d^{n-j}}{dt^{n-j}}\chi(t,\lambda)\right)[b_j(\lambda)x(\lambda)] \equiv$$

$$\equiv \frac{d^n}{dt^n}[\chi(t,\lambda)x(\lambda)] + \sum_{j=1}^{n} b_j(\lambda)\frac{d^{n-j}}{dt^{n-j}}[\chi(t,\lambda)x(\lambda)] \equiv L\left(\frac{d}{dt}, \lambda\right)[\chi(t,\lambda)x(\lambda)].$$

□

Assume that, in the initial conditions (2), $h_k \in \mathfrak{L}$ for each $k \in \{0, 1, \dots, n-1\}$. This implies that there exist on Λ linear operators $R_{h_k}(\lambda)$, $k \in \{0, 1, \dots, n-1\}$, depending on h_k, acting in $\mathfrak{H}$; measures $\mu(\lambda)$ such that

$$h_k = \int_{\Lambda} R_{h_k}(\lambda)x(\lambda)d\mu(\lambda), \quad k \in \{0, 1, \dots, n-1\}. \tag{6}$$

Theorem 1. *Let in problem (1), (2) $h_k \in \mathfrak{L}$ for each $k \in \{0, 1, \dots, n-1\}$, i.e. h_k for $k \in \{0, 1, \dots, n-1\}$ could be represented in form* (6)*. Then the formula*

$$U(t) = \sum_{m=0}^{n-1}\int_{\Lambda} R_{h_m}(\lambda)\left[T_m(t,\lambda)x(\lambda)\right]d\mu(\lambda), \tag{7}$$

where $T_0(t,\lambda), T_1(t,\lambda), \ldots, T_{n-1}(t,\lambda)$ *is a normal fundamental system of solutions of equation* (4)*, determines a formal solution of problem* (1), (2)*.*

Proof. By Lemma 1 and Remark 1, we have

$$L\left(\frac{d}{dt}, A\right) U(t) = L\left(\frac{d}{dt}, A\right) \left\{ \sum_{m=0}^{n-1} \int_{\Lambda} R_{h_m}(\lambda) \left[T_m(t,\lambda)x(\lambda)\right] d\mu(\lambda) \right\} =$$

$$= \sum_{m=0}^{n-1} \int_{\Lambda} R_{h_m}(\lambda) \left\{ L\left(\frac{d}{dt}, A\right) \left[T_m(t,\lambda)x(\lambda)\right] \right\} d\mu(\lambda) =$$

$$= \sum_{m=0}^{n-1} \int_{\Lambda} R_{h_m}(\lambda) \left\{ L\left(\frac{d}{dt}, \lambda\right) \left[T_m(t,\lambda)x(\lambda)\right] \right\} d\mu(\lambda) =$$

$$= \sum_{m=0}^{n-1} \int_{\Lambda} R_{h_m}(\lambda) \left\{ \left[L\left(\frac{d}{dt}, \lambda\right) T_m(t,\lambda) \right] x(\lambda) \right\} d\mu(\lambda) = \sum_{m=0}^{n-1} \int_{\Lambda} R_{h_m}(\lambda) \left\{ 0 \cdot x(\lambda) \right\} d\mu(\lambda).$$

Since the operators R_{h_m}, $m \in \{0,1,\ldots,n-1\}$, are linear, $L\left(\frac{d}{dt}, A\right) U(t) = 0$, i.e. $U(t)$ of form (7) formally satisfies equation (1).

Now we shall prove that the initial conditions (2) hold for function (7), making use of the initial conditions (5):

$$\left.\frac{d^k U}{dt^k}\right|_{t=0} = \sum_{m=0}^{n-1} \int_{\Lambda} R_{h_m}(\lambda) \left[\left.\frac{d^k T_m}{dt^k}\right|_{t=0} x(\lambda) \right] d\mu(\lambda) =$$

$$= \sum_{m=0}^{n-1} \int_{\Lambda} R_{h_m}(\lambda) \left[\delta_{km} x(\lambda)\right] d\mu(\lambda) = \int_{\Lambda} R_{h_k}(\lambda) x(\lambda) d\mu(\lambda).$$

Using (6), we obtain

$$\left.\frac{d^k U}{dt^k}\right|_{t=0} = h_k, \quad k \in \{0,1,\ldots,n-1\}.$$

This proves our theorem. ☐

Remark 2. Function (7) is considered to be a formal solution of problem (1), (2) since we do not justify for it the following equalities:

$$L\left(\frac{d}{dt}, A\right) \left\{ \sum_{m=0}^{n-1} \int_{\Lambda} R_{h_m}(\lambda) \left[T_m(t,\lambda)x(\lambda)\right] d\mu(\lambda) \right\} = \\ = \sum_{m=0}^{n-1} \int_{\Lambda} R_{h_m}(\lambda) \left\{ L\left(\frac{d}{dt}, A\right) \left[T_m(t,\lambda)x(\lambda)\right] \right\} d\mu(\lambda), \tag{8}$$

$$\frac{d^k}{dt^k}\left\{\sum_{m=0}^{n-1}\int\limits_{\Lambda} R_{h_m}(\lambda)\left[T_m(t,\lambda)x(\lambda)\right]d\mu(\lambda)\right\}=\sum_{m=0}^{n-1}\int\limits_{\Lambda} R_{h_m}(\lambda)\left[\frac{d^k}{dt^k}\left[T_m(t,\lambda)x(\lambda)\right]\right]d\mu(\lambda), \tag{9}$$

neither the existence of the corresponding Stieltjes integrals in the right-hand sides of formulas (7)–(9).

Naturally, the proposed method of solving a Cauchy problem for a differential-operator equation is quite general. In the case of a specific Banach space $\mathfrak{H}$, subspace $\mathfrak{L}$, operator A and the corresponding measure, one can refine the obtained results and prove theorems on the existence and uniqueness of the Cauchy problem solution in the corresponding functional spaces.

3. Examples of applications of the method.

Example 1. Let $\Lambda=\mathbb{R}$, $\mathfrak{H}=\mathfrak{L}=H^\infty(\Lambda)$, where $H^\infty(\Lambda)$ is the functional space introduced by Yu. A. Dubinskiy (cf. [2]):

$$H^\infty(\Lambda)=\left\{h\in L_2(\mathbb{R}):\widehat{h}\text{ is finitary in }\Lambda\right\}.$$

Here $\widehat{h}(\xi)=\frac{1}{\sqrt{2\pi}}\int_\Lambda\varphi(x)\exp[-ix\xi]dx$ is the Fourier transform of the function $h(x)$. For the operator A, we take the differentiation operator $D=-i\frac{d}{dx}$ with respect to the variable $x\in\mathbb{R}$, then $\exp[i\lambda x]$ is an eigenvector of the operator $A=D$. Assuming for any $h\in H^\infty(\Lambda)$ the measure $\mu(\lambda)$ to be the Lebesgue measure, we obtain $d\mu(\lambda)=d\lambda$.

Equality (3) in this case, for $h\in H^\infty(\Lambda)$, has the form

$$h(x)=\int\limits_\Lambda R_h(\lambda)\exp[i\lambda x]d\lambda. \tag{10}$$

From (10), we obtain that $R_h(\lambda)$, within the factor $\frac{1}{\sqrt{2\pi}}$, is the Fourier transform of the function $h(x)$, e.g. $R_h(\lambda)=\frac{1}{\sqrt{2\pi}}\widehat{h}(\lambda)$.

Problem (1), (2) for this case is a Cauchy problem for infinite order partial differential equation:

$$L\left(\frac{\partial}{\partial t},D\right)U(t,x)\equiv\frac{\partial^n U}{\partial t^n}+\sum_{j=1}^{n}b_j(D)\frac{\partial^{n-j}U}{\partial t^{n-j}}=0,\quad t\in\mathbb{R}_+,\ x\in\mathbb{R},$$

$$\left.\frac{\partial^k U}{\partial t^k}\right|_{t=0}=h_k(x),\quad k\in\{0,1,\dots,n-1\}. \tag{11}$$

Assuming that $h_k\in H^\infty(\Lambda)$ for each $k\in\{0,1,\dots,n-1\}$, with formula (7), we find the solution of problem (11):

$$U(t,x)=\frac{1}{\sqrt{2\pi}}\sum_{m=0}^{n-1}\int\limits_\Lambda\widehat{h}_m(\xi)T_m(t,\lambda)\exp[ix\lambda]d\lambda, \tag{12}$$

where, as earlier, $T_0(t,\lambda), T_1(t,\lambda), \ldots, T_{n-1}(t,\lambda)$ is the normal fundamental system of solutions of the equation

$$L\left(\frac{d}{dt}, \lambda\right) T = 0.$$

In [2] it has been proved that, if $h_k \in H^\infty(\Lambda)$ for each $k \in \{0, 1, \ldots, n-1\}$, then, in the class of continuously differentiable up to the n-th order inclusively with respect to the variable t functions $U(t,x)$, there exists a unique solution $U(t,x)$ of problem (11).

Example 2. Take for $\mathfrak{H}$ the space of entire analytic on $\mathbb{R}$ functions $h(x)$, i.e. $\mathfrak{H} = \mathrm{A}(\mathbb{R})$, and assume the operator A to be the differentiation operator $\frac{d}{dx}$, and $\Lambda = \mathbb{R}$. Then $\exp[\lambda x]$ is an eigenvector of the operator A. Problem (1), (2) in this case is a Cauchy problem for infinite order partial differential equation:

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) U(t,x) \equiv \frac{\partial^n U}{\partial t^n} + \sum_{j=1}^{n} b_j\left(\frac{\partial}{\partial x}\right) \frac{\partial^{n-j} U}{\partial t^{n-j}} = 0, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}, \tag{13}$$

$$\left.\frac{\partial^k U}{\partial t^k}\right|_{t=0} = h_k(x), \quad k \in \{0, 1, \ldots, n-1\}. \tag{14}$$

Assume the measure $\mu(\lambda)$ for $h \in \mathrm{A}(\mathbb{R})$ to be the Dirac measure, i.e. $d\mu(\lambda) = \delta(\lambda)$, where $\delta(\lambda)$ is a delta-function. Then equality (3) gets the form

$$h(x) = R_h(\lambda) \exp[\lambda x]\Big|_{\lambda=0}. \tag{15}$$

For $h \in \mathrm{A}(\mathbb{R})$, the operator $R_h(\lambda)$ is defined as the infinite order differential operator

$$R_h(\lambda) = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}\left(\frac{d}{d\lambda}\right)^k.$$

Then, from equality (15), we obtain

$$h(x) = \left\{\sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}\left(\frac{d}{d\lambda}\right)^k\right\} \exp[\lambda x]\Big|_{\lambda=0} =$$

$$= \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!}\left\{\left(\frac{d}{d\lambda}\right)^k \exp[\lambda x]\right\}\Big|_{\lambda=0} = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} x^k.$$

Thus, equality (3) gives a natural representation of entire analytic functions in the class $\mathrm{A}(\mathbb{R})$.

Formula (7) for this case determines a solution of problem (13), (14) in the form as follows:

$$U(t,x) = \sum_{m=0}^{n-1}\left(\sum_{k=0}^{\infty} \frac{h_m^{(k)}(0)}{k!}\left(\frac{d}{d\lambda}\right)^k \{T_m(t,\lambda)\exp[\lambda x]\}\right)\Bigg|_{\lambda=0},$$

or, taking into account that $\sum_{k=0}^{\infty} \frac{h_m^{(k)}(0)}{k!}\left(\frac{d}{d\lambda}\right)^k = h_m\left(\frac{d}{d\lambda}\right)$, in the form

$$U(t,x) = \sum_{m=0}^{n-1} h_m\left(\frac{d}{d\lambda}\right)\{T_m(t,\lambda)\exp[\lambda x]\}\Big|_{\lambda=0}. \tag{16}$$

Representation (16) of a solution of problem (13), (14) has been obtained in [4, 5] by means of the differential-symbol method.

For specifying a class of univalent solvability of problem (13), (14), we specify in the class A($\mathbb{R}$) a certain subclass $\mathfrak{L}$ which the initial functions $h_m(x)$, for $m \in \{0, 1, \ldots, n-1\}$, should belong to. This subclass $\mathfrak{L}$ is invariant with respect to the infinite order operators $h_m\left(\frac{d}{d\lambda}\right)$, for $m \in \{0, 1, \ldots, n-1\}$. The subclass $\mathfrak{L}$ could be characterized (cf. [5]) as the class of entire analytic on $\mathbb{R}$ functions which as functions of complex variable are entire with the order dual in the sense of Young to the order determined by the orders of the differential expressions $b_j\left(\frac{\partial}{\partial x}\right)$, $j \in \{1, 2, \ldots, n\}$, of equation (13).

4. Conclusions. In the present paper, we propose a method of solving Cauchy problem for homogeneous high order differential-operator equation. In a special class of functions, the Cauchy problem solution is expressed as a sum of Stieltjes integrals with respect to certain measures. Such a representation includes, as particular cases, the integral representation of the Cauchy problem solution (cf. [2]), obtained by means of the infinite order operators technique, and the differential representation of the mentioned solution, obtained by means of the differential-symbol method (cf. [5]).

The analogous method of solving Cauchy problem for inhomogeneous differential-operator equation of high order needs further development.

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Lviv Polytechnic National University, S.Bandery str. 12, Lviv, Ukraine
University of Rzeszów, Rejtana str. 17a, Rzeszów, Poland

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