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L. P. Plachta

ON ORIENTABILITY OF SINGULAR FOLIATIONS OF SURFACES IN CLOSED BRAID COMPLEMENTS

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Every essential closed oriented and smooth of class C^r surface in standard position in a closed braid complement admits a singular foliation in a natural way. We show that each such foliation of the surface is orientable, i.e. there is a flow $\mathcal{F}$ of the class C^{r-1} on this surface such that the trajectory of $\mathcal{F}$ yields the given singular foliation.

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Каждая существенная замкнутая ориентированная и гладкая класса C^r поверхность в стандартном положении в дополнении замкнутой косы допускает естественное сингулярное слоение. Мы показываем, что каждое такое слоение поверхности является ориентируемым, т.е. существует поток $\mathcal{F}$ класса C^{r-1} на этой поверхности такой, что траектория $\mathcal{F}$ индуцирует заданные сингулярные слоения.

1. Introduction. We think S^3 of as $\mathbb{R}^3$ union a point at ∞. Let L be a non-split link in S^3 represented as a closed braid with the axis A. Birman and Menasco ([2]) studied the geometric positions of incompressible surfaces S in $\mathbb{R}^3 \setminus L$ via the natural foliations on them induced by the open book decomposition of $\mathbb{R}^3$ by half-planes with boundary on the z-axis A (see also [1] for details). Such the foliations of a surface S are usually singular and can be standardized by using a sequence of smooth isotopies and exchange moves to the ones which have a typical structure (see Section 1 for precise definitions). The main purpose of this note is to show that such standardized singular foliations can be thought of as smooth flows on S.

Section 1 presents some preliminaries. Here we review the singular foliations of incompressible surfaces in closed braid complements and their combinatorial patterns.

Let $L \subset \mathbb{R}^3$ be an oriented closed braid with the axis A and let $\mathcal{H} = \{H_\theta : \theta \in [0, 2\pi]\}$ be the open book decomposition of $\mathbb{R}^3$ by half-planes with boundary on the axis A. Let S' be a closed orientable incompressible surface in $\mathbb{R}^3 \setminus L$. Assume that S' is in general position with respect to $\mathcal{H}$. The intersection of the H_θ's with S' induces a (singular) foliation $\mathcal{F}'$ on the surface S'.

Using isotopy in $\mathbb{R}^3 \setminus L$, one can standardize the foliation $\mathcal{F}'$ so that the resulting foliated surface S is essential and allows a decomposition into some typical foliated regions ([1],

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Theorem 1.1). Each such region contains a unique saddle point and admits a canonical embedding in $\mathbb{R}^3$ with respect to the z-axis A ([1]). The combinatorial decomposition $\mathcal{S}$ of a foliated surface S is supplied with some additional data called the decoration. This yields a combinatorial pattern for the foliated surface S.

In Section 2, we show that if S is a smooth closed orientable surface in a closed braid complement $S^3 \setminus L$, which is in standard position, then the natural (singular) foliation $\mathcal{F}$ of S is orientable. In this case we can speak about the class of smoothness of the foliation $\mathcal{F}$ and the corresponding flow on S. That is, if S is smoothly embedded in S^3 of class $C^r, r \geq 2$, then the singular foliation $\mathcal{F}$ is of class C^{r-1}, in some reasonable sense, and there is a flow $\mathcal{X}$ on S of class C^{r-1}, whose trajectories induce the given foliation $\mathcal{F}$. In our proof we use the method and techniques from [5].

2. Foliated surfaces in closed braid complements. All the surfaces embedded in $\mathbb{R}^3$ or in S^3 are assumed to be smooth of class C^r where $r \geq 2$. Throughout Sections 1 and 2, L denotes a non-split link in three-space.

First we review the Birman and Menasco approach to the study of incompressible surfaces in closed braid complements. For details see [2] and [1]. Choose cylindrical coordinates (r, θ, z) in 3-space $\mathbb{R}^3$ (remind that S^3 thought of as $\mathbb{R}^3$ union a point at ∞). Let $\mathcal{H} = \{H_\theta : \theta \in [0, 2\pi]\}$ be the open book decomposition of $\mathbb{R}^3$ by half-planes with boundary on the z-axis. A link L is a closed braid with the z-axis A as its *braid axis* if each component intersects every half-plane $H_\theta, \theta \in [0, 2\pi]$, through the axis transversely. Then $S^3 \setminus A$ is foliated by open half-panes H'_θ parameterized by the polar angle $\theta \in [0, 2\pi]$. This foliation is called the braid fibration of $S^3 \setminus A$ and is denoted by H. Let S' be a closed orientable surface in $\mathbb{R}^3 \setminus L$. Assume that S' is in general position with respect to $\mathcal{H}$. The intersection of the H_θ's with S' induces a smooth foliation $\mathcal{F}'$ of S' of class C^{r-1}. The foliation $\mathcal{F}'$ is, in general, singular. The singular points of $\mathcal{F}'$ are the points where S meets the braid axis A or is tangent to the half-planes H_θ, and are called the *vertices* of $\mathcal{F}'$ and the *singularities* of $\mathcal{F}'$, respectively. By general position, the number of singular points is finite and all singularities can be assumed to be of saddle type or center type. Since S' does not intersect L, each non-singular leaf is either an arc which has both its endpoints on the axis A, or a circle. As in [2], we shall call the leaves in the first case b-arcs, while in the second case the non-singular leaves are called c-circles. After appropriate isotopy in $S^3 \setminus L$, S' can be replaced with an *essential*, smoothly embedded surface S, so that the natural foliation $\mathcal{F}$ of S has the following properties (see Theorem 1.1 of [1]):

(i) The foliation is radial at the vertices. All singularities are of saddle type. The singularity together with its leaves (branches) is called then a singular leaf of $\mathcal{F}$;

(ii) The singularities fall into three types bb, bc and cc (see below);

(iii) The vertices are (circularly) ordered by their order on the braid axis A. Moreover, after appropriate isotopy, distinct singular leaves are on distinct singular half-planes (singular fibers) H_θ, so they are also circularly ordered.

In the following, we shall assume that an embedded essential surface S is chosen so that its natural foliation has properties (i)-(iii).

Consider now the different types of singularities which can occur in the foliation of an essential surface S of genus g. There are only two types of non-singular leaves, so it can occur at most three possible types of singularities, called bb, bc and cc-singularities, according to the types of leaves which are surgered when passing through a singularity. The number of branches of singular leaves coming from a vertex v of the foliation $\mathcal{F}$ is called the valence of v.

The dual point of view is instead of a foliation $\mathcal{F}$ and its singular leaves to consider a decomposition of a surface S into the regions, one for each saddle point of $\mathcal{F}$, by cutting the surface along appropriate b-arcs and c-circles. By Theorem 1.2 of [1], either S is a torus foliated by c-circles, or is decomposed into canonically embedded regions of type bb, bc, cc. The latter gives a combinatorial pattern $\mathcal{S}$ for a foliated surface S in $\mathbb{R}^3 \setminus L$. Next, each vertex of $\mathcal{F}$ has a sign and the singularities are also marked by signs. The sign of a vertex v is $+$ (respectively, $-$), if the positive normal vector to S at v agrees (respectively, disagrees) with the orientation of the axis A. The sign of a singularity s or the region contained s is $+$ (respectively, $-$) if the positive normal to S at the point s agrees (respectively, disagrees), with the normal to H_θ which points in the direction of increasing the angle θ. By Theorem 2.1 of [1], we may assume that any two vertices v and w which are adjacent to the same singularity s so that v, s and w lie on the boundary of a face D have the different signs. Moreover, all non-singular leaves of $\mathcal{F}$ are essential in the closed braid complement $\mathbb{R}^3 \setminus L$ with respect to the axis A (see [1]). S is called *essential* if all b-arcs and c-circles in $\mathcal{F}$ are essential.

We shall say that a foliation $\mathcal{F}$ of S is *standard* if the decomposed foliated surface $\mathcal{S}$ satisfies all conditions mentioned above. Note that all the steps in the procedure of standardization of a foliation $\mathcal{F}$ and the corresponding foliated surface S can be performed in the smooth category.

3. Proof of the main result. Let L be a closed braid with the axis A which represents a non-split link in S^3. Let $S \subset S^3 \setminus L$ be an essential smooth of class C^r surface which is in standard position. Fix an orientation on S and in $\mathbb{R}^3$. As we have seen in Section 1, the open book decomposition $\mathcal{H} = \{H_\theta : \theta \in [0, 2\pi]\}$ of $\mathcal{R}^3$ induces a foliation $\mathcal{F}$ of S so that all non-singular leaves and the branches of non-singular leaves are the curves of class C^r. We may suggest that the foliation $\mathcal{F}$ and the foliated surface S are standard. S is the Riemannian manifold with the metric inherited from $\mathbb{R}^3$.

We shall say that a singular foliation $\mathcal{F}$ on S is orientable if there is a vector field $\mathcal{X}$ on S whose (non-trivial) trajectories and critical points coincide with the leaves and singular points (which are the vertices and singularities in our case), respectively, of the given foliation.

Next, we say that the foliation $\mathcal{F}$ is coorientable ([5]) if there is a continuous vector field Y on $\mathbb{R}^3$ so that for each $y \in \mathbb{R}^3$ we have $\| Y_y \| = 1$ and $Y_x \perp E_x$ for each $x \in S$, where E_x denotes the tangent plane to S at the point x.

Note, that in the terminology of the papers [3] and [4] all singular points of standard foliations $\mathcal{F}$ on S are of orientable type. There is known, in principle, a procedure which allows to enhance S with the structure of a vector field or a flow whose trajectories induce the given foliation (see, for example, [3]). This can be done by passing first from the given singular foliation of S to a field of line elements on S. Here we indicate explicitly the construction of the desired smooth vector field on S.

Theorem 3.1. *Let S be an essential closed oriented smooth of class $C^r, r \geq 2$, surface in a closed braid complement $S^3 \setminus L$ and let $\mathcal{F}$ be a standard foliation of S induced by the braid fibration of $S^3 \setminus A$, where A is the axis. Then there is a smooth of class C^{r-1} vector field $\mathcal{X}$ on S whose trajectories and critical points coincide with the leaves and singular points, respectively, of $\mathcal{F}$.*

Proof. We shall construct a vector field $\mathcal{X}$ on S so that

(i) the vetrices and singularities of $\mathcal{F}$ correspond to the elliptic and saddle points, respectively, of $\mathcal{X}$;

(ii) the remaining trajectories of $\mathcal{X}$ are exactly the non-singular leaves or the branches of singular leaves of $\mathcal{F}$.

Consider on S the Riemannian metric inherited from the one of $\mathbb{R}^3$. It follows directly from definition that the foliation $H = \{H'_\theta : \theta \in [0, 2\pi]\}$ of $S^3 \backslash A$ is coorientable (the corresponding vector field $\mathcal{T}$ which is transverse to the fibers H_θ is defined as $(\partial x/\partial\theta, \partial y/\partial\theta, 0)$, which can be normalized).

For each point $x \in S$ denote by n_x the positive normal to the surface S at x of length 1. Let $\varphi_\mathcal{T}$ be the flow on $\mathbb{R}^3$ defined by the vector field $\mathcal{T}$. For each non-singular point $y \in S$ of the foliation $\mathcal{F}$ let u_y denote the tangent vector of length 1 at y to the (smooth) trajectory l_y of the flow $\varphi_\mathcal{T}$. Note that if y is a non-singular point in S, then the vectors n_y and t_y are non-colinear (i.e. they are not equal each to other in our case). Therefore, if y is a non-singular point, then there is a unique vector v_y of length 1 which is orthogonal to the plane spanned by n_y and u_y and so that the triple (n_y, u_y, v_y) defines the given orientation of $\mathbb{R}^3$. We thus define a smooth of class C^{r-1} vector field $\mathcal{X}_0$ on the punctured surface S', where $S' = S \setminus \{$all singular points of $\mathcal{F}\}$. It is easy to see that the trajectories of $\mathcal{X}_0$ coincide just with all leaves of the foliated surface $\mathcal{S}' = \mathcal{S} \setminus \{$all singular points of $\mathcal{F}\}$. If $V = S = \varnothing$, i.e. $\mathcal{S}$ is a circularly foliated torus, then we are done.

Suppose $V \neq \varnothing$. Then we have $V \neq \varnothing$. Now our aim is to define a suitable smooth flow or vector field in a neighborhood of each singular point v of $\mathcal{F}$. Let v be any vertex in $\mathcal{S}$. By the definition of standard foliation, a small neighborhood U_v of the vertex v on S is foliated radially. Therefore, under the choice of appropriate local coordinates (may be in a smaller neighborhood U'_v of v) around v, the leaves of $\mathcal{F}$ in U'_v can be considered as the trajectories of a gradient vector field $\mathcal{X}_u$ for a corresponding height function h. Clearly, h may be chosen to be of class C^r. Similarly, in a small neighborhood V_s of a singularity s one can choose local coordinates, so that the leaves of $\mathcal{F}$ in V_s will be the trajectories of a gradient vector field $\mathcal{X}_s$ modeled by the height function h in a neighborhood of a saddle point s on a hyperboloid in $\mathbb{R}^3$. Since S is smooth of class C^r, h can be also chosen to be of class C^r. Therefore in both the cases, $\mathcal{X}_v$ and $\mathcal{X}_s$ will be at least of class C^{r-1}. We may reach also without any efforts that the orientations of the trajectories t of $\mathcal{X}_v$ and $\mathcal{X}$, and $\mathcal{X}$ and $\mathcal{X}_s$ coincide in the common regions. Let V and T be the sets of vertices and singularities, respectively, of $\mathcal{F}$ and $|V| = l$ and $|T| = m$.

The collection of open sets $\{S', U_{v_1}, \dots, U_{v_l}, U_{s_1}, \dots, U_{s_m}\}$ forms a cover of the surface S. Let $\{\mu(S'), \mu(U_{v_1}), \dots, \mu(U_{v_l}), \mu(U_{s_1}), \dots, \mu(U_{s_m})\}$ be the corresponding partition of unity on S of class C^r. Put $\mathcal{X} = \mu(S')\mathcal{X}^0 + \Sigma_{v\in V}\ \mu(U_v)\cdot\mathcal{X}_v + \Sigma_{s\in S}\mu(U_s)\cdot\mathcal{X}_s$. By the construction, $\mathcal{X}$ is a smooth of class C^{r-1} vector field on S with the desired properties. Moreover, $\mathcal{X}$ has a finite number of critical elements, all they are the critical points of hyperbolic type. This completes the proof. □

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Institute of Applied Problems of Mechanics and Mathematics of NAS of Ukraine, Lviv

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