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SOME OPEN PROBLEMS ON THE CLASSICAL FUNCTION SPACE L_1

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The investigation of the geometry of $L_p(\mu)$-spaces has yielded a large quantity of interesting results on isomorphic, isometric and near isometric structure, the isomorphic classification of subspaces and complemented subspaces, geometric properties of operators, bases and unconditional structures and much more. $L_1(\mu)$ spaces offer many differences from $L_p(\mu)$, $p > 1$. Their study has led, for example, to the notion of stable Banach spaces. In this survey we present a number of open problems in L_1 which we believe merit further study, and whose solutions should lead to important further developments.

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На пути исследования геометрии пространств $L_p(\mu)$ получено огромное количество интересных результатов об изоморфной, изометрической и почти изометрической структуре, об изоморфной классификации подпространств и дополняемых подпространств, о геометрических свойствах операторов, о базисах и безусловных структурах и о многом другом. Пространства $L_1(\mu)$ имеют множество существенных отличий от пространств $L_p(\mu)$, $p > 1$. Изучение этих пространств привело, например, к понятию устойчивых банаховых пространств. В предлагаемом обзоре мы представляем открытые проблемы о пространстве L_1, которые, на наш взгляд, заслуживают внимания и решение которых приведет к дальнейшим открытиям.

1. Preliminaries and general properties.

Notation and terminology. Throughout the paper we deal with the space $L_1 = L_1[0,1]$. We use the standard terminology and usual notations as in [80], [50]. For a Banach space X we denote $S(X) = \{x \in X : \|x\| = 1\}$, $B(X) = \{x \in X : \|x\| \le 1\}$, by $\mathcal{L}(X,Y)$ and by $\mathcal{K}(X,Y)$ we denote the space of all bounded linear and respectively compact operators acting from X to Y. $\mathcal{L}(X)$ and $\mathcal{K}(X)$ stand for $\mathcal{L}(X,X)$ and respectively $\mathcal{K}(X,X)$. By χ_A we denote the characteristic function of a set A; Σ denotes the σ-algebra of all Lebesgue measurable subset of $[0,1]$; Σ^+ consists of all $A \in \Sigma$ of positive measure, and for $A \in \Sigma$ we denote $\Sigma(A) = \{B \in \Sigma : \ B \subseteq A\}$.

General structure and well known properties. The space L_1 is complemented in L_1^{**} [113, p.113]. Each separable Banach space is isomorphic to a quotient space of L_1. A Banach space is isomorphic simultaneously to a subspace of L_1 and a quotient space of $C[0,1]$ if and only if it isomorphic to ℓ_2 ([77]). L_1 possesses the Dunford-Pettis property: for every Banach

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space X every weakly compact operator[1] $T \in \mathcal{L}(L_1, X)$ takes weak Cauchy sequences in L_1 into norm Cauchy sequences in X ([40]). Recall that the Radon-Nikodým property (RNP) for a Banach space X means that for each finite measure space (Ω, Σ, μ) and each μ-continuous X-valued measure $G: \ \Sigma \to X$ of bounded variation there exists a $g \in L_1(\mu, X)$ such that $G(A) = \int_A g d\mu$ for all $A \in \Sigma$ ([25]). It is an easy exercise to show that $G(A) = \chi_A$ is an example of an L_1-valued such measure for which the function g does not exist; thus, L_1 does not have the RNP. Evidently, $B(L_1)$ contains no extreme point; in particular, L_1 does not have the Krein-Milman property (KMP) (a Banach space X is said to have the KMP if every closed bounded convex set in X coincides with the closed convex hull of its extreme points). Moreover, if L_1 G_δ-embeds in (a Banach space) X then X does not have the RNP and is not a Schur space[2] ([14]) (an injection $T \in \mathcal{L}(X, Y)$ is called a G_δ-embedding if TK is a G_δ-set in Y for any closed bounded set $K \subset X$). The fact that L_1 could not be embedded into a separable dual Banach space was known to I. M. Gelfand ([33]). L_1 is weakly sequentially complete [113, p.118] and hence contains no subspace isomorphic to c_0. Moreover, L_1 does not sign-embed in c_0 and c_0 does not semi-embed in L_1 ([14]) (for definitions see Sections 5 and 8).

Weak topology and weak convergence. A subset $M \subseteq L_1$ is said to be equi-integrable if

$$\lim_{\mu(A)\to 0} \sup_{x\in M} \int\limits_A |x| d\mu = 0.$$

L_1 satisfies the Dunford-Pettis condition: a bounded subset from L_1 is relatively weakly compact if and only if it is equi-integrable [23]. A sequence $(x_n)_{n=1}^\infty$ in L_1 is weakly null if and only if the following two conditions hold

$$(i) \qquad \lim_n \int\limits_{[0,t]} x_n d\mu = 0 \ \text{ for each } \ t \in [0,1];$$

(ii) $(x_n)_{n=1}^\infty$ is equi-integrable [6, p. 117].

Proximinality. By a theorem of M. G. Krein [70], L_1 contains no proper Chebyshev subspace of finite codimension. Hence there is no contractive projection of L_1 onto a finite-codimensional subspace of L_1 and L_1 is not isometric to any of its finite-codimensional subspaces (see the Douglas theorem in Section 4). In fact, the norm of any projection of L_1 onto a finite-codimensional subspace is at least 2 and the Banach-Mazur distance between L_1 and any finite-codimensional subspace is at least 2. These facts follow from the Daugavet property for L_1 [100, p.71]. Recall that a subset M of a Banach space X is called anti-proximinal if for each $x \in X \setminus M$ there is no nearest point to x in X. A closed bounded convex subset $M \subset X$ is anti-proximinal if and only if no nonzero functional attains its norm on M, i.e. $\Sigma(M) \cap \Sigma(B(X)) = \{0\}$ where $\Sigma(M) = \{f \in X^* : \ (\exists x \in M) \ f(x) = \sup f(M)\}$ [29], see also [32]. The following problem was communicated to us by V. P. Fonf.

Problem 1. *Does there exist an anti-proximinal closed bounded convex subset B in L_1? What if we assume* $\operatorname{int}(B) \neq \varnothing$*?*

[1] i.e. operator which takes bounded sets into weakly compact sets.

[2] a Schur space means that the weak convergence of sequences implies their norm convergence.

There is another simply stated question. Evidently, L_1 and $Y = L_1 \oplus_1 \mathbb{R}$ are isomorphic. Let $d = d(L_1, Y)$ be the Banach-Mazur distance between these spaces. What is d?

2. Subspaces and quotients of L_1.

Universality for 2-dimensional Banach spaces. Every two-dimensional Banach space embeds isometrically in L_1 while ℓ_∞^3 does not ([122]).

The Grothendieck theorem. Let $0 < p < \infty$. If a subspace $X \subseteq L_p$ is contained (as a subset) in L_∞ then X is finite dimensional ([41]) (see also [112]).

Each infinite dimensional subspace of L_1 contains ℓ_p for some $1 \leq p \leq 2$. For a subspace X of L_1 the following conditions are equivalent:

(i) X contains no subspace isomorphic to ℓ_1,

(ii) X contains no subspace isomorphic to ℓ_1 and complemented in L_1,

(iii) X is reflexive,

(iv) there is an $r \in (1, 2]$ such that X embeds in L_r,

(v) $B(X)$ is uniformly integrable (or equivalently, $B(X)$ is relatively weakly compact).

The equivalence (i)-(ii)-(iii)-(v) was obtained by M. I. Kadec and A. Pełczyński ([55]) and (i)-(iv) was proved by H. P. Rosenthal [105]. See also the recent monograph [76] for proofs of additional facts on subspaces of L_1.

In particular, every subspace of L_1 contains an unconditional basic sequence. Moreover each infinite dimensional subspace of L_1 contains a further subspace isomorphic to ℓ_p for some $p \in [1, 2]$. In fact, one can show this with $p = \sup\{r : X \text{ is of type } r\}$. The first part of this theorem was proved in [1] and the second in [43]. The proof in [1] led Krivine and Maurey to discover the notion of a stable Banach space and prove their wonderful result that stable spaces contain almost isometric copies of some ℓ_p ([71]).

Embedding ℓ_p and L_p in L_1. The space L_r isometrically embeds in L_p for $1 \leq p < r \leq 2$. This result was obtained in [16]. In [77] it was proved moreover, that if a separable Banach space X is finitely representable[3] in L_1 then X isometrically embeds in L_1. The finite representability of L_p in L_1 for $1 < p \leq 2$ follows from the results of [73] and also from the M. I. Kadec theorem ([54]) on the isometric embeddability of ℓ_p into L_1 for $1 < p \leq 2$. By the earlier result of R. E. A. C. Paley [95], the space ℓ_p does not embed into L_1 if $p > 2$. If $\ell_p(\ell_r)$ embeds in L_1 then $1 \leq p \leq r \leq 2$ ([104]).

Quotients. If a subspace $X \subseteq L_1$ has the RNP then L_1/X does not ([22]). There exists a subspace $X \subseteq L_1$ isomorphic to ℓ_2 such that L_1/X does not embed in L_1 ([99]), but it is unknown ([37]) whether one can take $X = \text{Rad}$, i.e. the subspace spanned by the Rademacher system. As noted to us by A. Plichko, the quotient space L_1/Rad is not isomorphic to L_1: if it were isomorphic, then the annihilator Rad^0 in L_∞ would thus be isomorphic to L_∞ and by the injectivity of L_∞ would be complemented in L_∞. On the other hand, L_∞/Rad^0 is isomorphic to a Hilbert space (as Rad^*) and cannot be complemented in L_∞. Any quotient space of L_1 by a reflexive subspace contains an isomorph of L_1. But the following problem is still unsolved.

[3] we recall that X is finitely representable in Y if for each $\varepsilon > 0$ and each finite dimensional subspace $F \subset X$ there is a subspace $G \subset Y$ of the same dimension such that the Banach-Mazur distance between F and G satisfies $d(F, G) \leq 1 + \varepsilon$.

Problem 2. ([37]) *Does there exist an infinite dimensional reflexive subspace R of L_1 such that L_1/R isomorphic to L_1?*

Failure of the three-space property. M. Talagrand in [118] constructed a subspace $X \subset L_1$ such that both X and L_1/X contain no subspace isomorphic to L_1. The strongest positive result in this direction is that for each subspace $X \subseteq L_1$ either L_1 embeds into X, or ℓ_1 embeds into L_1/X ([14]) (see also [17, p. 100] for the three-space problem for L_1).

Problem 3. (Kwapień, [119]) *Does every Banach subspace of L_0 embed in L_1?*

It is known that every Banach subspace of L_0 embeds into L_p for each $p < 1$ ([91]) and A Banach space X embeds into L_1 if and only if $\ell_1(X)$ embeds into L_0 ([64]). A. Koldobsky ([69]) noted that the isometric version of Problem 3 is solved in the negative: if $q > 2$ then ℓ_q^3 does not embed isometrically in L_p for any $0 < p \leq 2$, while L_0 contains isometrically every 3-dimensional normed space (see also [122]). Moreover, for each $0 < p < 1$ there exists a Banach space E_p which embeds isometrically in L_p but does not embed isometrically into any L_r with $p < r \leq 1$ ([65]). In that paper N. J. Kalton and A. Koldobsky posed the following version of Kwapień's problem: does every Banach space which embeds in L_p for some $0 < p < 1$ embed in L_1? An affirmative answer to this version would imply an affirmative answer to Kwapień's problem by Nikishyn's theorem.

Geometric properties of subspaces and some exotic subspaces. As it was shown by J. Bourgain and H. P. Rosenthal in [13], for subspaces of L_1 the RNP and PCP are equivalent: a Banach space X has the PCP if for any nonempty bounded closed subset $A \subset X$, the identity operator on A has a weak-to-norm point of continuity). For general Banach spaces, the RNP implies the PCP but the converse is not true ([13]). Accordingly to [12], a Banach space X has the *strong Schur property* if there is a constant $K < \infty$ such that for all $0 < \delta \leq 2$, every δ-separated sequence in B_X has a subsequence K/δ-equivalent to the unit vector basis of ℓ_1. By Rosenthal's ℓ_1 theorem ([106]), the Schur property is equivalent to the above condition, but without K, hence the strong Schur property implies the Schur property; the converse however is not true for subspaces of L_1 as it was mentioned in [12] with a reference to [52]. Moreover, J. Bourgain and H. P. Rosenthal constructed a subspace of L_1 having the strong Schur property which does not embed in ℓ_1 ([12]) answering a question implicitly raised by W. B. Johnson and E. Odell ([52]). On the other hand, there exists a hereditarily ℓ_1 subspace of L_1 without the Schur property ([102]). There is a subspace of L_1 which does not have an unconditional basis but has an unconditional decomposition onto two-dimensional subspaces (and hence embeds in a Banach space with an unconditional basis) ([68]). Moreover, for each integer $k \geq 2$ and each $1 \leq p < 2$, there exists a subspace of L_p having a symmetric decomposition into $\leq k$-dimensional subspaces but which has no unconditional decomposition into $\leq (k-1)$-dimensional subspaces ([7]).

There is an easily formulated problem due to R. W. Cross, M. I. Ostrovskii and V. V. Shevchik, concerning the placing of subspaces in L_1.

Problem 4. ([19]) *Does every infinite dimensional subspace $X \subset L_1$ admits a decomposition $L_1 = Y \oplus Z$ into subspaces Y and Z, both isomorphic to L_1 such that both $X \cap Y$ and $X \cap Z$ are infinite dimensional?*

3. Complemented subspaces of L_1 and $\mathcal{L}_1$-spaces.

Isomorphic classification of complemented subspaces. P. Enflo proved that L_1 is primary, i.e. if $L_1 = X \oplus Y$ then either X or Y is isomorphic to L_1 [31]. In fact, the Enflo theorem is valid for L_p with $1 \le p < \infty$ (see [3]). There are two obvious examples of non-isomorphic complemented subspaces of L_1, namely ℓ_1 and L_1 itself. One of the eldest and most interesting problems of the Banach space theory is whether these examples are the only possible ones.

Problem 5. ([79]) *Is every (infinite dimensional) complemented subspace of L_1 isomorphic to either L_1 or ℓ_1 (equivalently, to an $L_1(\nu)$-space)?*

It is a deep result that there is uncountably many mutually non isomorphic complemented subspaces of L_p for $1 < p < \infty$, $p \neq 2$ ([15]) while for $p < 1$ it is unknown whether every complemented subspace of L_p is isomorphic to L_p.

Isometric stability of complemented subspaces. A theorem of B. Randrianantoanina states that if X and Y are isometric subspaces of L_1 then their projection constants (i.e. the infimum of the norms of projections onto these subspaces) in L_1 (finite or infinite) coincide ([103]). In fact, this theorem was proved for subspaces of L_p with $1 \le p < \infty$ not equal to an even integer, for which counterexamples were provided [103].

Properties of complemented subspaces. The following theorems have been established for an arbitrary complemented subspace E of L_1. E^* is isomorphic to ℓ_∞ ([77]). If E contains a subspace isomorphic to L_1 then E is itself isomorphic to L_1 ([31]). E contains a subspace isomorphic to ℓ_1 and complemented in L_1 ([96]). If E has an unconditional basis then E is isomorphic to ℓ_1 ([77]). If E has the RNP then E is isomorphic to ℓ_1 ([75]). If E does not have the Schur property then $\bigoplus_{\ell_1}(\ell_2)$ embeds into E ([9])[4]. This theorem was obtained by J. Bourgain as a consequence of properties of non-Dunford-Pettis operators (see Section 8).

1-complemented subspaces. A classical theorem of Douglas asserts that a subspace $X \subset L_1$ is 1-complemented in L_1 if and only if X is isometric to an $L_1(\nu)$ space ([28]) (note that the «only if» part is due to A. Grothendieck ([42])). This theorem was extended later to L_p with $1 \le p < \infty$ in [5]. In [30] Enflo characterizes subsets of L_1 which are the ranges of contractive, possibly, non-linear, projections (i.e. $\|Px - Py\| \le \|x - y\|$ for all x, y).

Complemented and uncomplemented subspaces of L_1 isomorphic to $L_1(\nu)$. There exists a subspace $X \subset L_1$ isomorphic to ℓ_1 (or even isomorphic to L_1) and uncomple-mented in L_1 ([10]). But if X is a subspace of L_1 with $\mathrm{d}\big(X, L_1(\nu)\big) < \sqrt{2}$ for some $L_1(\nu)$ then X is $(2\lambda^{-2} - 1)^{-1}$-complemented in L_1 ([26]).

Problem 6. ([10]) *What is the maximal value of λ such that if X is a subspace of L_1 which is λ-isomorphic to some $L_1(\nu)$ then X is complemented in L_1?*

If $X \subset L_1$ is isomorphic to ℓ_1 and generated by a sequence of independent random variables then X is complemented in L_1 ([27]) (for another proof of this fact see [10]).

The $\mathcal{L}_1$-spaces. A Banach space X is called an $\mathcal{L}_{p,\lambda}$-space for $1 \le p \le \infty$ and $1 \le \lambda < \infty$ provided for each finite dimensional subspace $B \subset X$ there is a finite dimensional subspace $E \subset X$ containing B such that the Banach-Mazur distance $\mathrm{d}(E, \ell_p^n) \le \lambda$ where $n = \dim E$. X is called an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$-space for some $\lambda \ge 1$. Of course, $L_p(\mu)$ spaces are $\mathcal{L}_{p,\lambda}$-spaces for each $\lambda > 1$. The converse is also true: if X is a $\mathcal{L}_{p,\lambda}$-space for each $\lambda > 1$ and some $1 \le p < \infty$ then X is isometric to $L_p(\mu)$ for suitable μ[5]. For an arbitrary $\mathcal{L}_p$-space

[4] note that earlier H. P. Rosenthal in [107] proved that ℓ_2 embeds into E.

[5] for $p > 1$ this is proved only under the assumption of separability of X, see [77].

X with $1 \le p \le \infty$ there exists a constant $K < \infty$ such that for each finite dimensional subspace $B \subset X$ there is a finite dimensional K-complemented subspace $E \subset X$ containing B such that $d(E, \ell_p^n) \le K$ where $n = \dim E$ [79].

Each complemented subspace of $L_1(\mu)$ is an $\mathcal{L}_1$-space but an $\mathcal{L}_1$-space need not be isomorphic to a complemented subspace of an $L_1(\mu)$ space ([11]). The dual to a separable infinite-dimensional $\mathcal{L}_1$-space is isomorphic to ℓ_∞ ([77]). A Banach space X is an $\mathcal{L}_1$-space if and only if X^* is an $\mathcal{L}_\infty$-space ([79]). A complemented subspace of an $\mathcal{L}_1$-space is an $\mathcal{L}_1$-space (J. M. F. Castillo, private communication; this answers to a question of J. Lindenstrauss and A. Pełczyński ([77])). There are uncountable many non-isomorphic $\mathcal{L}_1$-spaces ([49]), and the class of all separable $\mathcal{L}_1$-spaces not containing an isomorphic copy of L_1 has no universal element ([11]). There exists an $\mathcal{L}_1$-space with the Radon-Nikodym property failing the Schur property ([49]).

Whether some natural subspaces are isomorphic to L_1? After constructing an uncomplemented subspace of L_1 isomorphic to ℓ_1, J. Bourgain asked whether the subspace spanned by the complement to the Rademacher functions in the Walsh system is isomorphic to L_1 or, at least is an $\mathcal{L}_1$-space. The answer to both questions is negative, as shown by N. J. Kalton and A. Pełczyński ([66]). Before ([66]) appeared, G. Schechtman (private communication) asked whether the subspace of L_1 spanned by a «half» of the Haar system $\{h_{2^{2k}+l} : \ k \in \{0,1,2,\ldots\},\ l \in \{1,2,\ldots,2^{2k}\}\}$ is isomorphic to L_1 (a standard technique shows that it is uncomplemented). The paper [66] leads us to believe that, very probably, the answer is no.

4. Near isometric theory.

Should «well» complemented subspaces of L_1 be «well» isomorphic to $L_1(\nu)$-spaces? As it was mentioned in the previous section, by Douglas' theorem, a subspace $X \subset L_1$ is isometric to an $L_1(\nu)$ space if and only if there is a projection of L_1 onto X of norm one. Most questions of this section have a common direction: to replace words «isometry» and «norm one» by «$(1+\delta)$-isomorphism for $0 < \delta \le \delta_0$» and «norm $\le 1+\varepsilon$ for $0 < \varepsilon \le \varepsilon_0$» respectively in this theorem. The Dor theorem (see below) gives an almost isometric version of «half» of the Douglas theorem. The second «half» constitutes the following problem of D. E. Alspach and W. B. Johnson.

Problem 7. ([4]) *(i) Does there exist a $\delta > 0$ such that every separable $\mathcal{L}_{1,1+\varepsilon}$-space is isomorphic to an $L_1(\nu)$ for $\varepsilon \le \delta$? If yes then does the isomorphism constant tend to 1 as $\varepsilon \to 0$?*

(ii) Does there exist a $\delta > 0$ such that every $(1+\varepsilon)$- complemented subspace of L_1 is isomorphic to an $L_1(\nu)$ for $\varepsilon \le \delta$? If yes then does the isomorphism constant tend to 1 as $\varepsilon \to 0$?

The authors proved that (ii) implies (i). Precisely, there is a constant K such that if X is an $\mathcal{L}_{1,1+\varepsilon}$-subspace of L_1 ($\varepsilon \le 0.1$) then X is $(1 + K\varepsilon|log\varepsilon|)$-complemented in L_1 ([4]).

A related result to (ii) is that if P is a projection of L_1 onto X with $\|I - P\| < 2$ then X is isomorphic to L_1 ([100]).

Near isometric stability of complemented subspaces. An interesting question naturally arises from the isometric stability theorem of B. Randrianantoanina (see Section 3).

Problem 8. [B. Randrianantoanina] *Does there exists an $\varepsilon_1 > 0$ such that whenever X is a complemented subspace of L_1 and Y is a subspace of L_1 with $\mathrm{d}(Y, X) < 1+\varepsilon$ with $\varepsilon \le \varepsilon_1$,* *then Y is complemented? If yes, how is the projection constant of Y related to that of X? Maybe, ε_1 depending on the projection constant K of X?*

Distortion type properties. By the well known James theorem, if a Banach space E is isomorphic to ℓ_1 then for each $\varepsilon > 0$ there exists a subspace $E_0 \subseteq E$ which is $(1+\varepsilon)$-isomorphic to ℓ_1 ([47]). The same is not true for L_1: for each $\lambda > 1$ there exists a Banach space B isomorphic to L_1 which contains no subspace λ-isomorphic to L_1 ([78]). But this does not happen for subspaces of L_1: let X be a subspace of L_1 isomorphic to L_1. Then for each $\varepsilon > 0$, there exists a further subspace $Y \subset X$ which is $(1+\varepsilon)$-isomorphic to L_1 ([111]).

Isometric and almost isometric copies of ℓ_1 and ℓ_1^n in L_1. A normalized sequence $(x_n)_{n=1}^N \subset L_1$ (N is finite or ∞) is isometrically equivalent to the unit vector basis of ℓ_1^N if and only if its elements have disjoint supports ([96]). A theorem of L. E. Dor ([26]) is as follows: let $(x_n)_{n=1}^\infty \subset L_1$ and $\theta \in (0,1]$ be such that for each integer n and each finite collection of scalars $(a_k)_{k=1}^n$ we have that

$$\left\| \sum_{k=1}^n a_k x_k \right\| \geq \theta \sum_{k=1}^n |a_k|.$$

Then there is a disjoint sequence $\{A_n\}_{n=1}^\infty$ of measurable subsets of $[0,1]$ such that for each n

$$\|x_n|_{A_n}\| \stackrel{def}{=} \int_{A_n} |x_n| d\mu \geq \theta^2.$$

5. Bases, basic sequences and decompositions.

General properties of bases for L_1. A uniformly bounded normalized sequence in L_1 could not be a basis for L_1. This was proved (assuming, in addition, orthogonality) in [92]. The general result appears in [72]. To the best of our knowledge, the strongest result of the type is Szarek's theorem which asserts that each normalized basis in L_1 contains a subsequence equivalent to the unit vector basis of ℓ_1 (or equivalently, is not equi-integrable) ([116]). Szarek asked:

Problem 9. [116] *Does there exist a normalized basis of L_1 which can be split into two subsequences, one of which forms a relatively weakly compact set and second is equivalent to the unit vector basis of ℓ_1?*

We denote the L_1-normalized Haar system by $(h_n)_{n=1}^\infty$. This basis for L_1 has the following property: $\lim_n \mu(\operatorname{supp} h_n) = 0$. Nevertheless, there are infinitely many mutually non-permutatively-equivalent normalized bases without the above property ([68]). In particular, the Olewskii system ([93]) is such a basis. A. Plichko and E. Tokarev have constructed a basis for L_1 which contains no almost disjoint subsequence[6] ([98]).

L_1 has no boundedly complete basis ([25, p.66]). According to [44], a semi-normalized basis $(x_n)_{n=1}^\infty$ of a Banach space X is said to be *monotonically boundedly complete* if, given a sequence of scalars $(a_n)_{n=1}^\infty$ with $a_n \searrow 0$, if $\sup_n \left\| \sum_{i=1}^n a_i x_i \right\| < \infty$ then $\sum_{n=1}^\infty a_n x_n$ converges.

[6] a sequence (y_n) of nonzero vectors in an r.i. function space E is called *almost disjoint* if there is a disjoint sequence (x_n) in E with $\|x_n - y_n\|/\|y_n\| \longrightarrow 0$ as $n \to \infty$.

Recently V. M. Kadets solved a problem of J. R. Holub by proving that the Haar system is a monotonically boundedly complete basis of L_1 ([56]).

Special bases. The Haar system is a precisely reproducible basis for L_1, i.e. if (e_n) is a basis in a Banach space X containing L_1 isometrically then for each $\varepsilon > 0$ there is a block-basis of (e_n) which is $(1+\varepsilon)$-equivalent to (h_n) ([94], [78]). This theorem is valid for L_1 replaced by any separable rearrangement invariant space E on $[0,1]$ ([81, p.158]). The trigonometric and the Walsh system are Markushevich bases for L_1 but do not form Shauder bases. The Walsh system is a Cesàro basis for L_1 ([89]).

Bases for subspaces. Every separable $\mathcal{L}_1$-space has a basis. In particular, every complemented subspace of L_1 has a basis ([53]). Every subspace of L_1 with an unconditional basis is isomorphic to a complemented subspace of a subspace of L_1 with a symmetric basis [88].

L_1 and spaces with unconditional bases. A classical theorem of A. Pełczyński is that L_1 does not embed in a Banach space with an unconditional basis ([97]). Moreover, L_1 does not G_δ-embed in a Banach space with an unconditional basis ([108]). This result was extended in the sense of the following definition. An injective operator $T \in \mathcal{L}(L_1, X)$ is called a sign-embedding if there is a $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for every sign $x \in L_1$ (i.e. x takes values only from $\{-1, 0, 1\}$ and $x \neq 0$) ([109], [110])[7].

The following theorem was stated in [109] and proved in an unpublished paper of H. P. Rosenthal «A new stopping time Banach space». A proof appears in [57]: L_1 does not sign-embed in a Banach space with an unconditional basis. This theorem is stronger than the previous since if L_1 G_δ-embeds in a Banach space X then L_1 sign embeds in X (see Section 8).

Non-existence of unconditional decompositions. A theorem of J. Lindenstrauss and A. Pełczyński states that if an $\mathcal{L}_1$-space X has an unconditional decomposition $X = \sum_{i\in I} X_i$, then X is isomorphic to $\left(\sum_{i\in I} X_i\right)_{\ell_1}$ ([77]). The following theorem, due to N. J. Kalton, P. Enflo and T. Starbird ([31]), is a generalization of Enflo's theorem that L_1 is primary. If L_1 is isomorphic to an unconditional sum of Banach spaces $\sum_{n=1}^{\infty} X_n$ then L_1 is isomorphic to X_n for some n. Another generalization was obtained by V. Kadets and R. Shvidkoy: each bounded below operator $T \in \mathcal{L}(L_1, X)$ cannot be represented as a point-wise unconditionally convergent series of compact operators ([60]).

Normalized weakly null and unconditional basic sequences. The family of all unconditional basic sequences in L_1 contains a complementably universal element ([114]). B. Maurey and H. P. Rosenthal have constructed a Banach space and an example of a normalized weakly null sequence with no unconditional subsequence ([87]). In that paper they asked whether such a sequence exists in L_1. Recently W. B. Johnson, B. Maurey and G. Schechtman answered this by constructing a normalized weakly null sequence in L_1 without an unconditional subsequence ([51]).

An unconditional basic sequence problem for non-separable spaces $L_1(\mu)$.

Problem 10. ([100, p.38]) *Does every subspace $X \subseteq L_1(\mu)$ contain an unconditional basic sequence of cardinality* dens$X$*?*

6. Operators from L_1.

[7] In this definition the injectivity assumption for T is essential.

Factorization theorems. There are a lot of theorems, saying that if a Banach space X is not similar (in some sense) to L_1 then each operator from L_1 to X is «small». For example, the classical Grothendieck theorem ([24, p. 63]) states that every operator from L_1 to ℓ_2 is absolutely summing[8]. Another classical result due to D. R. Lewis and C. Stegall ([75], [25, p. 66]) asserts that a Banach space X has the RNP if and only if each $T \in \mathcal{L}(L_1, X)$ factors through ℓ_1, i.e. there are an $S \in \mathcal{L}(L_1, \ell_1)$ and an $U \in \mathcal{L}(\ell_1, X)$ such that $T = U \circ S$.

Problem 11. ([77]) *Let $1 < s < p \leq \infty$. Can every $T \in \mathcal{L}(L_p, L_1)$ be factored through L_s?*

Kalton's Representation Theorem. A beautiful representation theorem for opera-tors in $\mathcal{L}(L_1)$ was obtained by N. J. Kalton ([63]): for each $T \in \mathcal{L}(L_1)$ there exists a weak*-measurable map $\nu : [0,1] \to \mathcal{M}$ where $\mathcal{M}$ is the space of all regular Borel measures on $[0,1]$ such that

$$Tx(t) = \int x(\tau) d\nu_t(\tau)$$

a.e. on $[0,1]$. Moreover,

$$\|T\| = \sup_{\mu(B)>0} \frac{1}{\mu(B)} \int |\nu_t|(B) d\mu.$$

Compact and weakly compact operators. An operator $T \in \mathcal{L}(L_1, X)$ is compact if and only if the restriction of T to the positive face $\mathcal{F} = \{x \in L_1 : x \geq 0 \text{ and } \|x\| = 1\}$ is uniformly weak-to-norm continuous [34, p. 58]. If $T \in \mathcal{L}(L_1)$ is weakly compact then T^2 is compact [25, p. 79]. Every weakly compact operator from L_1 to a symmetric sequence space is regular[9] [46]. A Banach space X has the Schur property if and only if each weakly compact operator from L_1 to X is compact [48].

7. Representable operators and the RNP.

Preliminary observations. According to [25, p.61], an operator $T \in \mathcal{L}(L_1, X)$ is said to be representable[10] if there exists $y \in L_\infty(X)$ such that

$$Tf = \int_{[0,1]} xy d\mu$$

for all $x \in L_1$. For a Banach space X the following properties are equivalent: (i) X has the RNP; (ii) every operator $T \in \mathcal{L}(L_1, X)$ is representable. For each Banach space X every weakly compact operator $T \in \mathcal{L}(L_1, X)$ is representable ([25, pp.75,77]). The set $\mathcal{R}$ of all representable operators from $\mathcal{L}(L_1)$ is a projection band in $\mathcal{L}(L_1)$ ([82]).

Dunford-Pettis operators. Recall that an operator $T \in \mathcal{L}(X, Y)$ is called a Dun-ford-Pettis operator (they also called completely continuous operators) provided T maps weak convergent sequences to norm convergent sequences. Of course, if either X or Y has the Shur property then each operator from $\mathcal{L}(X, Y)$ is Dunford-Pettis. For a reflexive space X the set of all Dunford-Pettis operators coincides with the compact operators space $\mathcal{K}(X, Y)$. The ideal of all Dunford-Pettis operators from $\mathcal{L}(L_1)$ is a sublattice ([8]). Every representable (and therefore, every compact, ([25, p.68])) operator from $\mathcal{L}(L_1, X)$ is Dunford-Pettis ([25, p.74]).

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[8] i.e maps unconditionally convergent series to absolutely convergent.
[9] i.e. equals a difference of two positive operators.
[10] or, Radon-Nikodým operator.

The converse is not true: the Volterra operator $V \in \mathcal{L}(L_1, C[0,1])$ defined as $(Vx)(t) = \int_{[0,t]} x d\mu$ for $0 \le t \le 1$ is Dunford-Pettis but not representable ([74]). Since every weakly compact operator from $\mathcal{L}(L_1, X)$ is representable ([25, p.75]), the Volterra operator also provides an example of a Dunford-Pettis operator which is not weakly compact. The following characterization of Dunford-Pettis operators with the domain space L_1 which can be deduced from [107] is proved in [8]: for $1 < p \le \infty$ denote by I_p the injection operator from L_p to L_1. Let X be a Banach space. For an operator $T \in \mathcal{L}(L_1, X)$ the following assertions are equivalent.

(i) T is Dunford-Pettis;

(ii) $T \circ I_p$ is compact for some $1 < p \le \infty$;

(iii) $T \circ I_\infty$ is compact.

An operator $T \in L_1 \to c_0$ is Dunford-Pettis if and only if it is regular ([45]). For some more facts on the connection between regular and Dunford-Pettis operators from L_1 to a symmetric sequence spaces we refer to [46]. The following theorem of J. Bourgain asserts that in the definition of the RNP via representable operators one may consider only Dunford-Pettis operators: a Banach space X has the RNP if and only if each Dunford-Pettis operator $T \in \mathcal{L}(L_1, X)$ is representable ([8]). There is a Banach space X without the RNP such that each operator $T \in \mathcal{L}(L_1, X)$ is Dunford-Pettis ([12]). Moreover there is a weakly sequentially complete Banach lattice with these properties ([18]).

Non-Dunford-Pettis operators and biased-coin convolution operators. Recall that an operator $T \in \mathcal{L}(X, Y)$ is strictly singular provided T fixes no copy of any infinite dimensional Banach space Z, i.e. the restriction $T|_{X_0}$ to each infinite dimensional subspace $X_0 \subset X$ is not an isomorphic embedding. Of course, each compact operator is strictly singular. Each strictly singular operator $T \in \mathcal{L}(L_1)$ is Dunford-Pettis. In fact, the following two theorems confirm that. First, H. P. Rosenthal proved that any non-Dunford-Pettis operator $T \in \mathcal{L}(L_1)$ fixes a copy of ℓ_2 ([107]). Then J. Bourgain obtained more: any non-Dunford-Pettis operator $T \in \mathcal{L}(L_1)$ fixes a copy of $\bigoplus_{\ell_1}(\ell_2)$ ([9]). One could not obtain much more in this direction: a non-Dunford-Pettis operator $T \in \mathcal{L}(L_1)$ need not fix a copy of any reflexive subspace of L_1 but ℓ_2. A counterexample was constructed by Rosenthal: for an integer $k \ge 0$ denote by W_k the set of all products of k distinct Rademacher functions on $[0,1]$ (W_0 consists of one function $\chi_{[0,1]}$). Then for each $\varepsilon > 0$ there exists a unique operator $T \in \mathcal{L}(L_1)$ such that $Tw = \varepsilon^k w$ for all integers k and $w \in W_k$ ([107]). The operator T from this theorem is called the *ε-biased-coin convolution operator*. Being evidently non-Dunford-Pettis, the ε-biased-coin convolution operator fixes no copy of L_1, nor even any reflexive subspace of L_1 other than ℓ_2. More precisely, let Z be a reflexive subspace of L_1 not isomorphic to ℓ_2. Then there exists $\varepsilon_0 \in (0,1)$ such that the ε-biased-coin convolution operator fixes no copy of Z for any $\varepsilon \in (0, \varepsilon_0]$ ([107]). In particular, this yields that for $\varepsilon \in (0, \varepsilon_0]$, the ε-biased-coin convolution operator is not an E-operator. In fact, this is true for every $\varepsilon \in (0,1)$ ([107]).

The Dunford-Pettis property. A Banach space X is said to have the *Dunford-Pettis property* if for each Banach space Y every weakly compact operator $T \in \mathcal{L}(X, Y)$ is Dunford-Pettis. Of course, L_1 has the Dunford-Pettis property. Applying one of the main result of [66] to L_1, we obtain for the Walsh system $(w_n)_{n=1}^\infty$ the following result. Suppose that for a set of the integers I the subsequence $(w_n)_{n \in I}$ in L_∞ is equivalent to the unit vector basis of ℓ_1 (for example, the Rademacher system). Then the subpace of L_1 spanned by $(w_n)_{n \notin I}$ has the Dunford-Pettis property but is not an $\mathcal{L}_1$-space ([66]).

Problem 12. ([66]) *Suppose X is a subspace of L_1 with L_1/X isomorphic to c_0. Does X have the Dunford-Pettis property?*

The L_1-singular and the Dunford-Pettis operators. Let X, Y and Z be Banach spaces. Recall that an operator $T \in \mathcal{L}(X,Y)$ *fixes a copy of* Z if there is a subspace X_0 of X isomorphic to Z such that the restriction $T|_{X_0}$ is an isomorphic embedding. If an operator $T \in \mathcal{L}(X,Y)$ does not fix a copy of Z then we say that T is *Z-singular*.

The following problem one may consider as a weak version of Problem 5.

Problem 13. ([107], [31] and [9]) *Is every L_1-singular projection $P \in \mathcal{L}(L_1)$ a Dunford-Pettis operator?*

8. Weak embeddings and narrow operators. We discuss some notions of «weak» embeddings of Banach spaces such as semi-embed-dings, G_δ-embeddings and sign-embed-dings. Two of these were already considered in Sections 1 and 5.

Semi-embeddings. An injective operator $T \in \mathcal{L}(X,Y)$ is called a semi-embedding if $TB(X)$ is closed ([83]). One of the main result in [14] states that if a separable Banach space semi-embeds into a Banach space with the RNP then it has the RNP. Hence, L_1 does not semi-embed into a Banach space with the RNP. But for L_1 one can trivially obtain the following stronger result from the definitions: L_1 does not semi-embed into a Banach space with the Krein-Milman property.

Semi-, G_δ- and sign-embeddings of L_1. The connections between these three notions could be described as follows. Every semi-embedding is a G_δ-embedding ([14]); the converse is not true. There is a semi-embedding (hence, a G_δ-embedding) of L_1 into L_1 that is not a sign-embedding and there is a sign-embedding of L_1 into L_1 which is not a semi-embedding (hence, is not a G_δ-embedding ([90])).

It is said that L_1 semi-embeds (or sign-embeds, or «other type» embeds) into a Banach space X provided there exists a semi-embedding (sign-embedding, or respectively «other type» embedding) $T \in \mathcal{L}(L_1, X)$. If L_1 G_δ-embeds into a Banach space X then L_1 also sign-embeds into X. More precisely, if $T \in \mathcal{L}(L_1, X)$ is an injective operator such that $T\mathcal{P}$ is a G_δ set then L_1 sign-embeds into X, where $\mathcal{P} = \{x \in B_{L_1} : \ x \geq 0\}$ is the positive face of the unit ball of L_1 ([109]). An obvious question - whether the existence of «weak» embeddings are in fact weaker than the existence of isomorphic embeddings - was posed in [14] for semi- and G_δ-embeddings and in [110] - for sign-embeddings. A complete answer to all of them was obtained by M. Talagrand in [118], where the author constructed a Banach space X into which L_1 semi-embeds (hence, G_δ-embeds and sign-embeds) but does not embed isomorphically. But for large enough class of spaces X the existence of a «weak» embedding of L_1 into X implies the the existence of an isomorphic embedding. In [14] it is shown that if L_1 G_δ embeds into X where X either is isomorphic to a dual space, or itself embeds into L_1, then L_1 embeds isomorphically into X. More in this direction is obtained in [35]: for the smallest class $\mathcal{G}$ of separable Banach spaces closed under G_δ-embeddings and containing L_1, if L_1 sign-embeds into $X \in \mathcal{G}$ then L_1 isomorphically embeds into X (in particular, $\mathcal{G}$ contains all separable duals). It seems to be unknown whether the existence of these three notions of weak embeddings are in fact distinct.

Problem 14. *A) Suppose that L_1 sign-embeds into X. Does L_1 G_δ-embed into X?*

B) Suppose that L_1 G_δ-embeds into X. Does L_1 semi-embed into X?

C) *Suppose that L_1 sign-embeds into X. Does L_1 semi-embed into X?*

The three space property. Sign-embeddings of L_1 have the following three-space property: if L_1 sign embeds into X and Y is a subspace of X then L_1 sign embeds either into Y or into X/Y ([109]). Thus, from the above Rosenthal's results we obtain that if X is a subspace of L_1 then either L_1 embeds into X or L_1 sign-embeds into L_1/X. In [118] it is proved that the isomorphic embeddings of L_1 do not have the three space property. To the best of our knowledge, it is not known whether semi- or G_δ-embeddings have the three-space property.

Problem 15. A) *Suppose that L_1 G_δ-embeds into a Banach space X and let Y be a subspace of X. Does L_1 G_δ-embed either into Y or into X/Y?*

B) *Suppose that L_1 semi-embeds into a Banach space X and let Y be a subspace of X. Does L_1 semi-embed either into Y or into X/Y?*

Weak embeddings of L_1 fix copies of ℓ_1. Each sign-embedding and each G_δ-embedding (hence, each semi-embedding) $T : L_1 \to X$ fixes a copy of ℓ_1 ([14]).

Images of the positive face. By $\mathcal{P}$ we denote the set of all non-negative elements of S_{L_1}. Suppose $T \in \mathcal{L}(L_1, c_0)$. If T is injective then $T\mathcal{P}$ is not G_δ. If $T|_\mathcal{P}$ is injective then $T\mathcal{P}$ is not closed ([14]). Let X be a Banach space and $T \in \mathcal{L}(L_1, X)$ be such that $T|_\mathcal{P}$ is injective and $T\mathcal{P}$ is closed. Then T fixes a copy of ℓ_1. If L_1 semi-embeds into X, then there exists an injective operator $T \in \mathcal{L}(L_1, X)$ such that $T\mathcal{P}$ is closed. The authors asked whether the converse holds.

Problem 16. [14] *Suppose that $T \in \mathcal{L}(L_1, X)$ is such that $T\mathcal{P}$ is closed. Must T be a semi-embedding?*

Narrow operators. The following notion is closely related to sign-embeddings. An operator $T \in \mathcal{L}(L_1, X)$ is called *narrow* if for every measurable set $A \subseteq [0, 1]$ and every $\varepsilon > 0$ there is an $x \in L_1$ such that $x^2 = \chi_A$, $\int x d\mu = 0$ and $\|Tx\| < \varepsilon$ ([100], [101]). Obviously, the notions of sign-embeddings and narrow operators are mutually exclusive, but $Tx = x\chi_{[0,\frac{1}{2}]}$ is an example of an operator in $\mathcal{L}(L_1)$, which is neither a sign-embedding nor narrow. On the other hand, for a Banach space X the following two assertions are equivalent: (i) L_1 does not sign-embed into X; (ii) every operator $T \in \mathcal{L}(L_1, X)$ is narrow. Indeed, (ii) trivially implies (i); the converse constitutes Lemma 3 of [110]. It is not hard to see that compact, absolutely summing, Dunford-Pettis, representable operators are narrow (see [100], [58]). Let $T \in \mathcal{L}(L_1, X)$ be any of these types of operators. Given $A \in \Sigma^+$ and $\varepsilon > 0$, consider a Rademacher system (r_n) on $L_1(A)$. Since $\liminf_n \|Tr_n\| = 0$ then there exists an $n \geq 1$ with $\|Tr_n\| < \varepsilon$. But there is a narrow projection in L_1 which is not L_1-singular ([100, p.57]). A G_δ-embedding cannot be narrow. This nice and deep result follows from Theorem II.11 of N. Ghoussoub and H. P. Rosenthal ([35]). Note that this fact can also be deduced from Lemma 2.1 of [14]. This yields the above fact that if L_1 G_δ-embeds into X then L_1 sign-embeds into X. If $T \in \mathcal{L}(L_1, X)$ fixes no copy of ℓ_1 then T is narrow ([14]).

Problem 17. ([100]) *Let E be a reflexive subspace of L_1. Suppose that $T \in \mathcal{L}(L_1, X)$ fixes no copy of E. Does it follow that T is narrow?*

This is unknown for any E.

Ideal properties of narrow operators. The set $\mathcal{N}$ of all narrow operators from $\mathcal{L}(L_1)$ is a projection band ([86]). The composition of a narrow operator (from the right) with a bounded operator (from the left) is trivially narrow and so $\mathcal{N}$ is a left-side ideal. Since there is a narrow projection P of L_1 onto a subspace PL_1 isomorphic to L_1 (see above) then the composition $P \circ T$ of P with an isomorphism $T : L_1 \to PL_1$ is not narrow.

Problem 18. ([59]) *For each Banach space X, is the sum of two narrow operators from L_1 to X also narrow?*

The Daugavet property.

The DP. A classical result of I. K. Daugavet ([21]) states that for every compact $T \in \mathcal{L}(C[0,1])$ the following equation is satisfied:

$$\|I + T\| = 1 + \|T\|. \tag{DE}$$

(DE) holds also for compact operators $T \in \mathcal{L}(L_1)$ (G. Ya. Lozanovskii ([84])). A Banach space X is said to have the *DP* (or more generally, the *DP for a class of operators* $\mathcal{M} \subseteq \mathcal{L}(X)$) provided that (DE) is satisfied for all compact $T \in \mathcal{L}(X)$ (for all $T \in \mathcal{M}$). X has the DP if and only if X has the DP for all rank 1 operators on X ([61]). That L_1 has the DP for all narrow operators is proved in [100, p. 68].

A new notion of narrow operators. In [62] V. M. Kadets, R. V. Shvidkoy and D. Werner introduced a new notion of narrow operators acting from a Banach space with the DP. Using Proposition 3.11 from [62], this can be formulated as follows. Let X be a Banach space possessing the DP and let Y be any Banach space. An operator $T \in \mathcal{L}(X,Y)$ is called *KSW-narrow* if for every $x, y \in S(X)$, $\varepsilon > 0$ and every slice $S(x^*, \varepsilon_1) = \{x \in B(X) : x^*(x) \geq 1 - \varepsilon_1\}$ containing y there is $v \in S(x^*, \varepsilon_1)$ such that $\|x + v\| > 2 - \varepsilon$ and $\|T(y - v)\| < \varepsilon$. A Banach space X with the DP has the DP for the class $\mathcal{N}_{KSW}(X)$ of all KSW-narrow operators on X which, in particular, contains all ℓ_1-singular operators from $\mathcal{L}(X)$, i.e. operators fixing no copy of ℓ_1 ([62]).

In [62, Theorem 6.1] (see also [115]) it is proved that for operators T from L_1 to a Banach space X the new notion of narrow operators is equivalent to the following one. An operator $T \in \mathcal{L}(L_1, X)$ is called KSW-narrow if for each $\varepsilon > 0$, $\varepsilon_1 > 0$ and every measurable $A \subset [0,1]$ there exists an $x \in L_1(A)$ such that: (i) $x(t) \geq -1$ a.e. on A; (ii) $\int x d\mu = 0$; (iii) $\mu\{t \in A : x(t) > -1\} \leq \varepsilon_1$; (iv) $\|Tx\| \leq \varepsilon$. It is easily verified that every narrow operator is KSW-narrow. But as was mentioned in [62] and [115], the converse is unknown.

Problem 19. ([62]) *Is every KSW-narrow operator from L_1 to X narrow?*

The answer is yes if X itself embeds into L_1. On the other hand, the answer is evidently affirmative for Banach spaces X for which every operator $T \in \mathcal{L}(L_1, X)$ is narrow. In the setting of both definitions, the sum of two narrow operators in $\mathcal{L}(L_1)$ is narrow [115].

Problem 20. ([59]) *For any Banach space X, is the sum of two KSW-narrow operators from $\mathcal{L}(L_1, X)$ also KSW-narrow (cf. Problem 18)?*

9. Tauberian operators and quotients of L_1 by reflexive subspaces. Tauberian operators stemmed in the mid 70's from summability theory ([67]), Fredholm theory ([121]) and factorization of operators ([20]. [39]). Following a definition of Kalton and Wilansky

([67]), an operator $T\colon X \longrightarrow Y$ is said to be *Tauberian* if T^{**} maps $X^{**} \setminus X$ into $Y^{**} \setminus Y$. Recall that if T^{**} maps $X^{**} \setminus X$ into Y, then T is *weakly compact*. Obviously, Tauberian operators and weakly compact operators exhibit a very opposite type of behavior except in the trivial case when their domain is a reflexive space.

It is immediate that any closed range operator with reflexive kernel is Tauberian. Therefore, for every reflexive subspace R of L_1, the quotient operator from L_1 onto L_1/R is Tauberian. Hence, L_1 is an important source of Tauberian operators since it contains many reflexive subspaces.

In the case when X is weakly sequentially complete, we have that an operator T is Tauberian if and only if $N(T) = N(T^{**})$. In the particular case when $X = L_1$, we have more precise characterizations.

Theorem 9.1. ([38]) *Let μ be a finite, non-purely atomic measure. For every operator $T \in \mathcal{L}(L_1(\mu), Y)$, the following statements are equivalent:*

(i) *T is Tauberian;*

(ii) $N(T) = N(T^{**})$;

(iii) $\liminf_n \|Tf_n\| > 0$ *for all normalized disjointly supported sequences (f_n) in $L_1(\mu)$;*

(iv) *for every normalized sequence (f_n) in $L_1(\mu)$ such that $\mu(\mathrm{supp} f_n) \underset{n}{\longrightarrow} 0$, we have* $\liminf_n \|Tf_n\| = 0$;

(v) *there exists $r > 0$ such that $\liminf_n \|Tf_n\| > r$ for all normalized disjointly supported sequences (f_n) in $L_1(\mu)$;*

(vi) *there exists $r > 0$ such that for every norm one function $f \in L_1(\mu)$ with $\mu(\mathrm{supp} f) < r$ we have $\|Tf\| > r$.*

The interest in Tauberian operators on $L_1(\mu)$ was raised by Weis and Wolff ([120]), who were mainly concerned with the study of the weak Calkin algebra $\mathcal{L}(L_1)/\mathcal{W}(L_1)$. They posed the following question.

Problem 21. *Is every Tauberian operator $T\colon L_1 \longrightarrow L_1$ a semi-Fredholm operator?*

Of course, a positive answer to Problem 21 means a negative answer to Problem 2. Several facts suggest that Problem 21 might admit a positive solution. Indeed, let $\Phi(X, Y)$ and $\mathcal{T}(X, Y)$ denote respectively the collections of all upper semi-Fredholm operators and all Tauberian operators acting between X and Y. It is well known that all the classes $\Phi(X, Y)$ are open, and that the $2n$-conjugates of the operators in $\Phi(X, Y)$ are also upper semi-Fredholm. In general, we cannot say the same about the classes $\mathcal{T}(X, Y)$ ([2]). Nevertheless, all components $\mathcal{T}(L_1, Y)$ are open, and $T \in \mathcal{L}(L_1, Y)$ is Tauberian provided $T^{*(2n)}$ is Tauberian ([117]). This last fact can be also deduced from operator finite representability techniques ([85]) and from the fact that every ultrapower of a Tauberian operator $T\colon L_1 \longrightarrow Y$ is also Tauberian ([36]).

An immediate consequence of Theorem 9.1 is that for every $T \in \mathcal{T}(L_1, Y)$ and every measurable subset A of $[0, 1]$ with $\mu(A) > 0$, there is a measurable subset C of A with $\mu(C) > 0$ such that $T|_{L_1(C)}$ is an isomorphism. Therefore, the class $\mathcal{T}(L_1, Y)$ is not empty if and only if Y contains a subspace isomorphic to L_1. In particular, if R is a reflexive subspace of L_1 then L_1/R contains a subspace isomorphic to L_1. Moreover, for each Tauberian operator $T \in \mathcal{L}(L_1, Y)$ there exists a finite partition of of $[0, 1]$ into measurable sets, $\{\Omega_j\}_{j=1}^{n}$, such that all the restrictions $T|_{L_1(\Omega_j)}$ are isomorphisms ([38]).

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