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ON SOME CARDINAL INVARIANTS OF HYPERSPACES

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We investigate relationships between some cardinal invariants of a T_1-space X and its hyperspace $\exp(X)$ of all non-empty closed subsets of X, endowed with the Vietoris topology. We shall prove that hyperspace construction exp preserves π-weight and calibers. We compare π-characters of various hyperspace constructions: $\exp_n$, $\exp_\omega$, $\exp_c$, exp.

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Исследуются соотношения между некоторыми кардинальными инвариантами T_1-пространства X и его гиперпространства $\exp(X)$ всех непустых замкнутых подмножеств в X, наделенного топологией Виеториса. Доказано, что конструкция гиперпространства сохраняет π-вес и калибры. Сравниваются π-характеры различных конструкций гиперпространства $\exp_n$, $\exp_\omega$, $\exp_c$, exp.

1. Introduction. We investigate relationships between some cardinal invariants of a topological space X and its hyperspace $\exp(X)$ of non-empty closed subsets of X, endowed with the Vietoris topology. In the paper we consider only T_1-spaces. In this case X can be identified with the subspace of $\exp(X)$ consisting of singletons.

We recall some basic cardinal characteristics of a topological space X:

- the *weight* $w(X)$ is the smallest size $|\mathcal{B}|$ of a base $\mathcal{B}$ of the topology of X;
- the *π-weight* $\pi w(X)$ of X is the smallest size $|\mathcal{B}|$ of a π-base $\mathcal{B}$ for X;
- the *density* $d(X)$ of X is the smallest size $|A|$ of a dense subset A of X;
- the *cellularity* or *Souslin number* $c(X) = \sup\{|\mathcal{U}| : \mathcal{U}$ is a disjoint family of non-empty open sets of $X\}$.

We shall also be interested in two local cardinal invariants. For a point x of a topological space X

- the *character* $\chi(x; X)$ of X at x is the smallest size of a neighborhood base at x;
- the *π-character* $\pi\chi(x; X)$ of X at x is the smallest size of a π-base at x.

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We recall that a family $\mathcal{B}$ of open subsets of a topological space X is called a *π-base* (at a point x) if each non-empty open set $U \subset X$ (containing the point x) contains some non-empty set $B \in \mathcal{B}$. The cardinal $\pi\chi(X) = \sup_{x\in X} \pi\chi(x;X)$ is called the *π-character* of X. By a general convention, if some cardinal invarinat of a topological space is finite then we replace it by $\aleph_0$.

It is known that

$$d(\exp(X)) = d(X) \text{ for any } X$$

and

$$w(\exp(X)) = w(X) \text{ for a compact } X.$$

The cellularity of the hyperspace $\exp(X)$ of a compact Hausdorff space X is related to the cellularity of finite powers of X via the formula

$$c(\exp(X)) = \sup_{n\in\mathbb{N}} c(X^n)$$

proved by V. V. Fedorchuk and S. Todorcevic in [4].

This formula shows, in particular, that the equality $c(\exp(X)) = c(X)$ is not true in general. For example, for the Souslin continuum S we have $\aleph_0 = c(S) < c(S \times S) = \aleph_1$. On the other hand, $\mathrm{MA}(\omega_1)$, the Martin Axiom for ω_1, implies that if $c(X) = \aleph_0$ for a Hausdorff compact space X, then $c(\exp(X)) = \aleph_0$ (see [4], Theorem 5.5). Nevertheless, there are ZFC-examples of compact spaces X such that $c(X) < c(X \times X)$ and, hence, $c(X) < c(\exp(X))$ ([7]).

Another example of a cardinal invariant φ such that $\varphi(X) \neq \varphi(\exp(X))$ is the character $\chi(X) = \sup_{x\in X} \chi(x;X)$ of X. Clearly, if $\exp(X)$ is a first countable space, then X is perfectly normal. Consequently,

$$\chi(X) < \chi(\exp(X))$$

for every first countable X which is not perfectly normal.

In the paper we shall be interested in three topological cardinal invariants: π-weight, π-character, and calibers. We prove that the hyperspace construction exp preserves π-weight (Theorem 1) and calibers (Theorem 1). Examples 1–3 concern π-character and π-base.

2. π-Weight and π-character of hyperspaces. For a topological space X by $\exp(X)$ we denote the space of all nonempty closed subsets of X endowed with the Vietoris topology generated by the base consisting of the sets

$$O\langle\mathcal{U}\rangle = \Big\{F \in \exp(X) : F \subset \bigcup \mathcal{U},\ \ \forall U \in \mathcal{U}\ F \cap U \neq \varnothing\Big\}$$

where $\mathcal{U}$ is a finite family of non-empty open sets in X. If $\mathcal{U} = \{U_1, \dots, U_n\}$ then we shall use the more traditional notation $O\langle U_1, \dots, U_n\rangle$ for $O\langle\mathcal{U}\rangle$.

For an ordinal $n \le \omega$, let

$$\exp_n(X) = \{F \in \exp(X) : |F| \le n\}$$

where $|F|$ denotes the cardinality of a set F. By $\exp_c(X)$ (resp. $\exp_s(X)$) we denote the subspace of $\exp(X)$ consisting of all non-empty compact (resp. closed separable) subsets of X. Clearly, $\exp_\omega(X)$ is dense in $\exp(X)$ and $\exp_\omega(X) \subset \exp_c(X) \cap \exp_s(X)$. The hyperspaces $\exp_n(X)$ for $n \in \{c, s, \omega\} \cup \mathbb{N}$ are particular cases of the hyperspace

$$\exp_{\mathcal{P}}(X) = \{F \in \exp(X) : F \text{ has property } \mathcal{P}\},$$

where $\mathcal{P}$ is a topological property. Such a property $\mathcal{P}$ will be called *finitely-hereditary* if each finite subspace of a space with the property $\mathcal{P}$ satisfies this property. In the sequel we shall assume that such a property $\mathcal{P}$ is non-trivial in the sense that at least one non-empty space satisfies property $\mathcal{P}$. Then the finite-heredity of $\mathcal{P}$ implies that all singletons satisfy this property. The structure of the hyperspaces $\exp_{\mathcal{P}}(X)$ is described by the following easy

Proposition 1. *Let $\mathcal{P}$ be a non-trivial finitely-hereditary topological property and X be a topological T_1-space. If each finite T_1-space satisfies property $\mathcal{P}$, then $\exp_\omega(X) \subset \exp_{\mathcal{P}}(X)$. Otherwise, $\exp_{\mathcal{P}}(X) = \exp_n(X)$ for some $n \in \mathbb{N}$.*

The π-weight and π-character of spaces and their hyperspaces relate as follows.

Theorem 1. *For any T_1-space X and a finitely-hereditary topological property $\mathcal{P}$ we have*

$$\pi\chi(X) = \pi\chi(\exp_{\mathcal{P}}(X) \cap \exp_s(X)) \leq \pi\chi(\exp_{\mathcal{P}}(X)) \leq \pi\chi(\exp(X)) =$$
$$= \pi\chi(\{X\}, \exp(X)) = \pi w(\exp(X)) = \pi w(\exp_{\mathcal{P}}(X)) = \pi w(X).$$

We divide the proof of this theorem into a series of lemmas in which we assume that X is a T_1-space X and $\mathcal{P}$ is a non-trivial finitely-hereditary topological property.

Lemma 1. *$\pi w(X) \leq \pi\chi(\{X\}, \exp(X))$ where $\pi\chi(\{X\}, \exp(X))$ is the π-character of $\exp(X)$ at the point $\{X\} \in \exp(X)$.*

Proof. First we prove that $\pi w(X) \leq \pi\chi(\{X\}, \exp(X))$. Let $\mathfrak{B}$ be a π-base of $\exp(X)$ at the point $\{X\}$ of size $|\mathfrak{B}| = \pi\chi(\{X\}, \exp(X))$. Without any loss of generality, we can assume that each set $B \in \mathfrak{B}$ is of the basic form $B = O\langle\mathcal{U}\rangle$ for some finite family $\mathcal{U}_B$ of open subsets of X. We claim that the union $\mathcal{B} = \bigcup_{B \in \mathfrak{B}} \mathcal{U}_B$ is a π-base for the space X. Indeed, given any non-empty open set $W \subset X$, consider the open neighborhood $O\langle W, X\rangle$ of $\{X\}$ in $\exp(X)$. Since $\mathfrak{B}$ is a π-base at $\{X\}$, there is an element $B \in \mathfrak{B}$ with $B \subset O\langle W, X\rangle$. By our convention, B is of the basic form $B = O\langle\mathcal{U}_B\rangle$. We claim that $U \subset W$ for some $U \in \mathcal{U}_B$. Otherwise, for every $U \in \mathcal{U}_B$ we could select a point $a_U \in U \setminus W$. Then the set $A = \{a_U : U \in \mathcal{U}_B\}$ belongs to $B = O\langle\mathcal{U}_B\rangle$ but not to $O\langle W, X\rangle \supset B$, which is not possible. This contradiction shows that $\mathcal{B}$ is a π-base for X and hence

$$\pi w(X) \leq |\mathcal{B}| \leq |\mathfrak{B}| \cdot \aleph_0 \leq \pi\chi(\{X\}, \exp(X)).$$

□

Lemma 2. $\pi w(X) = \pi w(\exp_{\mathcal{P}}(X))$.

Proof. Given a π-base $\mathcal{B}$ for X of size $|\mathcal{B}| = \pi w(X)$, consider the family

$$\mathfrak{B} = \{O\langle\mathcal{U}\rangle \cap \exp_{\mathcal{P}}(X) : \mathcal{U} \subset \mathcal{B},\ |\mathcal{U}| < \infty\}$$

having size $|\mathfrak{B}| \leq \aleph_0 \cdot |\mathcal{B}| = \pi w(X)$. We claim that $\mathfrak{B}$ is a π-base for $\exp_{\mathcal{P}}(X)$. Indeed, given a non-empty open set $O \subset \exp_{\mathcal{P}}(X)$, take any element $F \in O$ and find a basic neighborhood $O\langle\mathcal{U}\rangle$ in the hyperspace $\exp(X)$ such that $F \in O\langle\mathcal{U}\rangle \cap \exp_{\mathcal{P}}(X) \subset O$. Since $F \in O\langle\mathcal{U}\rangle$, there is a finite subset $\{x_1, \dots, x_n\} \subset F$ meeting each set $U \in \mathcal{U}$. We may assume that the points $x_1, \dots, x_n$ are pairwise distinct. For every $i \leq n$ pick an open set $U_i \in \mathcal{U}$ containing x_i. For every $i \leq n$ find a non-empty set $B_i \in \mathcal{B}$ with $B_i \subset U_i$ and pick a point $z_i \in B_i$. The set $Z = \{z_1, \dots, z_n\}$ is homeomorphic to a finite subset of the set F and hence satisfies property

$\mathcal{P}$ by the finite-heredity of $\mathcal{P}$. Hence $Z \in \exp_{\mathcal{P}}(X)$ and $O\langle B_1, \ldots, B_n\rangle \cap \exp_{\mathcal{P}}(X) \ni Z$ is a non-empty subset from the family $\mathfrak{B}$, lying in the set O. This witnesses that $\mathfrak{B}$ is a π-base for $\exp_{\mathcal{P}}(X)$ and hence $\pi w(\exp_{\mathcal{P}}(X)) \le |\mathfrak{B}| \le \pi w(X)$.

To prove the reverse inequality, fix a π-base $\mathfrak{B}$ for the hyperspace $\exp_{\mathcal{P}}(X)$ with $|\mathfrak{B}| \le \pi w(\exp_{\mathcal{P}}(X))$. We loose no generality assuming that each set $B \subset \mathfrak{B}$ is of the basic form $O\langle \mathcal{U}_B\rangle \cap \exp_{\mathcal{P}}(X)$ for some finite family $\mathcal{U}_B$ of open subsets of X. We claim that $\mathcal{B} = \bigcup\{\mathcal{U}_B : B \in \mathfrak{B}\}$ is a π-base for X. Indeed, given any non-empty open set $W \subset X$ consider the open set $\langle W\rangle = \{F \in \exp_{\mathcal{P}}(X) : F \subset W\}$ and find a set $B \in \mathfrak{B}$ with $B = O\langle \mathcal{U}_B\rangle \subset \langle W\rangle$. Since all the singletons satisfy property $\mathcal{P}$, we get $\bigcup \mathcal{U}_B \subset W$. Then there is a set $U \in \mathcal{U}_B \subset \mathcal{B}$ with $U \subset W$, i.e., $\mathcal{B}$ is a π-base for X having size $|\mathcal{B}| \le \pi w(\exp(X))$. This proves the inequality $\pi w(X) \le \pi w(\exp_{\mathcal{P}}(X))$. □

Lemma 3. $\pi\chi(X) = \pi\chi(\exp_{\mathcal{P}}(X) \cap \exp_s(X)) \le \pi\chi(\exp_{\mathcal{P}}(X))$.

Proof. The inequality $\pi\chi(X) \le \pi\chi(\exp_{\mathcal{P}}(X))$ can be proved by analogy with the proof of the inequality $\pi w(X) \le \pi w(\exp_{\mathcal{P}}(X))$ in Lemma 2. Since the separability is a finitely-hereditary topological property, we also get $\pi\chi(X) \le \pi\chi(\exp_{\mathcal{P}}(X) \cap \exp_s(X))$.

So it remains to prove the inequality $\pi\chi(\exp_{\mathcal{P}}(X) \cap \exp_s(X)) \le \pi\chi(X)$. To be short, denote $\exp_{s\mathcal{P}}(X) = \exp_s(X) \cap \exp_{\mathcal{P}}(X)$. Take any closed separable subspace $F \subset X$ with property $\mathcal{P}$ and fix a countable dense subset $D \subset F$. We may find a family $\mathcal{B}$ of open subsets of X of size $|\mathcal{B}| \le \pi\chi(X)$ that is a π-base at each point $x \in D$.

We claim that family

$$\mathfrak{B} = \{O\langle \mathcal{U}\rangle \cap \exp_{s\mathcal{P}}(X) : \mathcal{U} \subset \mathcal{B},\ |\mathcal{U}| < \infty\}$$

is a π-base at the point F in the space $\exp_{s\mathcal{P}}(X)$. Indeed, given a non-empty open set $O \subset \exp_{s\mathcal{P}}(X)$ containing F, find a basic neighborhood $O\langle \mathcal{U}\rangle$ in the hyperspace $\exp(X)$ such that $F \in O\langle \mathcal{U}\rangle \cap \exp_{s\mathcal{P}}(X) \subset O$. Since $F \in O\langle \mathcal{U}\rangle$, there is a finite subset $\{x_1, \ldots, x_n\} \subset D$ meeting each set $U \in \mathcal{U}$. We may assume that the points $x_1, \ldots, x_n$ are pairwise distinct. For every $i \le n$ pick an open set $U_i \in \mathcal{U}$ containing x_i. Since $\mathcal{B}$ is a π-base at each point x_i, we can find a non-empty set $B_i \in \mathcal{B}$ with $B_i \subset U_i$. Proceeding as in the proof of Lemma 2, we can show that $O\langle B_1, \ldots, B_n\rangle \cap \exp_{s\mathcal{P}}(X) \in \mathfrak{B}$ is a non-empty subset lying in the set O. This witnesses that $\mathfrak{B}$ is a π-base for $\exp_{s\mathcal{P}}(X)$ at the point F. □

With Lemmas 1–3 in hand, the proof of Theorem 1 becomes quite easy. We get a chain of inequalities

$$\pi w(X) \le \pi\chi(\{X\}, \quad \exp(X)) \le \pi\chi(\exp(X)) \le \pi w(\exp(X)) \le \pi w(X)$$

where the first and the last inequalities follow from Lemmas 1 and 2, respectively. Combining Lemmas 2 and 3, we get

$$\pi w(X) = \pi w(\exp_{\mathcal{P}}(X)) \ge \pi\chi(\exp_{\mathcal{P}}(X)) \ge \pi\chi(\exp_{s\mathcal{P}}(X)) = \pi\chi(X)$$

which completes the proof of Theorem 1. □

Example 1. For any uncountable cardinal $\kappa \le \mathfrak{c}$ there exists a compact Hausdorff space X_1 such that

$$\aleph_0 = \chi(\exp_\omega(X_1)) = \pi\chi(\exp_\omega(X_1)) < \pi\chi(\exp_c(X_1)) = \pi\chi(\exp(X)) = \kappa.$$

As the space X_1 one can take an arbitrary first countable compact Hausdorff space of density κ, for example, a suitable subspace in the Alexandroff duplicate of the interval $I = [0, 1]$. By Theorem 1, the space X_1 satisfies the required properties.

Example 2. For arbitrary uncountable cardinal number λ there exists a metrizable space X_2 such that

$$\aleph_0 = \chi(\exp_c(X_2)) = \pi\chi(\exp_c(X_2)) < \pi\chi(\exp(X_2)) = \pi w(X_2) = \lambda.$$

For the space X_2 we can take a discrete set of cardinality λ and then apply Theorem 1 to show that it has the required properties.

Unifying Examples 1 and 2 we get

Example 3. For any uncountable cardinal numbers $\kappa \le \lambda$ with $\kappa \le \mathfrak{c}$ there exists a paracompact p-space (a discrete union of compact Hausdorff space and a metrizable space) $X_3 = X_1 \sqcup X_2$ such that

$$\aleph_0 = \chi(\exp_c(X_3)) = \pi\chi(\exp_\omega(X_3)) < \pi\chi(\exp_c(X_3)) = \kappa \le \lambda = \pi\chi(\exp(X_3)).$$

3. Calibers of hyperpsaces. Let us recall that a cardinal number κ is said to be a *caliber* of a space X if any family $\mathcal{U}$ of non-empty open subsets of X with $|\mathcal{U}| = \kappa$ contains a subfamily $\mathcal{V} \subset \mathcal{U}$ with $|\mathcal{V}| = \kappa$ and $\bigcap \mathcal{V} \ne \varnothing$. It is well-known (and easily seen) that each regular cardinal $\kappa > d(X)$ is a caliber for X. By $K(X)$ we shall denote the class of all calibers of a given topological spaces. It is easy to see that for each dense subspace Y of a topological space X we get $K(Y) \subset K(X)$.

The class $K(X)$ induces two cardinal invariants called *Shanin's numbers* of X:

- $\mathrm{sh}(X) = \sup\{\kappa : \kappa \notin K(X) \text{ is a regular cardinal}\}$;
- $\mathrm{sh}_0(X) = \min\{\kappa : \kappa^+ \in K(X)\}$.

where κ^+ stands for the successor cardinal to κ. It is clear that $\mathrm{sh}_0(X) \le \mathrm{sh}(X)$.

Theorem 2. *For a T_1-space X and a finitely-hereditary topological property $\mathcal{P}$ the following equalities hold:*

$$K(X) = K(\exp_{\mathcal{P}}(X)) = K(\exp(X)).$$

Proof. Let κ be a caliber of X. Then by Shanin's theorem ([6]), τ is a caliber of the product X^n. Since $\exp_n(X)$ is the image of X^n under the continuous map $(x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}$, the space $\exp_n(X)$ has κ as a caliber. Also the space $\exp_\omega(X) = \bigcup_{n\in\omega} \exp_n(X)$, being the countable union of spaces with caliber κ, has caliber κ. Now we see that $\exp_{\mathcal{P}}(X)$ has caliber κ because $\exp_{\mathcal{P}}(X)$ either contains $\exp_\omega(X)$ as a dense subspace or else $\exp_{\mathcal{P}}(X)$ equals $\exp_n(X)$ for some $n \in \omega$, see Proposition 1.

Now assume conversely that κ is a caliber of $\exp_{\mathcal{P}}(X)$. To show that κ is a caliber of X, take any family $\mathcal{U}$ of nonempty open subsets of X with $|\mathcal{U}| = \kappa$. Each set $U \in \mathcal{U}$ induces a non-empty open set $O\langle U\rangle = \{F \in \exp_{\mathcal{P}}(X) : F \subset U\}$ in $\exp_{\mathcal{P}}(X)$. Since κ is a caliber of $\exp_{\mathcal{P}}(X)$, there is a subfamily $\mathcal{V} \subset \mathcal{U}$ with $|\mathcal{V}| = \kappa$ such that $\bigcap_{U\in V} O\langle V\rangle$ contains some set $F \in \exp_{\mathcal{P}}(X)$. Then the intersection $\cap\mathcal{V} \supset F$ is not empty, witnessing that κ is a caliber of X. □

Theorem 2 implies

Corollary 1. *Let X be a T_1-space and $\mathcal{P}$ is a finitely-hereditary topological property. Then* $\mathrm{sh}(X) = \mathrm{sh}(\exp_{\mathcal{P}}(X)) = \mathrm{sh}(\exp(X))$ *and* $\mathrm{sh}_0(X) = \mathrm{sh}_0(\exp_{\mathcal{P}}(X)) = \mathrm{sh}_0(\exp(X))$.

If φ is a cardinal invariant of topological spaces, then by $h\varphi$ we denote a new (hereditary) cardinal invariant which is defined by $h\varphi(X) = \sup\{\varphi(Y) : Y \subset X\}$. Some cardinal invariants are hereditary by definition. For example *spread* of X (the notation is $s(X)$):

$$s(X) = \sup\{|Y|, Y \subset X, Y \text{ is discrete}\}.$$

It is known that $d(X) = d(\exp(X))$ for any T_1-space X. We have proven that $\pi w(X) = \pi w(\exp(X))$ and $\mathrm{sh}(X) = \mathrm{sh}(\exp(X))$. But the hereditary versions of these inequalities fail even for first countable compact Hausdorff spaces.

Example 4. There exists a first countable compact Hausdorff space X such that

1) $hd(\exp(X)) \neq d(X)$,
2) $h\pi w(\exp(X)) \neq h\pi w(X)$,
3) $hsh(\exp(X)) \neq hsh(X)$,
4) $hc(\exp(X)) \neq hc(X)$,
5) $s(\exp(X)) \neq s(X)$.

In fact, for X we can take the space known as *double arrows* of Alexandroff. Then all "right" cardinal numbers are equal to $\aleph_0$. On the other hand, $\exp_2(X)$ contains a discrete subset of cardinality of the continuum. Hence, all "left" cardinal numbers are equal to $\mathfrak{c}$.

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