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CELLULAR BALLEANS

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We prove that a ballean is cellular if and only if its asymptotic Gromov dimension is 0. We construct also a universal countable metrizable ballean and show that every separable non-Archimedean metric space is asymptotically embeddable into a Hilbert space.

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Доказано, что боллеан является сотовым тогда и только тогда, когда его асимптотическая размерность по Громову равна 0. Построен универсальный счётный метризуемый боллеан и доказано, что каждое сепарабельное неархимедово метрическое пространиство асимтотически вложимо в гильбертово пространство.

1. Ball structures and balleans. A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called the *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radiuses*.

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \quad B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure $\mathbb{B} = (X, P, B)$ is called

- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B^*(x, \alpha') \subseteq B(x, \alpha), \quad B(x, \beta') \subseteq B^*(x, \beta);$$

- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- *lower multiplicative* if, for any α, $\beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);$$

- *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

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Let $\mathcal{B} = (X, P, B)$ be a lower symmetric, lower multiplicative ball structure. Then the family

$$\left\{ \bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha) : \alpha \in P \right\}$$

is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, U) = \{y \in X : (x, y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

A ball structure is said to be a *ballean* if it is upper symmetric and upper multiplicative. The balleans arouse independently in coarse geometry [8] under name coarse structure and in combinatorics [6]. For good motivation to study the balleans related to metric spaces see the survey [1].

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f\colon X_1 \to X_2$ is called a $\prec$*-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f\big(B_1(x, \alpha)\big) \subseteq B_2(f(x), \beta).$$

By the definition, $\prec$-mappings can be considered as the asymptotic counterparts of the uniformly continuous mappings between the uniform topological space.

If $f\colon X_1 \to X_2$ is a bijection such that f and f^{-1} are the $\prec$-mappings, we say that the balleans $\mathcal{B}_1$ and $\mathcal{B}_2$ are *asymorphic*. If $X_1 = X_2$ and the identity mapping $\mathrm{id}\colon X_1 \to X_2$ is a $\prec$-mapping, we write $\mathcal{B}_1 \prec \mathcal{B}_2$. If $\mathcal{B}_1 \prec \mathcal{B}_2$ and $\mathcal{B}_2 \prec \mathcal{B}_1$, we write $\mathcal{B}_1 = \mathcal{B}_2$.

A pair (f_1, f_2) of $\prec$-mappings $f_1\colon X_1 \to X_2$, $f_2\colon X_2 \to X_1$ is called a *quasi-asymorphism* between the balleans $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ if there exist $\alpha \in P_1, \beta \in P_2$ such that, for all $x \in X_1, y \in X_2$,

$$f_2 f_1(x) \in B_1(x, \alpha), f_1 f_2(y) \in B_2(y, \beta).$$

The notion of quasi-asymorphism is a generation of the notion of course equivalence of metric spaces [1].

2. Cellularity. Given any ballean $\mathcal{B} = (X, P, B), x \in X, y \in X$ and $\alpha \in P$, we say that x, y are α*-path connected* if there exists a sequence $x_0, x_1, ..., x_n,\ x_0 = x,\ x_n = y$ such that

$$x_i \in B(x_{i+1}, \alpha),\ \ x_{i+1} \in B(x_i, \alpha)$$

for every $i \in \{0, 1, ..., n-1\}$. For any $x \in X$ and $\alpha \in P$, we put

$$B^{\square}(x, \alpha) = \{y \in X : x, y \text{ are } \alpha\text{-path connected}\}.$$

The ballean $\mathcal{B}^{\square} = (X, P, B^{\square})$ is called the *cellularization* of $\mathcal{B}$. A ballean $\mathcal{B}$ is called *cellular* if $\mathcal{B} = \mathcal{B}^{\square}$. These notions were introduced in [7]. Every ballean which is quasi-asymorphic to some cellular ballean is cellular.

Example 1. Let X be a set and let $\mathcal{P}$ be a partition of X. For $x, y \in X$, we say that $y \in B(x, 1)$ if and only if x, y are in the same cell of the partition $\mathcal{P}$. By $\mathcal{B}(X, \mathcal{P})$ we denote the ballean $(X, \{1\}, B)$ and call it the *partition ballean*. Clearly, every partition ballean is cellular.

Let (X, d) be a metric space, $\mathbb{R}^+ = \{r \in \mathbb{R} : r \geq 0\}$. For any $x \in X$ and $r \in \mathbb{R}^+$, denote

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

By $\mathcal{B}(X, d)$ we denote the ballean $(X, \mathbb{R}^+, B_d)$ and say that $\mathcal{B}(X, d)$ is determined by (X, d). A ballean $\mathcal{B}$ is called *metrizable* if $\mathcal{B}$ is asymorphic to $\mathcal{B}(X, d)$ for some metric space (X, d). For criterion of metrizability see [7] or [6, Theorem 9.1]. Using this criterion it is easy to show that a ballean quasi-isomorphic to some metrizable ballean is metrizable.

Example 2. A metric d on a set X is called *non-Archimedean* (or *ultrametric*) if

$$d(x, y) \leq \max\{d(x, z),\ d(y, z)\}$$

for all $x, y, z \in X$. If (X, d) is a non-Archimedean metric space, then the ballean $B(X, d)$ is cellular. By [6, Theorem 9.3], every metrizable cellular ballean is asymorphic to $\mathcal{B}(X, d)$ for an appropriate non-Archimedean metric space (X, d).

Example 3. Let X be a set and let φ be a filter on X. For any $x \in X,\ F \in \varphi$, we put

$$B(x, F) = \begin{cases} X \setminus F, & \text{if } x \notin F; \\ \{x\}, & \text{if } x \in F; \end{cases}$$

denote by $\mathcal{B}(X, \varphi)$ the ballean (X, φ, B) and say that $\mathcal{B}(X, \varphi)$ is a *filter ballean*. Since $B^{\square}(x, F) = B(x, F)$ for all $x \in X,\ F \in \varphi$, every filter ballean is cellular. A ballean $\mathcal{B} = (X, P, B)$ is called *pseudodiscrete* if, for every $\alpha \in P$, there exists a bounded subset $Y \subseteq X$ such that $B(x, \alpha) = \{x\}$ for every $x \in X \setminus Y$. A subset $Y \subseteq X$ is called *bounded* if $Y \subseteq B(x, \alpha)$ for some $x \in X,\ \alpha \in P$. A ballean $\mathcal{B}$ is pseudodiscrete if and only if $\mathcal{B}$ is asymorphic to some filter ballean. Hence, every pseudodiscrete ballean is cellular.

Example 4. Let G be an infinite group with the identity e, k be an infinite cardinal such that $k \leq |G|$. We denote by $\mathcal{F}(G, k)$ the family $\{F \subset G : |F| < k,\ e \in F\}$ and, for any $g \in G,\ F \in \mathcal{F}(G, k)$, put

$$B_l(g, F) = gF,\ B_r(g, F) = Fg.$$

Thus, we get two balleans

$$\mathcal{B}_l(G, k) = (G, \mathcal{F}(G, k), B_l),\ \mathcal{B}_r(G, k) = (G, \mathcal{F}(G, k), B_r).$$

Clearly, the mapping $x \mapsto x^{-1}$ is an asymorphism between $\mathcal{B}_l(G, k)$ and $\mathcal{B}_r(G, k)$.

A ballean $\mathcal{B}_l(G, k)$ is cellular if and only if either $k > \aleph_0$ or $k = \aleph_0$ and every finite subset of G is contained in some finite subgroup.

Let $\mathcal{B} = (X, P, B)$ be a ballean. A family $\mathcal{F}$ of subsets of X is called *uniformly bounded* if there exists $\alpha \in P$ such that, for any $F \in \mathcal{F}$ and $x \in F$, we have $F \subseteq B(x, \alpha)$.

Let $\{\mathcal{B}_\lambda = (X, P_\lambda, B_\lambda) : \lambda \in \Lambda\}$ be a family of balleans with common support X. We suppose that, for any $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda \in \Lambda$ such that $\mathcal{B}_{\lambda_1} \prec \mathcal{B}_\lambda,\ \mathcal{B}_{\lambda_2} \prec \mathcal{B}_\lambda$. For any $\lambda \in \Lambda$, we choose a copy $P'_\lambda = f_\lambda(P_\lambda)$ of P_λ such that the family $\{P'_\lambda : \lambda \in \Lambda\}$ is disjoint, and put $P = \bigcup_{\lambda \in \Lambda} P_\lambda$. For all $x \in X,\ \beta \in P,\ \beta \in P'_\lambda$, we put $B(x, \beta) = B_\lambda(x, f^{-1}(\beta))$. A ballean $\mathcal{B} = (X, P, B)$ is called an *inductive limit* of the family $\{\mathcal{B}_\lambda : \lambda \in \Lambda\}$. Clearly, $\mathcal{B}_\lambda \prec \mathcal{B}$ for every $\lambda \in \Lambda$, and $\mathcal{B}$ is the smallest ballean on X with this property.

Theorem 1. *For every ballean $\mathcal{B} = (X, P, B)$, the following statements are equivalent:*

(i) $\mathcal{B}$ is cellular;

(ii) for every uniformly bounded in $\mathcal{B}$ family $\mathcal{F}$ of subsets of X and every $\alpha \in P$, the family $\mathcal{F}^{\square}(\alpha) = \{B^{\square}(F, \alpha) : F \in \mathcal{F}\}$ is uniformly bounded in $\mathcal{B}$;

(iii) $\mathcal{B}$ is an inductive limit of some family $\{\mathcal{B}_\lambda : \lambda \in \Lambda\}$ of partition balleans.

Proof. (i)⇒(ii). Since $\mathcal{B}$ is cellular, we may suppose that $B^{\square}(x, \lambda) = B(x, \lambda)$ for all $x \in X,\ \lambda \in P$. Choose $\beta \in P$ such that $F \subseteq B(x, \beta)$ for all $F \in \mathcal{F}$ and $x \in F$. Choose $\gamma \in P$ such that $B(B(x, \beta), \alpha) \subseteq B(x, \gamma)$ for every $x \in X$. Then $F \subseteq B(x, \gamma) = B^{\square}(x, \gamma)$ for all $F \in \mathcal{F}$ and $x \in F$.

(ii)⇒(iii). For all $\alpha \in P$ and $x, y \in X$, we put $x \sim_\alpha y$ if and only if $y \in B^{\square}(x, \alpha)$. Denote by $\mathcal{F}_\alpha$ the partition of X determined by $\sim_\alpha$, and by B_α the partition ballean $\mathcal{B}(X, \mathcal{F}_\alpha)$. Let $\mathcal{B}'$ be an inductive limit of the family $\{\mathcal{B}_\alpha : \alpha \in P\}$. Since $\mathcal{F}_\alpha$ is uniformly bounded, we have $\mathcal{B}_\alpha \prec \mathcal{B}$ for every $\alpha \in P$, so $\mathcal{B}' \prec \mathcal{B}$. On the other hand $B(x, \alpha) \prec B^{\square}(x, \alpha)$ for all $x \in X$ and $\alpha \in P$, so $\mathcal{B} \prec \mathcal{B}'$.

(iii)⇒(i). Let $\mathcal{B}$ be an inductive limit of the family $\{\mathcal{B}_\lambda(X, \mathcal{P}_\lambda) : \lambda \in \Lambda\}$ of partition balleans. Since $\mathcal{B}_\lambda(X, \mathcal{P}_\lambda) = (X, \{\lambda\}, B_\lambda)$ and $B^{\square}(x, \lambda) = B^{\square}_\lambda(x, \lambda) = B_\lambda(x, \lambda) = B(x, \lambda)$, $\mathcal{B}$ is cellular. □

3. Cellularity and dimension. Let $\mathcal{B} = (X, P, B)$ be a ballean, $\mathcal{F}$ be a family of subsets of X, $\alpha \in P$. We say that $\mathcal{F}$ is *α-disjoint* if every ball $B(x, \alpha)$ intersects at most one member of the family $\mathcal{F}$. Given $n \in \mathbb{N} \cup \{0\}$, we say that $\operatorname{asdim} \mathcal{B} \leq n$ if, for every $\alpha \in P$, there exists a uniformly bounded covering $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup ... \cup \mathcal{F}_n$ of X such that every family $\mathcal{F}_i,\ i \in \{0, 1, ..., n\}$ is α-disjoint. In this case we denote by $\operatorname{asdim} \mathcal{B}$ the minimal number m such that $\operatorname{asdim} \mathcal{B} \leq m$. If the statement $\operatorname{asdim} \mathcal{B} \leq n$ does not hold for every $n \in \mathbb{N}$, we put $\operatorname{asdim} \mathcal{B} = \infty$. We omit a routine verification of invariance of asdim under quasi-asymorphisms.

This definition of asdim is a direct generalization of the Gromov dimension of metric spaces, see the survey [1].

Theorem 2. *A ballean $\mathcal{B} = (X, P, B)$ is cellular if and only if* $\operatorname{asdim} \mathcal{B} = 0$.

Proof. Assume that $\mathcal{B}$ is cellular and $B^{\square}(x, \alpha) = B(x, \alpha)$ for all $x \in X,\ \alpha \in P$. Given an arbitrary $\alpha \in P$, denote by $\mathcal{F}_\alpha$ the covering of X determined by the equivalence $\sim_\alpha$ from the proof of Theorem 1. If $F \in \mathcal{F}_\alpha$ and $x \in F$, then $B^{\square}(x, \alpha) = F$ so $\mathcal{F}_\alpha$ is α-disjoint. Clearly, $\mathcal{F}_\alpha$ is uniformly bounded, so $\operatorname{asdim} \mathcal{B} = 0$.

Assume that $\operatorname{asdim} \mathcal{B} = 0$. Given an arbitrary $\alpha \in P$, we take an α-disjoint uniformly bounded covering $\mathcal{F}_\alpha$ of X. Let $F \in \mathcal{F}_\alpha$ and $x \in F$. We show that $B^{\square}(F, \alpha) = F$. Suppose the contrary and choose $y \in F, z \in B^{\square}(F, \alpha) \setminus F$. Then there exists a sequence $x_0, x_1, ..., x_k, x_{k+1}, ..., x_n$ in X such that $x_0 = y,\ x_n = z,\ x_i \in B(x_{i+1}, \alpha),\ i \in \{0, ..., n-1\}$ and $x_k \in F,\ x_{k+1} \in B(x_k, \alpha) \setminus F$. Then the ball $B(x_k, \alpha)$ intersects at least two members of the family $\mathcal{F}_\alpha$. Since $\mathcal{F}_\alpha$ is uniformly bounded, there exists $\beta \in P$ such that $F \subseteq B(y, \beta)$ for every $y \in F$. Given any $x \in X$ we take $F \in \mathcal{F}_\alpha$ such that $x \in F$. Then $B^{\square}(x, \alpha) = F \subseteq B(x, \beta)$ so $\mathcal{B} = \mathcal{B}^{\square}$. □

4. Universal balleans. Let $\mathcal{B} = (X, P, B)$ be a ballean, $Y \subseteq X$. We say that the ballean $\mathcal{B}_Y = (Y, P, B_Y)$, where $B_Y(y, \alpha) = B(y, \alpha) \cap Y$, is a *subballean* of $\mathcal{B}$.

Let $\mathcal{K}$ be a class (with respect to asymorphisms) of balleans. A ballean $\mathcal{B} \in \mathcal{K}$ is called *universal* if every ballean from $\mathcal{K}$ is asymorphic to some subballean of $\mathcal{B}$. By [2], there exists a metric space M_0 (which is an asymptotic counterpart of Cantor set) such that the metric ballean determined by M_0 is universal in the class $\mathcal{K}_0$ of balleans defined as follows: a ballean $\mathcal{B} = (X, P, B)$ belongs to $\mathcal{K}_0$ if and only if $\mathcal{B}$ is countable metrizable cellular and, for every $r > 0$, there exists a natural number $c(r)$ with $|B(x, r)| < c(r)$ for every $x \in X$.

Now we are going to construct a ballean which is universal in the class of all countable metrizable cellular balleans.

Let $\{Z_n : n \in \omega\}$ be a family of nonempty sets. For every $n \in \omega$, we fix some element $e_n \in Z_n$ and say that the family $\{(Z_n, e_n) : n \in \omega\}$ is pointed. Let us consider the direct product $Z = \bigotimes_{n\in\omega}(Z_n, e_n)$. Every element $z \in Z$ is a sequence $(z_n)_{n\in\omega}$ such that $z_n \in Z_n$, $n \in \omega$ and $z_n = e_n$ for all but finitely many $n \in \omega$. For every $n \in \omega$, we put $z_n = \mathrm{pr}_n z$ and define a metric ρ on Z by the rule: $\rho(z, z) = 0$ and if $z \neq z'$ let $\rho(z, z') = \min\{n \in \omega : \mathrm{pr}_i z = \mathrm{pr}_i z'$ for every $i \geq n\}$. Clearly, (Z, ρ) is a non-Archimedean metric space, so the ballean $\mathcal{B}(Z, \rho)$ is cellular.

Theorem 3. *Let $\{(Z, e_n) : n \in \omega\}$ be a pointed family such that $|Z_n| = \aleph_0$ for every $n \in \omega$, $Z = \bigotimes_{n\in\omega}(Z_n, e_n)$. Then every countable cellular metrizable ballean $\mathcal{B} = (X, P, B)$ is asymorphic to some subballean of $\mathcal{B}(Z, \rho)$.*

Proof. By Theorem 1, there exists a family $\{\mathcal{P}_n : n \in \omega\}$ of partitions of X such that $\mathcal{B}$ is an inductive limit of the family of partition balleans $\{\mathcal{B}(\mathcal{P}_n) : n \in \omega\}$ and every partition $\mathcal{P}_{k+1}$ is an enlargement of $\mathcal{P}_{n+1}$, i.e. every cell of $\mathcal{P}_{n+1}$ is union of the cell of $\mathcal{P}_n$. For every $n \in \omega$, we define an equivalence $\sim_n$ on Z by the rule: $z \sim_n z'$ if and only if $\rho(z, z') \leq n + 1$. Every equivalence $\sim_n$ defines some partition $\mathcal{R}_n$ of Z. Clearly, every cell of $\mathcal{R}_n$ is countable and every cell of $\mathcal{R}_{n+1}$ is a union of countable many cells of $\mathcal{R}_n$. Therefore it suffices to define an injective mapping $f : X \to Z$ such that, for all $n \in \omega$ and $x, y \in X$, we have x, y are in the same cell of $\mathcal{P}_n$ if and only if $f(x), f(y)$ are in the same cell of $\mathcal{R}_n$.

We fix some element $x_0 \in X$ and, for every $n \in \omega$, denote by X_n the cell of $\mathcal{P}_n$ containing x_0. We denote by Y_n the cell of $\mathcal{R}_n$ containing the element $e = (e_n)_{n\in\omega}$. Let $f_0 : X_0 \to Y_0$ be some injective mapping. Assume that we have defined the sequence $f_0, f_1, ..., f_n$ of mappings $f_i : X_i \to Y_i$ such that f_{i+1} is an extension of f_i and, for every $i \in \{0, 1, ..., n\}$, $x, y \in X_n$ are in the same cell of $\mathcal{P}_i$ if and only if $f(x), f(y)$ are in the same cell of $\mathcal{R}_i$. Since every cell of $\mathcal{R}_n$ is countable and every cell of $\mathcal{R}_{n+1}$ is a union of countably many cells of $\mathcal{R}_n$, we can extend f_n to $f_{n+1} : X_{n+1} \to Y_{n+1}$ with above property. After ω steps we define f as an inductive limit of the family $\{f_n : n \in \omega\}$. □

Remark 1. Let $\{(Z_n, e_n) : n \in \omega\}$ be a pointed family of finite sets such that $|Z_n| > 1$ for every $n \in \omega$, $Z = \bigotimes_{n\in\omega}(Z_n, e_n)$. It can be shown that the metric ballean $\mathcal{B}(Z, \rho)$ is universal in the class $\mathcal{K}_0$ determined above.

Applying Lemma 10.8 from [6], we get a family of cardinality $2^{\aleph_0}$ of pairwise non-asymorphic universal balleans in $\mathcal{K}_0$.

Remark 2. It is natural to ask if there exists a universal ballean in the class of all countable cellular balleans. It is well-known that there exist 2^c pairwise non-equivalent ultrafilters on a countable set. Using Example 3, we get a family of cardinality 2^c of pairwise non-asymorphic countable cellular balleans. Since a family of all subballeans of a countable ballean is of cardinality c, we get a negative answer to this question.

Theorem 4. *For every countable cellular metrizable ballean $\mathcal{B}$, there exists a subspace Y of Hilbert space ℓ_2 such that $\mathcal{B}$ is asymorphic to the metric ballean determined by Y.*

Proof. In view of Theorem 3 it suffices to find Y for the ballean of metric space (Z, ρ) from Theorem 3. Let d be a standard metric on ℓ_2 and, for all $x \in \ell_2,\ r \in \mathbb{R}^n$, let $B(x, r)$ be the corresponding ball. We can construct inductively an increasing sequence $(r_n)_{n\in\omega},\ r_0 = 1$ of natural numbers and, for every $n \in \omega$, a sequence $(x_{nm})_{m\in\omega},\ x_{n0} = 0$ of elements of ℓ_2 such that

(i) $B(x_{nm}, r_n) \subset B(0, r_{n+1})$ for all $n, m \in \omega$;

(ii) $d(x_{nm}, x_{nk}) > 2r_n + n$ for all $n, m, k \in \omega,\ m \neq k$.

Then we fix some countable subset $Y_0 \subset B(0, r_0)$ and define inductively a chain $Y_0 \subset Y_1 \subset ...$ of subsets of ℓ_2 by the rule

$$Y_{n+1} = \bigcup_{m\in\omega} (x_{nm} + Y_n),$$

and put $Y = \bigcup_{n\in\omega} Y_n$. By the construction, the ballean $\mathcal{B}(Y, d)$ is cellular. To show that $\mathcal{B}(Y, d)$ is asymorphic to $\mathcal{B}(Z, \rho)$ we may use the arguments from the proof of Theorem 3. ☐

Let $(X_1, d_1), (X_2, d_2)$ be metric spaces. We say that a mapping $f\colon (X_1, d_1) \to (X_2, d_2)$ is an asymorphic embedding (or uniform embedding in the terminology of [4]) if, for every $R > 0$, there exists $S > 0$ such that

$$d_1(x_1, x_2) \leq R \Rightarrow d_2(f(x_1), f(x_2)) \leq S,$$

$$d_2(f(x_1), f(x_2)) \leq R \Rightarrow d_1(x_1, x_2) \leq S.$$

In [3] Gromov has drawn attention to the problem of asymptotic embedding of separable metric space into Hilbert space. The mainstream of this problem is in asymptotic embedding of Cayley graphs of finitely generated groups into Hilbert space, for the references see [4]. Recently, Higson constructed a counterexample to Gromov's problem (see [8, Proposition 11.29] or [9]).

Theorem 5. *Every separable non-Archimedean metric space is asymptotically embeddable into the Hilbert space.*

Proof. In view of Theorem 4 it suffices to show that every separable metric space (X, d) is asymptotically embeddable into some countable metric space. Let Y be a countable dense subset of X. For every $x \in X$, we take an arbitrary $y \in Y$ such that $d(x, y) \leq 1$ and put $f(x) = Y$. Clearly, f is an asymptotic embedding. ☐

Remark 3. Following [1], we say that a metric space (X, d) is of *bounded geometry* if there exists $r > 0$ such that, for every $n \in \mathbb{N}$, there exists $c(n) \in \mathbb{N}$ such that every r-discrete subset in every ball $B(x, n)$ is of cardinality $\leq c(n)$. Let Y be a maximal r-discrete subset of X. For every $x \in X$, we take an arbitrary $y \in Y$ such that $d(x, y) \leq 2$ and put $f(x) = y$. Clearly, Y is countable and f is an asymptotic embedding. If X is non-Archimedean, then Y belongs to $\mathcal{K}_0$. Hence, every cellular metrizable ballean of bounded geometry is asymptotically embeddable into every universal ballean from $\mathcal{K}_0$.

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