

УДК 515.12

L. S. ZDOMSKYY

***o*-BOUNDEDNESS OF FREE OBJECTS OVER A TYCHONOFF SPACE**

L. S. Zdomskyy. *o*-Boundedness of free objects over a Tychonoff space, Matematychni Studii, **25** (2006) 10–28.

In this paper we characterize various sorts of boundedness of the free (abelian) topological group $F(X)$ ($A(X)$) as well as the free locally-convex linear topological space $L(X)$ in terms of properties of a Tychonoff space X . These properties appear to be close to so-called selection principles, which permits us to show, that (it is consistent with ZFC that) the property of Hurewicz (Menger) is l -invariant. This gives a method of construction of OF -undetermined topological groups with strong combinatorial properties.

Л. С. Здомский. *o*-Ограниченность свободных объектов над пространством Тихонова // Математичні Студії. – 2006. – Т.25, №1. – С.10–28.

В данной работе характеризуются различные типы ограниченности свободной (абелевой) топологической группы $F(X)$ ($A(X)$), а также свободного локально-выпуклого линейного топологического пространства $L(X)$ в терминах свойств тихоновского пространства X . Эти свойства оказываются близкими к селекционным принципам, что позволяет нам показать что (это совместимо с ZFC) что свойства Гуревича (Менгера) l -инвариантны. Это дает метод построения OF -неопределенных топологических групп с сильными комбинаторными свойствами.

Introduction. Main objects and related notions. The starting impulse for writing this paper came from [15], where the problem of characterization of Tychonoff spaces X whose free (abelian) topological group $F(X)$ ($A(X)$) is [strictly] o -bounded was posed. In fact, this problem consists of four subproblems. Three of them (except for the characterization of o -boundedness of $F(X)$) are solved here. Throughout the paper group operations on abelian groups are denoted by $+$ and “topological space” means “Tychonoff space”.

Thus the main objects considered in this paper are free (abelian) topological groups over a space X , i.e. a (abelian) topological group G that contains X as a set of generators and satisfies the following condition: each continuous function $\varphi: X \rightarrow H$ of X to an arbitrary (abelian) topological group H admits a unique extension to a continuous homomorphism $\tilde{\varphi}: G \rightarrow H$, see [13] or [31] for basic properties of free topological groups. As usually we denote by $C_p(X)$ the space of continuous real-valued functions on X , endowed with topology inherited from the Tychonoff product \mathbb{R}^X . It is well-known [3, Ch. 0] that the correspondence $x \mapsto \psi_x$, where $\psi_x(f) = f(x)$ for all $f \in C_p(X)$, is a closed embedding of X into $C_p C_p(X)$ such that the image of X is linearly independent. In what follows we denote by $L_p(X)$ the linear hull of X in $C_p C_p(X)$ with the subspace topology. The space $L_p(X)$ is the free object

2000 *Mathematics Subject Classification*: 54D20, 22A05, 20E05, 91A44.

over X in the category of linear topological spaces with the weak topology, see [36]. The free object over X in the category of all (locally-convex) linear topological spaces will be denoted by $L_s(X)$ (resp. $L(X)$). (The topology on $L_s(X)$ is the strongest linear topology inducing the original topology on the space $X \subset L_s(X)$. This justifies the choice of the notation for $L_s(X)$.)

The spaces X and Y are called

- *M-equivalent*, if the topological groups $F(X)$ and $F(Y)$ are topologically isomorphic;
- *A-equivalent*, if the topological groups $A(X)$ and $A(Y)$ are topologically isomorphic;
- *l-equivalent*, if $C_p(X)$ and $C_p(Y)$ are isomorphic as linear topological spaces.

It was shown in [3] that a space X is *l-equivalent* to a space Y if and only if $L_p(X)$ is linearly homeomorphic to $L_p(Y)$. We shall use this as an alternative definition of the *l-equivalence* relation.

We say that a topological property P is φ -invariant, where φ runs over M , A and l , if a space X has this property whenever so does any space Y φ -equivalent to X . It is known [2] that *M-equivalence* implies *A-equivalence*, and *A-equivalence* implies *l-equivalence*, and consequently each *l*-invariant property is *A*-invariant, and each *A*-invariant property is *M*-invariant. For various examples of φ -invariant properties see, e.g. [3, Ch. 2], [31], and [32]. In this paper we prove the *l-equivalence* of selection principles defined below. It is worth to mention here that these principles are not multiplicative by [16, Th. 2.12], which makes it impossible to use that the free (nonabelian) group over a space X can be represented as countable union of continuous images of finite powers of X . They are also not hereditary [7], and therefore the corresponding proofs cannot be reduced to the classical result of V. Pestov [20] asserting that X is a countable union of subspaces each of which is homeomorphic to a subspace of Y provided X and Y are *M-equivalent*.

Selection principles on topological spaces and groups. The notion of an *o*-bounded topological group was introduced by O. Okunev and M. Tkachenko with the purpose of characterizing subgroups of σ -compact groups, see [14], [15], and [30] for the discussion of these properties. Recall that a topological group G is *o-bounded*, if for every sequence $(U_n)_{n \in \omega}$ of nonempty open subsets of G , there exists a sequence $(F_n)_{n \in \omega}$ of finite subsets of G such that $G = \bigcup_{n \in \omega} F_n \cdot U_n$. It is clear that every σ -compact group is *o-bounded*. Properties of topological spaces X appearing as duals of the *o*-boundedness of free groups and $L_p(X)$ are closely related to so-called selection principles. The duality between properties of X and $F(X)$, $A(X)$, and $C_p(X)$ is represented by many results, see [31] and [3]. The oldests of selection principles, namely the covering properties of Menger and Hurewicz¹, were introduced at the beginning of the 20th century, see [24] or [34] for their history and basic properties. In nearly seventy years after their appearance M. Scheepers systematized existing and introduced new properties of this kind. Among them the property $\bigcup_{\text{fin}}(\mathcal{O}, \Omega)$ is of extreme importance for us, and we shall refer to it as the Scheepers property. To define the above three selection principles used in this paper, we have to recall from [11] the definitions of some classes of covers: family $\{U_n : n \in \omega\}$ of subsets of X is said to be

- an ω -cover of X , if for every finite subset F of X there exists $n \in \omega$ such that $F \subset U_n$;
- a γ -cover of X , if for every $x \in X$ the set $\{n \in \omega : x \notin U_n\}$ is finite.

¹The properties $\bigcup_{\text{fin}}(\mathcal{O}, \mathcal{O})$ and $\bigcup_{\text{fin}}(\mathcal{O}, \Gamma)$ in terms of M. Scheepers [24].

Let B be a subset of a set X and u be a cover of X . We say that B is u -bounded, if $B \subset \bigcup c$ for some finite subfamily c of u . A topological space X is said to have the *Menger* (resp. *Scheepers*, *Hurewicz*) property, if for every sequence $(u_n)_{n \in \omega}$ of open covers of X there exists a $(\omega$ -, γ -) cover $\{B_n : n \in \omega\}$ of X such that each B_n is u_n -bounded. The Menger and Scheepers properties differ under the Continuum Hypothesis by [16, Theorem 2.8] and coincide under $(\mathfrak{u} < \mathfrak{g})$ according to Corollary 2 of [39]. Note that the \mathfrak{o} -boundedness (of all finite powers) of a topological group is nothing else but the Menger (Scheepers) property applied to the family of uniform covers with respect to the left uniformity on it, see Lemma 17 and Proposition 12. From now on we denote by νX and μX the Hewitt and Dieudonné completions of a space X , see [10] for their definitions and basic properties. We are in a position now to present the characterization of the \mathfrak{o} -boundedness of free abelian topological group.

Theorem 1. *For a space X the following conditions are equivalent:*

- (1) $A(X)$ is \mathfrak{o} -bounded;
- (2) $A(X)^n$ is \mathfrak{o} -bounded for all $n \in \mathbb{N}$;
- (3) $L_p(X)$ is \mathfrak{o} -bounded;
- (4) $L(X)$ is \mathfrak{o} -bounded;
- (5) $L_s(X)^n$ is \mathfrak{o} -bounded for all $n \in \mathbb{N}$;
- (6) every continuous metrizable image of X has the Scheepers property;
- (7) $A(\nu X)$ is \mathfrak{o} -bounded;
- (8) $A(\mu X)$ is \mathfrak{o} -bounded.

Selection games and multicovered spaces. \mathfrak{o} -Boundedness as well as the Menger property have natural game counterparts. In case of a σ -compact group G a sequence $(F_n)_{n \in \omega}$ witnessing the \mathfrak{o} -boundedness of G may be constructed by the second player in the process of an infinite game, called OF . This game is played by two players, say I and II. Player I selects an open subset U_0 of G , and player two responds choosing some finite subset F_0 of G . In the second turn, player I selects some open subset U_1 of G , and II responds choosing a finite subset F_2 of G , and so on. At the end of this game we obtain the sequences $(U_n)_{n \in \omega}$ and $(F_n)_{n \in \omega}$. Player II is declared the winner, if $\bigcup_{n \in \omega} F_n \cdot U_n = G$. Otherwise, player I wins. A group G is *strictly \mathfrak{o} -bounded*, if the second player (= player II) has a winning strategy in the game OF on G . If none of the players has a winning strategy, then G is called *OF -undetermined*. It is clear that each σ -compact group is strictly \mathfrak{o} -bounded and thus \mathfrak{o} -bounded. Examples distinguishing the σ -compactness, strict \mathfrak{o} -boundedness and \mathfrak{o} -boundedness may be found in [5], [14], [30], and [33].

As we shall see later, the strict \mathfrak{o} -boundedness of the free objects over a space X has no characterization in terms of continuous metrizable images of X in spirit of Theorem 1. This constrained us to use the language of multicovered spaces, which seems to be the most appropriate one for description of the corresponding property of X . By a *multicovered space*² we understand a pair (X, λ) , where X is a set and λ is a *multicover* of X , i.e. a family of covers of X . There are many natural examples of multicovered spaces:

The notion of a multicovered space and some other notions related to multicovered spaces, as well as Corollaries 23 and 24 are due to T.Banach. Multicovered spaces, which seem to be the most general objects where properties looking similar to (strict) \mathfrak{o} -boundedness can be considered, are discussed in [6].

- Each topological space X can be considered as a multicovered space (X, \mathcal{O}) , where \mathcal{O} denotes the family of all open covers of X ;
- Every metric space (X, ρ) admits a natural multicover λ_ρ consisting of covers by ε -balls: $\lambda_\rho = \{\{B_\rho(x, \varepsilon) : x \in X\} : \varepsilon > 0\}$, where $B_\rho(x, \varepsilon) = \{y \in X : \rho(y, x) < \varepsilon\}$;
- Every uniform space (X, \mathcal{U}) has a multicover $\lambda_{\mathcal{U}}$ consisting of uniform covers, i.e. $\lambda_{\mathcal{U}} = \{\{U(x) : x \in X\} : U \in \mathcal{U}\}$, where $U(x) = \{y \in X : (x, y) \in U\}$;
- In particular, each topological group G admits four natural multicovers $\lambda_L(G)$, $\lambda_R(G)$, $\lambda_{L \vee R}(G)$ and $\lambda_{L \wedge R}(G)$ corresponding to its left, right, two-side and Rölcke uniformities, see [22] for more information on these uniformities;
- In case of an abelian topological group G all of the above uniformities coincide, and we denote them by $\mathcal{U}(G)$. The family $\{\{(x, y) : x - y \in U\} : 0 \in U \in \mathcal{O}(G)\}$ is a base of $\mathcal{U}(G)$. Therefore the corresponding multicovers coincide as well, and we denote them by $\lambda(G)$.

By analogy with the game OF on a topological group G we can introduce the game CB (abbreviated from Cover-Bounded) on a multicovered space (X, λ) as follows: two players, I and II, step by step choose a cover $u_n \in \lambda$ and a u_n -bounded subset B_n of X , respectively. The player II is declared the winner, if $X = \bigcup_{n \in \omega} B_n$. Otherwise the player I wins. A multicovered space (X, λ) is said to be *winning*, if the second player has a winning strategy in the game CB on (X, λ) . It is clear that that the game OF on a topological group G is equivalent to the game CB on the multicovered space (G, λ_L) in the sense that one of the players has a winning strategy in one of these games if and only if he has a winning strategy in the other one. It also should be mentioned here that the game CB on a multicovered space $(X, \mathcal{O}(X))$ is nothing else but the game $H(X)$ introduced by R. Telgarsky in [29], see also [26] and references there in.

Let X be a Tychonoff space. Recall from [10] that the uniformity \mathcal{U} on X is called *universal*, if it generates the topology of X and contains all uniformities on X with this property. Throughout this paper the universal uniformity of a topological space X will be denoted by $\mathcal{U}(X)$. The reader is referred to the next section for the definition of the product of multicovered spaces. We are in a position now to present the main result of this paper.

Theorem 2. *For a space X the following conditions are equivalent:*

- (1) $(F(X), \lambda_{L \wedge R})$ is winning;
- (2) $F(X)$ is strictly o -bounded;
- (3) $(F(X)^n, \lambda_{L \vee R}^n)$ is winning for all $n \in \omega$;
- (4) $A(X)$ is strictly o -bounded;
- (5) $L_p(X)$ is strictly o -bounded;
- (6) $L(X)$ is strictly o -bounded;
- (7) $L_s(X)^n$ is strictly o -bounded for all $n \in \mathbb{N}$;
- (8) $(X, \lambda_{\mathcal{U}(X)})$ is winning.

The equivalent properties of in the above theorem are not preserved by finite powers. To describe the corresponding space we have to introduce some notions related to multicovered spaces. A multicovered space (X, λ) is called

- *totally-bounded*, if X is u -bounded for every $u \in \lambda$;
- ω -*bounded*, if each cover $u \in \lambda$ has a countable subcover.

These notions generalize the ω -boundedness of uniform spaces introduced by I. Guran in [12] and the total boundedness in sense that a uniform space (X, \mathcal{U}) has one of the above properties if and only if so does the multicovered space $(X, \lambda_{\mathcal{U}})$. For example, $(X, \mathcal{O}(X))$ is totally-bounded (ω -bounded) if and only if X is compact (Lindelöf). Let X be a countably-compact spaces X such that there exists a continuous pseudometric ρ on X^2 such that the space X^2 is not Lindelöf, see Example 26. Then the uniform space $(X, \mathcal{U}(X))$ as well as the multicovered space $(X, \lambda_{\mathcal{U}(X)})$ are totally-bounded, and consequently $(X, \lambda_{\mathcal{U}(X)})$ is winning. But the uniform space $(X^2, \mathcal{U}(X^2))$ obviously fails to be ω -bounded, consequently so does the multicovered space $(X^2, \lambda_{\mathcal{U}(X^2)})$, and hence X^2 does not satisfy the conditions of Theorem 2.

The properties of Menger, Scheepers, and Hurewicz can be also naturally carried out in the realm of multicovered spaces: a multicovered space (X, λ) has the Menger (resp. Scheepers, Hurewicz) property if for every sequence $(u_n)_{n \in \omega} \in \lambda^\omega$ there exists a sequence $(B_n)_{n \in \omega}$ of subsets of X such that each B_n is u_n -bounded and $\{B_n : n \in \omega\}$ is a cover (resp. ω -cover, γ -cover) of X . It is a simple exercise to show that each Menger multicovered space is ω -bounded. A crucial observation here is that a topological group G is o -bounded if and only if the multicovered space $(G, \lambda_R(G))$ is Menger. The o -boundedness of free objects may be also described in terms of properties of the multicovered space $(X, \lambda_{\mathcal{U}(X)})$ as well, which extends Theorem 1.

Theorem 3. *Let X be a Tychonoff space. Then $A(X)$ is o -bounded if and only if the multicovered space $(X, \lambda_{\mathcal{U}(X)})$ is Scheepers.*

The Hurewicz property is selfdual.

Theorem 4. *For a space X the following conditions are equivalent:*

- (1) $(A(X), \lambda(A(X)))$ is Hurewicz;
- (2) $(A(X)^n, \lambda(A(X)^n))$ is Hurewicz for all $n \in \omega$;
- (3) $(L_p(X), \lambda(L_p(X)))$ is Hurewicz;
- (4) $(L(X), \lambda(L(X)))$ is Hurewicz;
- (5) $L_s(X)^n$ is Hurewicz for all $n \in \mathbb{N}$;
- (6) every continuous metrizable image of X is Hurewicz;
- (7) $(A(X), \lambda(A(\nu X)))$ is Hurewicz;
- (8) $(A(X), \lambda(A(\mu X)))$ is Hurewicz.
- (9) $(X, \lambda_{\mathcal{U}(X)})$ is Hurewicz.

For a Lindelöf topological space X the multicovers $\lambda_{\mathcal{U}(X)}$ and $\mathcal{O}(X)$ are equivalent in the sense defined in the next section, see Corollary 15. In combination with Proposition 12 this gives us the following

Proposition 5. *The multicovered space $(X, \mathcal{O}(X))$ is winning (resp. Menger, Scheepers, Hurewicz) if and only if so is $(X, \lambda_{\mathcal{U}(X)})$ and X is Lindelöf.*

Note, that the multicovered space $(X, \mathcal{O}(X))$ has the Menger (resp. Scheepers, Hurewicz) property if and only if so does the topological space X . Concerning the winning property of the multicovered space $(X, \mathcal{O}(X))$, there are many equivalent statements to it. At the beginning of 80-ies R. Telgarsky introduced the game $H(X)$ (implicitly existing in earlier works of W. Hurewicz) on a topological space X , which coincides with the game OF on the multicovered space $(X, \mathcal{O}(X))$, and proved that the second player has a winning strategy in this game if and only if X is C -like, which means that the first player has a winning strategy in the compact-open game on X , see [26] and references there in. In the current terminology the game $H(X)$ is called the Menger game on X , see [24]. This yields the following reformulation of Proposition 5.

Corollary 6. *A Tychonoff space X is C -like (resp. Menger, Scheepers, Hurewicz) if and only if it is Lindelöf and the multicovered space $(X, \lambda_{\mathcal{U}(X)})$ is winning (resp. Menger, Scheepers, Hurewicz).*

Since the Lindelöf property is l -invariant [38] (see also [8], where it is shown that the Lindelöf number is l -invariant), and the properties of the multicovered space $(X, \lambda_{\mathcal{U}(X)})$ considered in the above corollary have counterparts among the properties of $L_p(X)$ obviously preserved by linear homeomorphisms, we get the following

Corollary 7. *The properties of Scheepers, Hurewicz and being C -like are l -invariant (and hence A - and M -invariant). Consequently the Menger property is l -invariant under $\mathbf{u} < \mathbf{g}$ being equivalent to the Scheepers one.*

In view of this it is worth to mention Question II.2.8 of [3] whether the Menger property is t -invariant, which seems to be still unsolved.

As we could see in Theorems 1, 3, and 4, the Scheepers and Hurewicz properties of the multicovered space $(X, \lambda_{\mathcal{U}(X)})$ admit a characterization in terms of metrizable images of X . Therefore one may try to prove some similar characterization of the winning property of $(X, \lambda_{\mathcal{U}(X)})$ of the following kind: $(X, \lambda_{\mathcal{U}(X)})$ is winning if and only if every metrizable image of X has some “strong” property P . But even the property P of being countable, which seems to be the strongest among those one could consider, does not work. Let us recall that a topological space X is a P -space, if every G_δ -subset of X is open. R. Telgarsky in [28] observed that a Lindelöf P -space Y constructed by R. Pol in [21] fails to be C -like (and hence $(X, \lambda_{\mathcal{U}(X)})$ is not winning by Corollary 6). It suffices to note that the size of an arbitrary metrizable image of a P -space X does not exceed the Lindelöf number of X .

Corollary 8. *Pol’s space Y has the following properties:*

- (1) *The groups $F(Y)$, $A(Y)$, $L_p(Y)$, $L(X)$, and $L_s(X)$ are OF -undetermined;*
- (2) *all metrizable images of Y^n , $F(Y)$, $A(Y)$ are countable, where $n \in \mathbb{N}$.*

Proof. The second item obviously follows from the facts that every finite power of a Lindelöf P -space is again Lindelöf P -space, and each metrizable image of a Lindelöf P -space is countable. Concerning the first one, it simply follows from Corollary 10 and the observation that each Lindelöf P -space is Hurewicz. \square

Corollary 8 is closely related to the result of A. Krawczyk and H. Michalewski [17] who used the space Y to construct an OF -undetermined P -group G . Similar ideas are also used in Theorem 3.1 of [15].

The problem of construction of OF -undetermined groups was posed in [30] and solved in [17] and [4] (and, probably, somewhere else) independently. Theorems 2 and 4 supply us with a method of constructing OF -undetermined groups: it suffices to take a topological space which does not satisfy condition (8) of Theorem 2 but still has some strong property guaranteeing the first player having no winning strategy on free objects considered in this paper. Such the properties are given by the subsequent proposition, which easily follows from [16, Theorem 27].

Proposition 9. *Let G be a topological group such that the underlying topological space is Menger. Then the first player has no winning strategy in the game OF on G .*

Finally, we present (nonmetrizable) examples of OF -undetermined groups.

Corollary 10. *Let X be a non- σ -compact metrizable space such that all finite powers of X are Menger (Hurewicz). Then all finite powers of $F(X)$, $A(X)$, $L_p(X)$, $L(X)$, and $L_s(X)$ are OF -undetermined being Menger (Hurewicz) groups which fail to be strictly o -bounded.*

Proof. Let X be a non- σ -compact metrizable space whose all finite powers of X are Hurewicz (Menger), and G be one of the groups $F(X)$, $A(X)$, $L_p(X)$, $L(X)$, $L_s(X)$. Since G is the countable union of continuous images of finite powers of $Y = X \times \mathbb{R}$ (see the proof of Theorem 1, where it is shown that $L_s(X)$ is a continuous image of $A(X \times \mathbb{R})$), so is G^n for all $n \in \mathbb{N}$. Applying Lemma 21 and Corollary 15, we conclude that each finite power of Y is Hurewicz (Menger). Since the Hurewicz (Menger) property is preserved by continuous images and countable unions by Lemma 20 (this was also pointed out in [16]), G^n is Hurewicz (Menger) for all $n \in \omega$. Applying Proposition 9, we conclude that for every $n \in \omega$ the first player has no winning strategy in the game OF on G^n .

As it was shown by R. Telgarsky, every winning (= C -like) metrizable topological space is σ -compact, see [23]. Therefore X fails to be winning, and thus G is not strictly o -bounded by Theorem 2. Consequently G^n is not strictly o -bounded for all $n \in \mathbb{N}$. From the above it follows that G^n is OF -undetermined for all $n \in \mathbb{N}$. \square

Observation 11. *Every topological group with the Hurewicz property is strictly o -bounded provided it is metrizable.*

Proof. Let G be a topological group whose underlying topological space G is Hurewicz and $\{U_n : n \in \omega\}$ be a countable local base at the identity of G . Set $u_n = \{gU_n : g \in G\}$. The Hurewicz property of G yields a sequence $(F_n)_{n \in \omega}$ of finite subsets of G such that $G = \bigcup_{n \in \omega} \bigcap_{k \geq n} F_n U_n$, consequently G is strictly o -bounded as a countable union of its totally-bounded subspaces. \square

Spaces X with such properties as in Corollary 10 were constructed in [7], [9], and [35].

Proofs. In our proofs of results announced in Introduction we shall exploit a number of auxiliary statements about multicovered spaces. As a matter of the fact, all of these results are (consequences of more general ones) proven in [6]. But for the sake of completeness we present their proofs. Their formulations involve some additional notions and notations. For uniform spaces (X_1, \mathcal{U}_1) and (X_2, \mathcal{U}_2) we shall identify the uniformity on their product $X_1 \times X_2$ generated by \mathcal{U}_1 and \mathcal{U}_2 with the product $\mathcal{U}_1 \times \mathcal{U}_2$. Let u and λ be a cover and a multicover of a set X respectively, and $Z \subset X$. Then $u|Z$ denotes the family $\{U \cap Z : U \in u\}$ and $\lambda|Z = \{u|Z : u \in \lambda\}$. Every subset Z of X with the induced multicover $\lambda|Z$ is called

a subspace of the multicovered space (X, λ) . By the product of multicovers λ and ν of sets X and Y we understand the multicover $\eta = \{u \cdot v : u \in \lambda, v \in \nu\}$ of $X \times Y$, where $u \cdot v = \{U \times V : U \in u, V \in v\}$. Again, we identify η with the product $\lambda \times \nu$.

Next, we shall also use the preorder \prec on the family of all covers of a set X , where $u \prec v$ means that each v -bounded subset is u -bounded. In other words, $u \prec v$ if and only if for every finite subset c of v there exists a finite subset d of u with $\cup d \supset \cup c$. Note, that all multicovers λ considered in this paper are *centered*, which means that each finite subset c of λ has an upper bound in λ with respect to \prec . Let us also observe that $u \prec v$ provided v is a refinement of u in the sense that each $V \in v$ lies in some $U \in u$. The preorder \prec on the family of all covers of X generates the following preorder on the family of all multicovers of X , which is also denoted by \prec : $\lambda \prec \nu$ if and only if for every $u \in \lambda$ there exists $v \in \nu$ such that $u \prec v$. We say that multicovers λ and ν of a set X are *equivalent* (and write $\lambda \cong \nu$) if $\lambda \prec \nu$ and $\nu \prec \lambda$. Given any multicovers λ and ν of X and Y respectively, we call a function $f : X \rightarrow Y$

- *uniformly bounded*, if for every $v \in \nu$ there exists $u \in \lambda$ such that for every u -bounded subset A of X its image $f(A)$ is v -bounded;
- *perfect*, if for every $u \in \lambda$ there exists $v \in \nu$ such that for every v -bounded subset B of Y the preimage $f^{-1}(B)$ is u -bounded.

In the subsequent simple statement we collect some straightforward properties of the notions introduced before.

Proposition 12. (1) *Let (X_1, \mathcal{U}_1) and (X_2, \mathcal{U}_2) be uniform spaces. Then the multicovers $\lambda_{\mathcal{U}_1 \times \mathcal{U}_2}$ and $\lambda_{\mathcal{U}_1} \times \lambda_{\mathcal{U}_2}$ of $X_1 \times X_2$ are equivalent.*

(2) *Let G and H be topological groups and $T \in \{L, R, L \vee R, L \wedge R\}$. Then $\lambda_T(G \times H)$ is equivalent to $\lambda_T(G) \times \lambda_T(H)$.*

(3) *Let G be a topological group G . Then $\lambda_L(G) \cong \{\{gU : g \in G\} : e \in U \in \mathcal{O}(G)\}$,
 $\lambda_R(G) \cong \{\{Ug : g \in G\} : e \in U \in \mathcal{O}(G)\}$,
 $\lambda_{L \vee R}(G) \cong \{\{gU \cap Ug : g \in G\} : e \in U \in \mathcal{O}(G)\}$,
 $\lambda_{L \wedge R}(G) \cong \{\{UgU : g \in G\} : e \in U \in \mathcal{O}(G)\}$.*

(3') *For an abelian group G the multicover $\lambda(G)$ is equivalent to $\{\{g + U : g \in G\} : e \in U \in \mathcal{O}(G)\}$.*

(4) *Let λ and ν be multicovers of a set X and $\lambda \prec \nu$. Then (X, λ) is Menger (resp. Scheepers, Hurewicz, winning) provided so is (X, ν) .*

(5) *If multicovers λ and ν of X are equivalent, then (X, λ) is Menger (resp. Scheepers, Hurewicz, winning) if and only if so is (X, ν) .*

(6) *$\lambda \prec \nu$ if and only if the identity map id_X is perfect with respect to λ and ν .*

(7) *If $f : X \rightarrow Y$ is perfect with respect to multicovers λ and ν of X and Y respectively, then (X, λ) is Menger (resp. Scheepers, Hurewicz, winning) provided so is (Y, ν) .*

(8) *If $f : X \rightarrow Y$ is uniformly bounded with respect to multicovers λ and ν , then (Y, ν) is winning (resp. Hurewicz, Scheepers, Menger) provided so is (X, λ) .*

Proof. Because of simplicity of all of the items, we shall only proof the “winning” part of the seventh one. For this purpose we have to consider more formally the notion of a winning strategy in the game CB on a multicovered space (X, λ) . By a strategy of the second

player we understand a map $\Theta: \lambda^{<\omega} \rightarrow \mathcal{P}(X)$ assigning to each finite sequence of covers $(u_0, \dots, u_n) \in \lambda^{<\omega}$ a u_n -bounded subset $\Theta(u_0, \dots, u_n)$ of X , where $\lambda^{<\omega} = \bigcup_{n \in \omega} \lambda^n$. Such a strategy is *winning*, if the family $\{\Theta(u_0, \dots, u_n) : n \in \omega\}$ is a cover of X for any sequence $(u_n)_{n \in \omega}$.

Now, assume that Θ_Y is a winning strategy of the second player in the game CB on a multicovered space (Y, ν) . Construct a map $\phi: \lambda \rightarrow \nu$ such that $f^{-1}(B)$ is u -bounded for every $\phi(u)$ -bounded subset B of Y . It suffices to observe that $\Theta_X: (u_0, \dots, u_n) \mapsto f^{-1}(\Theta_Y(\phi(u_0), \dots, \phi(u_n)))$ is a winning strategy of the second player in the game CB on (X, λ) . \square

We shall exploit the following important result of V. Pestov, see [19] or [31, 2.8].

Proposition 13. *Let X be a Tychonoff space. Then the natural uniformity on $A(X)$ generates the universal uniformity $\mathcal{U}(X)$ on X .*

We shall also use the following straightforward consequence of Lemma 1.0 of [18].

Corollary 14. *If X is a Lindelöf regular space and u is an open cover of X , then there exists a pseudometric d on X such that a subset Y of X is u -bounded provided $\text{diam}_d(Y) < \infty$ (Here, as usual, $\text{diam}_d(Y) = \sup\{d(y_1, y_2) : y_1, y_2 \in Y\}$).*

Corollary 15. *Let X be a Lindelöf regular space. Then the multicovers $\mathcal{O}(X)$ and $\lambda_{\mathcal{U}(X)}$ are equivalent.*

Proof. Since every uniform cover has an open uniform refinement, we conclude that $\lambda_{\mathcal{U}(X)} \prec \mathcal{O}(X)$.

To prove that $\mathcal{O}(X) \prec \lambda_{\mathcal{U}(X)}$, fix an arbitrary open cover u of X and find a pseudometric d on X such as in Lemma 14. Then for the uniform cover $v = \{B_d(x, 1) : x \in X\} \in \lambda_{\mathcal{U}(X)}$ we obviously have $u \prec v$, which finishes our proof. \square

Let A be a subset of the Cartesian product $X \times Y$. From now on we shall use the following notations: $A^{-1} = \{(y, x) \in Y \times X : (x, y) \in A\}$, $A(x) = \{y \in Y : (x, y) \in A\}$, where $x \in X$. Recall from [12] that a uniform space (X, \mathcal{U}) is ω -bounded, if each uniform cover contains a countable subcover. In particular, any topological group G is ω -bounded, if so is the uniform space (G, \mathcal{U}) , where \mathcal{U} is the left uniformity of G .

Lemma 16. *Let X be a space such that $(X, \mathcal{U}(X))$ is ω -bounded and $G \supset X$ be an abelian topological group such that every continuous map $\phi: X \rightarrow \mathbb{R}$ can be extended to a continuous homomorphism $\tilde{\phi}: G \rightarrow \mathbb{R}$. Then the maps $\psi_n: X^n \rightarrow G$, where $\psi_n: (x_1, x_2, \dots, x_n) \mapsto x_1 + x_2 + \dots + x_n$, are perfect with respect to $\lambda_{\mathcal{U}(X)^n}$ and $\lambda(G)$ for all $n \in \mathbb{N}$.*

Proof. Given any $u_0 \cdot \dots \cdot u_{n-1} \in \lambda_{\mathcal{U}(X)^n}$, find $U \in \mathcal{U}(X)$ such that the uniform cover $u = \{U(x) : x \in X\}$ is an upper bound of the family $\{u_i : i < n\}$ with respect to \prec . Let ρ be a pseudometric on X such that $\{(x, y) \in X^2 : \rho(x, y) < 1\} \subset U$. Since $(X, \mathcal{U}(X))$ is ω -bounded, the space (X, ρ) is Lindelöf. Applying Lemma 14 to the regular Lindelöf space (X, ρ) and the cover $w = \{B_\rho(x, 1) : x \in X\}$, we can find a continuous pseudometric d on X such that each $Y \subset X$ is w -bounded provided $\text{diam}_d(Y) < \infty$. Fix arbitrary $x_0 \in X$ and define a map $f: X \rightarrow \mathbb{R}$ letting $f(x) = d(x, x_0)$. From the above it follows that $f^{-1}(-r, r)$ is w -bounded, and hence u -bounded for every $r \in \mathbb{R}$. Let $\hat{f}: G \rightarrow \mathbb{R}$ be a continuous homomorphism extending f and O be an open neighborhood of the identity of G such that $\hat{f}(O) \subset (-1, 1)$.

Let us fix arbitrary finite subset K of G . Our proof will be completed as soon as we shall show that $B = \psi_n^{-1}(O + K)$ is w^n -bounded. By our choice of O there exists $r > 0$ such that $\hat{f}(O + K) \subset (-r, r)$. Therefore, $B \subset \psi_n^{-1}(\hat{f}^{-1}(-r, r))$. Let us note, that $\hat{f} \circ \psi_n(x_1, \dots, x_n) = f(x_1) + \dots + f(x_n)$, consequently $0 \leq f(x_i) < r$ for every $(x_1, \dots, x_n) \in B$ and $i \leq n$, and finally B is w^n -bounded being a subset of $(f^{-1}(-r, r))^n$. \square

Next, we shall deal with preservation of selection principles by operations of finite products and countable unions.

Lemma 17. *A multicovered space (X, λ) is Scheepers if and only if (X^n, λ^n) is Menger for all $n \in \omega$. Consequently the class of Scheepers multicovered spaces is closed under taking finite powers of its elements.*

Proof. Suppose that (X^n, λ^n) is Menger for every $n \in \omega$. To see that (X, λ) is Scheepers, fix any sequence $(u_n)_{n \in \omega} \in \lambda^\omega$. For every $n \in \omega$ we can apply the Menger property of X^n to find a cover $\{B_{n,k}^n : k \geq n\}$ of X^n by powers of u_k -bounded sets $B_{n,k} \subset X$.

For every $k \in \omega$ let $B_k = \bigcup_{n \leq k} B_{n,k}$. We claim that $\{B_n : n \in \omega\}$ is an ω -cover of X . Indeed, fix any finite subset $F = \{x_1, \dots, x_n\}$ of X . Since the family $\{B_{n,k}^n : k \geq n\}$ covers X^n , $(x_1, \dots, x_n) \in B_{n,k}^n$ for some $k \geq n$. Consequently, $F \subset B_{n,k} \subset B_k$, which completes the proof of the “if” part.

To prove the “only if” part, suppose that (X, λ) is Scheepers. To show that the powers of X are Menger, fix some $n \in \omega$ and a sequence of covers $(w_k)_{k \in \omega}$ in λ^n . For every $k \in \omega$ we can write w_k in the form $w_k = u_{k1} \cdot \dots \cdot u_{kn}$, where $u_{ki} \in \lambda$, $i \in \{1, \dots, n\}$. Since (X, λ) is centered, for every $k \in \omega$ we can find $u_k \in \lambda$ such that $u_k \succ u_{ki}$ for all $i \in \{1, \dots, n\}$. Using the Scheepers property of X , find an ω -cover $\{B_k : k \in \omega\}$ by a u_k -bounded subsets $B_k \subset X$. We claim that $X^n = \bigcup_{k \in \omega} B_k^n$, which clearly implies the mengersness of (X^n, λ^n) . Indeed, fix any $x = (x_1, \dots, x_n) \in X^n$ and find $k \in \omega$ such that $\{x_1, \dots, x_n\} \subset B_k$. Then $x \in B_k^n$. \square

We need the following auxiliary notion: a family $\{A_n : n \in \omega\}$ is called a *proper ω -cover* of a set X , if for every finite subset K of X the set $\{n \in \omega : K \subset A_n\}$ is infinite.

Lemma 18. *Let (X, λ) be a Scheepers multicovered space. Then for each sequence $(u_n)_{n \in \omega} \in \lambda^\omega$ there exists a proper ω -cover $\{B_n : n \in \omega\}$ of X such that B_n is u_n -bounded for all $n \in \omega$.*

Proof. Let $(u_n)_{n \in \omega} \in \lambda^\omega$ be a sequence of covers of X . Using the Scheepers property of (X, λ) , for every $k \in \omega$ we can find a sequence $(A_{k,n})_{n \geq k}$ of u_n -bounded subsets $A_{k,n} \subset X$ such that the family $\{A_{k,n} : n \geq k\}$ is an ω -cover of X . For every $n \in \omega$ consider the u_n -bounded subset $B_n = \bigcup_{k \leq n} A_{k,n}$ of X and note that $\{B_n : n \in \omega\}$ is a proper ω -cover of X , which finishes our proof. \square

Lemma 19. *The product $(X \times Y, \lambda_X \cdot \lambda_Y)$ of Hurewicz multicovered spaces (X, λ_X) and (Y, λ_Y) is Hurewicz. Consequently the class of Hurewicz multicovered spaces is closed under taking finite products of its elements.*

Proof. Let us fix a sequence $(w_n)_{n \in \omega} \in (\lambda_X \cdot \lambda_Y)^\omega$. For every $n \in \omega$ find $u_n \in \lambda_X$ and $v_n \in \lambda_Y$ such that $w_n = u_n \cdot v_n$. By the definition of the Hurewicz property, there are sequences $(A_n)_{n \in \omega}$ and $(B_n)_{n \in \omega}$ of subsets of X and Y respectively such that each A_n (B_n) is u_n - (v_n -) bounded, and the families $\{A_n : n \in \omega\}$ and $\{B_n : n \in \omega\}$ are γ -covers of Y . For every $n \in \omega$ put $C_n = A_n \times B_n$. It is a simple matter to verify that the family $\{C_n : n \in \omega\}$ is a γ -cover of $X \times Y$ and each C_n is w_n -bounded, which finishes our proof. \square

Lemma 20. *Let A_n , $n \in \omega$, be subspaces of a multicovered space (X, λ) . If every subspace A_n , $n \in \omega$, is winning (resp. Menger, Hurewicz), then so is their union $A = \bigcup_{n \in \omega} A_n$.*

Proof. 1. Assume that all the subspaces A_n , $n \in \omega$, are winning. For every $n \in \omega$ fix a winning strategy $\Theta_n: \lambda^{<\omega} \rightarrow \mathcal{P}(X)$ of the second player in the game CB on A_n . Define a strategy $\Theta: \lambda^{<\omega} \rightarrow \mathcal{P}(X)$ of the second player in the game CB on $A = \bigcup_{n \in \omega} A_n$ letting $\Theta(u_0, \dots, u_n) = \bigcup_{k \leq n} \Theta_k(u_k, \dots, u_n)$ for $(u_0, \dots, u_n) \in \lambda^{<\omega}$. The u_n -boundedness of the sets $\Theta_k(u_k, \dots, u_n)$, $k \leq n$, implies the u_n -boundedness of their union $\Theta(u_0, \dots, u_n)$.

We claim that $A \subset \bigcup_{n \in \omega} \Theta(u_0, \dots, u_n)$ for any infinite sequence $(u_n)_{n \in \omega} \in \lambda^\omega$. Fix any $k \in \omega$. Regarding the sequence $(u_n)_{n \geq k}$ as the moves of the first player in the Menger game on A_k , we see that $A_k \subset \bigcup_{n \geq k} \Theta_k(u_k, \dots, u_n)$ (according to the choice of Θ_k as a winning strategy). Then

$$A = \bigcup_{k \in \omega} A_k \subset \bigcup_{k \in \omega} \bigcup_{n \geq k} \Theta_k(u_k, \dots, u_n) = \bigcup_{n \in \omega} \bigcup_{k \leq n} \Theta_k(u_k, \dots, u_n) = \bigcup_{n \in \omega} \Theta(u_0, \dots, u_n)$$

and hence Θ is a winning strategy of the second player in the Menger game on $A = \bigcup_{n \in \omega} A_n$.

2. Next, assume that all the subspaces A_n , $n \in \omega$, are Menger (Hurewicz). To show that the union $A = \bigcup_{n \in \omega} A_n$ is Menger (Hurewicz), fix an infinite sequence of covers $(u_n)_{n \in \omega} \in \lambda^\omega$. By the Menger (Hurewicz) property of A_n , $n \in \omega$, for every $k \in \omega$ there is a (γ) -cover $\{B_n^k : n \geq k\}$ of A_k such that each set B_n^k , $n \geq k$, is u_n -bounded. Letting $B_n = \bigcup_{k \leq n} B_n^k$, we see that each set B_n , $n \in \omega$, is u_n -bounded and $\{B_n : n \in \omega\}$ is a (γ) -cover of A . This proves that the union $A = \bigcup_{n \in \omega} A_n$ is Menger (Hurewicz). \square

Concerning the Scheepers property, the situation with unions is much more delicate. As it is shown in [6], the class of Scheepers multicovered spaces is closed under finite unions if and only if two arbitrary ultrafilters are coherent, i.e. the NCF principle holds, see [37] for corresponding definitions.

Lemma 21. *Let X be a topological space such that the multicovered space $(X, \lambda_{\mathcal{U}(X)})$ is winning (resp. Hurewicz, Scheepers, Menger). Then so is the product $(X \times Y, \lambda_{\mathcal{U}(X \times Y)})$ for every σ -compact space Y .*

Proof. Given arbitrary σ -compact space Y , write it as a union $\bigcup \{K_n : n \in \omega\}$ of a countable family of its compact subspaces. Without loss of generality, $K_n \subset K_{n+1}$ for all $n \in \omega$. Let us denote by h_n the restriction to $X \times K_n$ of the projection $\text{pr}_X: X \times Y \rightarrow X$. We claim that h_n is perfect with respect to multicovers $\lambda_{\mathcal{U}(X \times Y)}|(X \times K_n)$ and $\lambda_{\mathcal{U}(X)}$ respectively. Indeed, let $u \in \lambda_{\mathcal{U}(X \times Y)}$ and d be a pseudometric on $X \times Y$ such that $w = \{B_d(z, 1) : z \in X \times Y\}$ is inscribed into u . For every $n \in \omega$ define a function $d_n: X^2 \rightarrow \mathbb{R}$ letting $d_n(x_1, x_2) = \sup\{d((x_1, y), (x_2, y)) : y \in K_n\}$ and observe that d_n is a continuous pseudometric on X . Let us fix arbitrary $x \in X$. The perfectness of h_n follows from w -boundedness of $h_n^{-1}(B_{d_n}(x, 1/3))$, which can be proven by a standard argument involving compactness of K_n and the definition of d_n .

Applying Proposition 12(7), we conclude that $(X \times K_n, \lambda_{\mathcal{U}(X \times Y)}|(X \times K_n))$ is winning (resp. Hurewicz, Scheepers, Menger) for all $n \in \omega$. Thus Lemma 20 completes our proof in winning, Hurewicz, and Menger cases. For the Scheepers property we need some auxiliary arguments. Assuming that $(X, \lambda_{\mathcal{U}(X)})$ is Scheepers, fix a sequence $(u_n)_{n \in \omega} \in \lambda_{\mathcal{U}(X \times Y)}^\omega$. For every $n \in \omega$ find $v_n \in \lambda_{\mathcal{U}(X)}$ such that $h_n^{-1}(B)$ is u_n -bounded for every v_n -bounded subset B of X . Then Lemma 18 yields a proper ω -cover $\{B_n : n \in \omega\}$ of X such that each B_n

is v_n -bounded. It suffices to show that $\{h_n^{-1}(B_n) : n \in \omega\}$ is an ω -cover of $X \times Y$. For this purpose fix a finite subset $C = \{(x_i, y_i) : i \leq m\}$ of $X \times Y$ and find $n \in \omega$ such that $\{x_i : i \leq m\} \subset B_n$ and $\{y_i : i \leq m\} \subset K_n$. Then $C \subset B_n \times K_n = h_n^{-1}(B_n)$, which means that $\{h_n^{-1}(B_n) : n \in \omega\}$ is an ω -cover of $X \times Y$ and thus finishes our proof. \square

Remark 1. It is well-known that under additional set-theoretic assumptions there exists a Hurewicz subspace S of \mathbb{R} such that S^2 is not Menger, see [25, Theorem 43]. But this does not contradict Lemmas 17 and 19. In order to explain this, let us consider Tychonoff spaces X and Y . Then the topological space $X \times Y$ is Menger if and only if so is the multicovered space $(X \times Y, \mathcal{O}(X \times Y))$, while the product $(X, \mathcal{O}(X)) \times (Y, \mathcal{O}(Y))$ is Menger if and only if so is the multicovered space $(X \times Y, \mathcal{O}(X) \times \mathcal{O}(Y))$. It is easy to see, that $\mathcal{O}(X) \times \mathcal{O}(Y) \subset \mathcal{O}(X \times Y)$, and these multicovers coincide if and only if $|X| = 1$ or $|Y| = 1$, and consequently $(X \times Y, \mathcal{O}(X \times Y))$ and $(X \times Y, \mathcal{O}(X) \times \mathcal{O}(Y))$ are different multicovered spaces. But in light of Proposition 12(5) it is more interesting to find out when the multicovers $\mathcal{O}(X) \times \mathcal{O}(Y)$ and $\mathcal{O}(X \times Y)$ are isomorphic. A direct verification shows that this is so when both of them are locally-compact or Lindelöf P -spaces, while from the above mentioned Theorem 43 of [25], Proposition 12(5), and Lemma 19 we conclude that $\mathcal{O}(S)^2$ is not equivalent to $\mathcal{O}(S^2)$. \square

Lemma 22. *Let (X, λ) be a winning multicovered space. Then there exists a winning strategy Θ_1 of the second player in the game CB such that $\{\Theta_1(u_0, \dots, u_n) : n \in \omega\}$ is a γ -cover of X for all sequences $(u_n)_{n \in \omega} \in \lambda^\omega$.*

Proof. Let us fix some winning strategy Θ of the second player in the game CB on (X, λ) .

We claim that the map $\Theta_1 : \lambda^{<\omega} \rightarrow \mathcal{P}(X)$,

$$\Theta_1 : (u_0, u_1, \dots, u_n) \mapsto \bigcup_{0 \leq i_0 \leq i_1 \leq \dots \leq i_k = n} \Theta(u_{i_0}, u_{i_1}, \dots, u_{i_k})$$

is a winning strategy of the second player in the game CB on (X, λ) with the required property.

Suppose, to the contrary, that there exists a sequence of covers $(u_n)_{n \in \omega} \in \lambda^\omega$, a subsequence $(i_k)_{k \in \omega} \in \omega^\omega$, and $x \in X$ such that $x \notin \bigcup_{k \in \omega} \Theta_1(u_0, u_1, \dots, u_{i_k})$. But Θ is a winning strategy in the Menger game on (X, λ) , which together with definition of Θ_1 gives us $X = \bigcup_{k \in \omega} \Theta(u_{i_0}, u_{i_1}, \dots, u_{i_k}) \subset \bigcup_{k \in \omega} \Theta_1(u_0, u_1, \dots, u_{i_k})$, a contradiction. \square

Corollary 23. *The class of winning multicovered spaces is closed under finite products of its elements.*

Proof. Let (X, λ) and (Y, ν) be two winning multicovered spaces and Θ_X and Θ_Y be winning strategies of the second player in the game CB on (X, λ) and (Y, ν) respectively having the property from Lemma 22. A direct verification shows that the strategy

$$\Theta : (u_0 \cdot v_0, \dots, u_n \cdot v_n) \mapsto \Theta_X(u_0, \dots, u_n) \times \Theta_Y(v_0, \dots, v_n)$$

is winning in the game CB on the product $(X \times Y, \lambda_X \times \lambda_Y)$. \square

The next corollary answers [15, Problem 1] in negative.

Corollary 24. *The product of finitely many strictly o -bounded topological groups is strictly o -bounded.*

Proof. Follows from the observation that a topological group G is strictly o -bounded if and only if the multicovered space $(G, \lambda_L(G))$ is winning, see Corollary 23, and Proposition 12(2,5). \square

The following lemma is the central part of the paper.

Lemma 25. *Let G be a topological group and $X \subset G$ be a set of its generators. If the multicovered space $(X \cup X^{-1}, \lambda_R(G)|X \cup X^{-1})$ is winning, then so is $(G, \lambda_R(G))$.*

If, additionally, G is abelian and $(X, \lambda(G)|X)$ is Hurewicz (Scheepers), then so is $(G, \lambda(G))$.

Proof. 1. We start by proving the “winning” part. Assuming that $(X \cup X^{-1}, \lambda_R(G)|X \cup X^{-1})$ is winning, find a strategy $\Theta : \lambda_R^{<\omega} \rightarrow \mathcal{P}(G)$ such that $\{\Theta(u_0, \dots, u_k) : k \in \omega\}$ is a γ -cover of $X \cup X^{-1}$ for every sequence $(u_n)_{n \in \omega} \in \lambda_R^\omega$. Let \mathcal{B} be the family of all open neighborhoods of the identity of G .

Next, for every $s \in \lambda_R^{<\omega}$ we shall construct a sequence $w(s) = (w(s)_n)_{n \in \omega} \in \lambda_R^\omega$. Let $s = (u_0, \dots, u_m)$, $U \in \mathcal{B}$ be such that $u_m \prec \{Uz : z \in G\}$, and $U_0 \in \mathcal{B}$ be such that $U \supset U_0^2$. Put $w(s)_0 = \{U_0z : z \in G\}$ and $A_0(s) = \Theta(s \hat{w}(s)_0)$. Assume that for some $n \in \omega$ and for all $k \leq n$ we have already constructed $w(s)_k = \{U_kz : z \in G\} \in \lambda_R$ and $A_k(s) \subset G$ such that the following conditions are satisfied:

- (i) $A_k(s) = \Theta(s \hat{w}(s)_0 \hat{\cdots} \hat{w}(s)_k)$;
- (ii) $U_k \supset U_l^2$ for all $k < l \leq n$;
- (iii) $A_k(s)B$ is $w(s)_{k-1}$ -bounded for every $w(s)_l$ -bounded subset B of G , where $k < l \leq n$ and $w(s)_{-1} = u_m$.

Since $A_n(s)$ is $\{U_nz : z \in G\}$ -bounded, there exists a finite subset K of G such that $A_n(s) \subset U_nK$. Let us find $U_{n+1} \in \mathcal{B}$ such that $zU_{n+1}z^{-1} \subset U_n$ for all $z \in K$ and $U_{n+1}^2 \subset U_n$, and set $w(s)_{n+1} = \{U_{n+1}z : z \in G\}$. Given any $k < n + 1$ and an $w(s)_{n+1}$ -bounded subset B of G , consider the product $C = A_k(s)B$. If $k < n$, then the $w(s)_{k-1}$ -boundedness of C follows from (iii) and the equation $w(s)_n \prec w(s)_{n+1}$. Thus, it suffices to consider the case $k = n$. Let L be a finite subset of G such that $B \subset U_{n+1}L$. Then

$$C = A_n(s)B \subset U_nKU_{n+1}L \subset U_nU_nKL \subset U_n^2KL \subset U_{n-1}KL,$$

which yields the $w(s)_{n-1}$ -boundedness of C , and thus completes our inductive construction of the sequence $(w(s)_n)_{n \in \omega}$ satisfying (i) – (iii) for all $n \in \omega$. Observe, that condition (iii) implies that the product $A_0(s)A_2(s) \cdots A_{2n}(s)$ is $u_m = w(s)_{-1}$ -bounded for all $n \in \omega$.

Given any $s = (u_0, \dots, u_{n-1}) \in \lambda_R^{<\omega}$, construct a finite sequence $(q_0(s), \dots, q_{2n-2}(s)) \in (\lambda_R^{<\omega})^{<\omega}$ as follows:

$$q_0(s) = (u_0), \quad q_{2k+1}(s) = q_{2k}(s) \hat{w}(q_{2k}(s)) | (2k+1), \quad q_{2k+2}(s) = q_{2k+1}(s) \hat{u}_{k+1}.$$

Let $\Theta_1(s) = A_0(q_{2n-2}(s))A_2(q_{2n-2}(s)) \cdots A_{2n-2}(q_{2n-2}(s))$. We claim that Θ_1 is a winning strategy of the second player in the game CB on (G, λ_R) . Indeed, from the above it follows that $\Theta_1(s)$ is $w_{-1}(q_{2n-2}(s)) = u_{n-1}$ -bounded, which implies that Θ_1 is a strategy of the second player. To show that it is winning, consider arbitrary $z = x_0x_1 \cdots x_m \in G$, where $x_i \in X \cup X^{-1}$ for all $i \leq m$. Let $t = (u_n)_{n \in \omega} \in \lambda_R^\omega$ be a sequence of covers of G . Our proof will be completed as soon as we show that there exists $n \in \omega$ such that $\Theta_1(t|n) \ni z$. For this aim consider the sequence $(v_k)_{k \in \omega} \in \lambda_R^\omega$ such that for every $n \in \omega$ there exists $k_n \in \omega$ such

that $q_{2n-2}(t|n) = (v_0, \dots, v_{k_{n-1}})$ (the definition of $q_-(-)$ easily yields such a sequence, and $k_n = k_{n-1} + (2n - 2) + 1$). From the above it follows that

$$\begin{aligned}\Theta_1(t|n) &= A_0(q_{2n-2}(t|n))A_2(q_{2n-2}(t|n)) \cdots A_{2n-2}(q_{2n-2}(t|n)) = \\ &= \Theta(v_0, \dots, v_{k_{n-1}})\Theta(v_0, \dots, v_{k_n}, v_{k_{n-1}+1}, v_{k_{n-1}+2}) \cdots \Theta(v_0, \dots, v_{k_{n-1}}, \dots, v_{k_{n-1}+2n-2}).\end{aligned}$$

By our choice of Θ , the family $\{\Theta(v_0, \dots, v_k) : k \in \omega\}$ is a γ -cover of $X \cup X^{-1}$, consequently there exists $l \in \omega$ such that $\{x_0, \dots, x_m\} \subset \Theta(v_0, \dots, v_k)$ for all $k \geq l$. Let $n > m$ be such that $k_{n-1} > l$. Then $\{x_0, \dots, x_m\} \subset \Theta(v_0, \dots, v_{k_{n-1}}, \dots, v_{k_{n-1}+2i})$ for all $i \in \{0, 2, \dots, 2n-2\}$, which implies $z \in \Theta_1(t|n)$.

2. Let us assume that the multicovered space $(X, \lambda(G)|X)$ is Scheepers and set $\lambda = \lambda(G)$. Given a sequence $(u_n)_{n \in \omega} \in \lambda^\omega$, find a sequence $(O_n)_{n \in \omega}$ of open neighborhoods of the neutral element e such that $u_n \prec \{g + O_n : g \in G\}$, $-O_n = O_n$, and $2O_{n+1} \subset O_n$ for all $n \in \omega$. By the definition of the Scheepers property applied to $(X, \lambda|X)$ there exists a sequence $(K_n)_{n \in \omega}$ of finite subsets of G such that the family $v = \{K_n + O_n : n \in \omega\}$ is a proper ω -cover of X . Without loss of generality, $K_n = -K_n$ and $K_n + K_n \subset K_{n+1}$ for all $n \in \omega$. We claim that $v_1 = \{K_{2n} + O_n : n \in \omega\}$ is an ω -cover of G .

Indeed, from the above it follows that $K + O_n \supset X$ for every $n \in \omega$, where $K = \bigcup_{n \in \omega} K_n$. Consequently for every $x \in X$ we can define a nondecreasing number sequence $z(x)$ letting $z(x)_n = \min\{m \in \omega : x \in K_m + O_n\}$. Since v is a proper ω -cover of X , for every finite subset S of X the set $I_S = \{n \in \omega : z(x)_n \leq n \text{ for all } x \in S\}$ is infinite. Now, consider an arbitrary finite subset A of G and find some finite subset S of X and $m \in \mathbb{N}$ such that $A \subset m(S - S)$. Let us fix arbitrary $l \in I_S \cap [3m, +\infty)$. Then

$$S - S \subset K_l + O_l - K_l - O_l \subset K_{l+1} + O_{l-1}, \quad 2(S - S) \subset 2(K_{l+1} + O_{l-1}) \subset K_{l+2} + O_{l-2}$$

and so on. Proceeding in this fashion, we obtain

$$A \subset m(S - S) \subset K_{l+m} + O_{l-m}.$$

Since $l \geq 3m$, there exists $n \in \omega$ such that $n \leq l - m$ and $2n \geq l + m$, which yields $O_n \supset O_{l-m}$ and $K_{2n} \supset K_{l+m}$. From the above it follows that $K_{l+m} + O_{l-m} \subset K_{2n} + O_n$, which proves that the multicovered space (G, λ) is Scheepers.

3. The proof of the ‘‘Hurewicz’’ part is similar to that of the ‘‘Scheepers’’ one. Let λ , $(u_n)_{n \in \omega} \in \lambda^\omega$, and $(O_n)_{n \in \omega}$ be such as in the previous item. Since the multicovered space $(X, \lambda|X)$ is Hurewicz, there exists a sequence $(K_n)_{n \in \omega}$ of finite subsets of G such that the family $v = \{K_n + O_n : n \in \omega\}$ is a γ -cover of X . Without loss of generality, $K_n = -K_n$ and $K_n + K_n \subset K_{n+1}$ for all $n \in \omega$. We claim that $v_1 = \{K_{2n} + O_n : n \in \omega\}$ is a γ -cover of G . Let K and $z(x) \in \omega^{\uparrow \omega}$ be such as in the second item. Since v is a γ -cover of X , for every finite subset S of X the set $I_S = \{n \in \omega : z(x)_n \leq n \text{ for all } x \in S\}$ is cofinite, i.e. the complement $\omega \setminus I_S$ is finite. Now, consider arbitrary $z \in G$ and find some finite subset S of X and $m \in \mathbb{N}$ such that $z \in m(S - S)$. Let us fix some $l \geq 3m$ such that $[l, +\infty) \subset I_S$. Then for every $p \geq l$ we have

$$S - S \subset K_p + O_p - K_p - O_p \subset K_{p+1} + O_{p-1}, \quad 2(S - S) \subset 2(K_{p+1} + O_{p-1}) \subset K_{p+2} + O_{p-2}$$

and so on. Proceeding in this fashion, we obtain $z \in m(S - S) \subset K_{p+m} + O_{p-m}$. Since $p \geq l \geq 3m$, $p + m \leq 2(p - m)$, and consequently $K_{p+m} \subset K_{2(p-m)}$, which yields $z \in K_{2(p-m)} + O_{p-m}$. Since $p \geq l$ was chosen arbitrary, we conclude that $z \in O_n + K_{2n}$ for all $n \geq l - m$, which means that v_1 is a γ -cover of G . \square

Remark 2. The winning property of any abelian topological group H containing X as a set of its generators can be derived from the winning property of $(X, \lambda(H)|X)$ much easier than in the general case considered in Lemma 25. Given any finite sequence $j = (j_0, \dots, j_{n-1}) \in \{-1, 1\}^{<\omega}$, define a map $\psi_j: X^n \rightarrow G$ letting $\psi_j(x_0, \dots, x_{n-1}) = j_0x_0 + \dots + j_{n-1}x_{n-1}$. A direct verification shows that ψ_j is uniformly-bounded with respect to multicovers $(\lambda(H)|X)^n$ and $\lambda(H)$ (here commutativity is essentially used), and hence $(\psi_j(X^n), \lambda(H)|\psi_j(X^n))$ is winning for each $j \in \{-1, 1\}^{<\omega}$. Now it suffices to use Corollary 23, Proposition 12(8), and Lemma 20.

The same arguments work for the Hurewicz property. In case of the Scheepers property we additionally have to prove that the countable union of uniformly-bounded images of finite powers of a Scheepers space $(X, \lambda(H)|X)$ is Scheepers (the union of Scheepers multicovered spaces could be not Scheepers, see the discussion following Lemma 20). \square

Proofs of Theorems 1 and 3. We shall prove these theorems by showing that conditions (1)-(8) of Theorem 1 are equivalent to the Scheepers property of $(X, \lambda_{\mathcal{U}(X)})$ (note that Theorem 3 states that (1) is equivalent to the Scheepers property of $(X, \lambda_{\mathcal{U}(X)})$), and the last condition will be denoted by (9). The implication (2) \Rightarrow (1) is obvious. The implications (5) \Rightarrow (4) and (4) \Rightarrow (3) follow from the continuity of linear maps $\varphi: L_s(X) \rightarrow L(X)$ and $\psi: L(X) \rightarrow L_p(X)$ extending the identity map id_X , and the simple fact that the \mathcal{o} -boundedness is preserved by continuous homomorphic images, see, e.g., [30].

In addition, we shall prove the subsequent implications: (1) \Rightarrow (9), (9) \Rightarrow (2), (9) \Rightarrow (5), (3) \Rightarrow (9), (6) \Leftrightarrow (9), (7) \Leftrightarrow (1), (8) \Leftrightarrow (1).

(1) \Rightarrow (9). Since $A(X)$ is \mathcal{o} -bounded, it is ω -bounded, and thus the uniform space $(X, \mathcal{U}(X))$ as well as the multicovered space $(X, \lambda_{\mathcal{U}(X)})$ are ω -bounded by Proposition 13. Therefore X and $G = A(X)$ satisfy the conditions of Lemma 16, and consequently for every $n \in \mathbb{N}$ the map ψ_n defined there is perfect with respect to $\lambda_{\mathcal{U}(X)}^n$ and $\lambda(A(X))$. As it was stressed in Introduction, the \mathcal{o} -boundedness of the group $A(X)$ is equivalent to the Menger property of the multicovered space $(A(X), \lambda(A(X)))$. Applying Proposition 12(7), we conclude that $(X^n, \lambda_{\mathcal{U}(X)}^n)$ is Menger for all $n \in \mathbb{N}$, and consequently $(X, \lambda_{\mathcal{U}(X)})$ is Scheepers by Lemma 17.

(9) \Rightarrow (2). Assume that $(X, \lambda_{\mathcal{U}(X)})$ is Scheepers. Then so is $(X, \lambda(A(X))|X)$. Applying Lemma 25, we conclude that $(A(X), \lambda(A(X)))$ is Scheepers too, and thus $(A(X)^n, \lambda(A(X))^n)$ is Menger for all $n \in \mathbb{N}$ by Lemma 17, which means that $A(X)^n$ is \mathcal{o} -bounded for all $n \in \mathbb{N}$.

(9) \Rightarrow (5). Let us note that we have already proven the equivalence of items (1), (2), and (9). Let X be a topological space satisfying (9). Then $(X, \lambda_{\mathcal{U}(X)})$ is Scheepers, and hence so is the multicovered space $(X \times \mathbb{R}, \lambda_{\mathcal{U}(X \times \mathbb{R})})$ by Lemma 21. Consequently $A(X \times \mathbb{R})^n$ is \mathcal{o} -bounded for all $n \in \mathbb{N}$. Consider the map $h: X \times \mathbb{R} \rightarrow L_s(X)$, $h(x, r) = rx$. Since $L_s(X)$ is a linear topological space, h is continuous, and hence it admits a continuous extension to a homomorphism $\tilde{h}: A(X \times \mathbb{R}) \rightarrow L_s(X)$. A direct verification shows that \tilde{h} is surjective. Therefore $L_s(X)$ is a continuous homomorphic image of $A(X \times \mathbb{R})$, and consequently $L_s(X)^n$ is a continuous homomorphic image of $A(X \times \mathbb{R})^n$ for all $n \in \mathbb{N}$.

(3) \Rightarrow (9). It suffices to use the fact that X and $G = L_p(X)$ satisfy the conditions of Lemma 16, see [3, Chapter 0], and apply the same argument as in the proof of the implication (1) \Rightarrow (9).

(9) \Leftrightarrow (6). Assuming that $(X, \lambda_{\mathcal{U}(X)})$ is Scheepers, fix a continuous surjective function $f: X \rightarrow Y$ onto a metrizable space Y . Then f is uniformly bounded with respect to multicovers $\lambda_{\mathcal{U}(X)}$ and $\lambda_{\mathcal{U}(Y)}$ being uniformly continuous with respect to uniformities $\mathcal{U}(X)$ and

$\mathcal{U}(Y)$. Therefore $(Y, \lambda_{\mathcal{U}(Y)})$ is Scheepers by Proposition 12(8). In particular, this implies that Y is Lindelöf and hence $\lambda_{\mathcal{U}(Y)}$ is equivalent to $\mathcal{O}(Y)$ by Corollary 15. Then the multicovered space $(Y, \mathcal{O}(Y))$ is Scheepers by Proposition 12(5).

Next, assume that $(X, \lambda_{\mathcal{U}(X)})$ is not Scheepers and find a sequence $(u_n)_{n \in \omega} \in \lambda_{\mathcal{U}(X)}^\omega$ witnessing for this. For every n find an entourage $U_n \in \mathcal{U}(X)$ such that $w_n = \{U_n(x) : x \in X\}$ is inscribed into u_n . Let d be a continuous pseudometric on X such that $v_n = \{B_d(x, 2^{-n}) : x \in X\}$ is inscribed into w_n . Then the identity map id_X is perfect with respect to multicovers $\lambda_1 = \{u_n : n \in \omega\}$ and $\lambda_2 = \{v_n : n \in \omega\}$ of X . Since (X, λ_1) is not Scheepers, so is (X, λ_2) by Proposition 12(7). Consequently X endowed with the topology generated by d is not Scheepers, and hence there are non-Scheepers metrizable images of X .

(7) \Leftrightarrow (1). Let us note that we have already proven the equivalence (1) \Leftrightarrow (6). It is well known that every Lindelöf space is Hewitt-complete and every continuous map $f: X \rightarrow Y$ from a space X into a Hewitt-complete space Y extends to a continuous map $\hat{f}: \nu X \rightarrow Y$, see [10, Th. 3.11.12, 3.11.16]. Let X be such that $A(X)$ is o -bounded and Y be a continuous metrizable image of νX under a map f . Then Y is Lindelöf containing a dense Lindelöf (even Scheepers) subspace $Z = f(X)$. Therefore Y as well as Z are Hewitt-complete, and hence the map $f|_X$ extends to a continuous map $g: \nu X \rightarrow Z$. Since f and g coincide on the dense subset X of νX , we get $f = g$, and hence $Y = Z = f(X)$. Thus we have already proven that each continuous metrizable image of νX is Scheepers, which implies the o -boundedness of $A(\nu X)$.

Now, assume that $A(\nu X)$ is o -bounded. It follows that each metrizable image of νX is Scheepers. The same argument as in the previous paragraph gives that each metrizable image of X is Scheepers as well, and hence $A(X)$ is o -bounded.

(8) \Leftrightarrow (1). It is well-known [10, 8.5.8(b)] that there are natural embeddings of νX and μX into the Stone-Čech compactification βX such that $X \subset \mu X \subset \nu X \subset \beta X$. This permits us to apply the same argument as in the proof of the equivalence of (1) and (7) and conclude that X and μX have the same continuous metrizable images, and then apply already proven equivalence (1) \Leftrightarrow (6). \square

Proof of Theorem 2. A part of the proof of this theorem runs fairly in a similar way as that of Theorem 1. Namely, the implications (4) \Rightarrow (8), (8) \Rightarrow (2), (4) \Rightarrow (7), and (5) \Rightarrow (8) can be proven similarly to the implications (1) \Rightarrow (9), (9) \Rightarrow (2), (2) \Rightarrow (5), (3) \Rightarrow (9) of Theorem 1 respectively (one has to additionally use that the product of winning multicovered spaces is winning, and $(X \cup X^{-1}, \lambda_R(X \cup X^{-1}))$ is winning provided so is $(X, \lambda_{\mathcal{U}(X)})$ by Lemma 20).

The implications (3) \Rightarrow (2), (2) \Rightarrow (1), and (7) \Rightarrow (6), (6) \Rightarrow (5) immediately follow from the corresponding definitions. Thus we are left with the task of proving the implications (1) \Rightarrow (4) and (2) \Rightarrow (3). Concerning the implication (1) \Rightarrow (4), it follows from Proposition 12(8) and the fact that the homomorphism $f: F(X) \rightarrow A(X)$ extending the identity map on X is uniformly continuous with respect to the uniformities $\mathcal{U}_{L \wedge R}(F(X))$ and $\mathcal{U}(A(X))$, and hence is uniformly bounded with respect to the multicovers $\lambda_{L \wedge R}(F(X))$ and $\lambda(A(X))$ of $F(X)$ and $A(X)$ respectively.

(2) \Rightarrow (3). Let us note that in light of Corollary 23 it suffices to prove that the multicovered space $(F(X), \lambda_{L \vee R})$ is winning. Set $\Delta_{F(X)} = \{(x, x) : x \in F(X)\}$. Then the map $i: X \ni x \mapsto (x, x) \in \Delta_{F(X)}$ as well as its inverse are obviously perfect with respect to multicovers $\lambda_{L \vee R}$ and $\lambda_L \times \lambda_R|_{\Delta_{F(X)}}$ of $F(X)$ and $\Delta_{F(X)}$ respectively. Since $F(X)$ is strictly o -bounded, both of the multicovered space $(F(X), \lambda_R)$ and $(F(X), \lambda_L)$ are winning, and hence so is the product $(F(X)^2, \lambda_R \times \lambda_L)$, and finally the multicovered spaces

$(\Delta_{F(X)}, \lambda_R \times \lambda_L | \Delta_{F(X)})$ and $(F(X), \lambda_{L \vee R})$ are winning as well. \square

Example 26. *There exists a countably-compact space Z and a pseudometric d on Z^2 such that the pseudometric space (Z^2, d) is not Lindelöf.*

Proof. To begin with, let us note that it suffices to construct two countably-compact spaces X and Y and a pseudometric d on their product $X \times Y$ such that the corresponding pseudometric space is not Lindelöf, and then the topological sum Z of X and Y obviously admits a required pseudometric. Let D be a discrete space of size $|D| = \aleph_1$. Similarly to Example 3.10.19 of [10] we define a function f assigning to each countable subset A of βD some element $f(A) \in \overline{A} \setminus A$. Let $X_0 = D$ and

$$X_\alpha = \left(\bigcup_{\gamma < \alpha} X_\gamma \right) \cup f \left(\left[\bigcup_{\gamma < \alpha} X_\gamma \right]^{\aleph_0} \right)$$

for $0 < \alpha < \omega_1$, where $[A]^{\aleph_0}$ stands for the family of all countable subsets of a set A . Thus we have already defined a transfinite sequence $(X_\alpha)_{\alpha < \omega_1}$ of subsets of βD . The space $X = \bigcup_{\alpha < \omega_1} X_\alpha$ is obviously countably-compact (every countable subset A of X is contained in some X_α , and hence is not closed in X). It is easy to prove that $|X| \leq \mathfrak{c}$. Set $Y = D \cup (\beta D \setminus X)$. According to Theorem 3.6.14 of [10], $|\overline{A}| = 2^{\mathfrak{c}}$ for every countable $A \subset \beta D$, and hence Y is countably-compact as well. It suffices to observe that $X \times Y$ contains an open discrete subspace $\Delta_D = \{(x, x) : x \in D\}$ of size \aleph_1 , and hence admits a non-Lindelöf pseudometric. \square

Proof of Theorem 4. The proof of this theorem is quite similar to that of Theorem 1 and is left to the reader. \square

Remark 3. The characterization of spaces X such that $F(X)$ is o -bounded is the same as in the abelian case. But its proof requires a technique of (semi)filter games investigated by C. Laflamme, and is not within the methods used here. This problem is to be considered in [6] from a more general point of view. \square

Acknowledgments. The author wishes to express his thanks to Taras Banakh and Igor Guran for many stimulating conversations.

REFERENCES

1. Архангельский А.В. *Об отображениях, связанных с топологическими группами*, ДАН СССР **181** (1968), 1303–1306.
2. Архангельский А.В. *О соотношениях между инвариантами топологических групп и их подпространств*, Успехи Мат. Наук **35** (1980), 3–22.
3. Архангельский А.В. *Топологические пространства функций*. – М.: Изд. Моск. Ун-та, 1989. – 222 с.
4. Banakh T., *On index of total boundedness of (strictly) o -bounded groups*, Topology Appl. **120** (2002), 427–439.

5. Banach T., Nikolas P., Sanchis M. *Filter games and pathologic subgroups of \mathbb{R}^ω* , to appear in J. Aust. Math. Soc..
6. Banach T., Zdomsky L. *Selection Principles and Games on multicovered spaces and their applications in topological algebra*, in preparation.
7. Bartoszynski T., Tsaban B. *Hereditary Topological Diagonalizations and the Menger-Hurewicz Conjectures*, to appear in Proc. Amer. Math. Soc.,
<http://arxiv.org/abs/math.LO/0208224>
8. Bouziad A. *Le degré de Lindelöf est l-invariant*, Proc. Amer. Math. Soc. **129** (2001), 913–919.
9. Chaber J., Pol R. *A remark on Fremlin–Miller theorem concerning the Menger property and Michael concentrated sets*, preprint.
10. Энгелькинг Р. Общая топология. – М.:“Мир”, 1986. – 751 с.
11. Gerlits J., Nagy Zs. *Some properties of $C(X)$, I*, Topology Appl. **14** (2) (1982), 151–163.
12. Гуран И.И. *Топологические группы и свойства их подпространств*, Канд. дис., Москва, 1981. – 69 с.
13. Гуран І.Й., Зарічний М.М. *Елементи теорії топологічних груп*. – Київ: УМК ВО, 1991. – 76 с.
14. Hernandez C. *Topological groups close to being σ -compact*, Topology Appl. **102** (2000), 101–111.
15. Hernandez C., Robbie D., Tkachenko M. *Some properties of o -bounded and strictly o -bounded topological groups*, Appl. Gen. Topol. **1** (2000), 29–43.
16. Just W., Miller A. W., Scheepers M., Szeptycki P. J. *The combinatorics of open covers II*, Topology Appl. **73** (1996), 241–266.
17. Кравчук А., Michalewski H. *An example of a topological group*, Topology Appl. **127** (2003), 325–330.
18. Lelek A. *Some cover properties of spaces*, Fund. Math. **LXIV** (1969), 209–218.
19. Пестов В.Г. *Некоторые свойства свободных топологических групп*, Вестн. Москов. Ун-та. **1** (1982), 35–37.
20. Пестов В.Г. *Некоторые топологические свойства, сохраняемые отношением M -эквивалентности*, Успехи Мат. Наук **39** (1984), no.6, 203–304.
21. Pol R. *A function space $C(X)$ which is weakly Lindelöf but not weakly compactly generated*, Studia Math. **64** (1979), 279–285.
22. Roelcke W., Dierolf S. *Uniform structures on topological groups and their quotients*, McGraw-Hill International Book Company, 1981.
23. Scheepers M. *A direct proof of a theorem of Telgarsky*, Proc. Amer. Math. Soc. **123**(11) (1995), 3483–3485.
24. Scheepers M. *Combinatorics of open covers I: Ramsey Theory*, Topology Appl. **69** (1996), 31–62.
25. Scheepers M., Tsaban B. *The combinatorics of Borel covers*, Topology Appl. **121** (2002), 357–382.
26. Telgarsky R. *Topological games: on the 50th anniversary of the Banach-Mazur game*, Rocky Mountain J. Math., **17** (1987), no.3, 227–276.
27. Telgarsky R. *On sieve-complete and compact-like spaces*, Topology Appl. **16** (1983), 63–68.
28. Telgarsky R. *Spaces defined by topological games*, Fund. Math. **CXVI** (1983), 189–207.
29. Telgarsky R. *On games of Topsøe*, Math. Scand. **54** (1984), 170–176.
30. Tkachenko M. *Introduction to topological groups*, Topology Appl. **86** (1998), 179–231.
31. Tkachenko M. *Topological groups for topologists: part II*, Bol. Soc. Mat. Mexicana **3** (2000), no.3, 1–41.
32. Tkachuk V. *Some non-multiplicative properties are l-invariant*, Comment. Math. Univ. Carolin. **38** (1997), 169–175.
33. Tsaban B. *o -Bounded groups and other topological groups with strong combinatorial properties*, Proc. Amer. Math. Soc., to appear.
<http://arxiv.org/abs/math.GN/0307225>

34. Tsaban B. *Selection principles in mathematics: A milestone of open problems*, Note Mat. **22** (2003/2004), no. 22, 179–208.
<http://arxiv.org/abs/math.GN/0312182>
35. Tsaban B., Zdomskyy L. *Scales, Fields, and a problem of Hurewicz*, submitted to J. Amer. Math. Soc..
<http://arxiv.org/abs/math.GN/0507043>.
36. Успенский В.В. *О топологии свободного локально-выпуклого пространства*, ДАН СССР **270** (1983), 1334–1337.
37. Vaughan J. *Small uncountable cardinals and topology*, in: Open problems in topology (eds. J. van Mill, G.M. Reed), Elsevier Sci. Publ., 1990, 197–216.
38. Velichko N. V. *The Lindelöf property is l -invariant*, Topology Appl. **89** (1998), 277–283.
39. Zdomskyy L. *A semifilter approach to selection principles*, to appear in Comment. Math. Univ. Carolin.
<http://arxiv.org/abs/math.GN/0412498>

Ivan Franko Lviv National University
Department of Mechanics and Mathematics
lzdmsky@rambler.ru

Received 13.09.2005