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AN ANALOG OF POISSON-JENSEN FORMULA FOR ANNULI

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In this paper we prove an analog of the Poisson-Jensen theorem for meromorphic functions on annuli.

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Доказывается аналог теоремы Пуассона-Иенсена для функций мероморфных в плоском круговом кольце.

1. Introduction. The value distribution of meromorphic functions in multi-connected domains was studied by many authors ([1-10]). Recently Kondratyuk A.A. and Khrystiyanyn A. Ya. ([19], [20]) have proposed a new approach to the Nevanlinna value distribution theory for meromorphic functions in doubly connected domains. They obtained an analog of the Jensen theorem, introduced the Nevanlinna and Ahlfors-Shimizu characteristics, studied their properties and proved the First and Second Fundamental Theorems for annuli.

By the Doubly Connected Mapping Theorem ([11]) each doubly connected domain is conformally equivalent to an annulus $\{z : r < |z| < R\}$, $0 \le r < R \le +\infty$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 < r < R < +\infty$. In the last case the homothety $z \longmapsto z/\sqrt{rR}$ reduces the given domain to the annulus $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R}{r}}$. So, in two cases every annulus is invariant with respect to the inversion $z \longmapsto \frac{1}{z}$.

2. Definitions and notations. For $t \in (1, R_0)$, $\tau \in (1/R_0, 1)$, $1 < R_0 \le \infty$ we denote $A^t = \{z : 1 < |z| < t\}$, $A_\tau = \{z : \tau < |z| < 1\}$, $A(R) = \{z : 1/R < |z| < R\}$, $R < R_0$.
Let f be a meromorphic function on $A = \{z : 1/R_0 < |z| < R_0\}$, $1 < R_0 \le +\infty$. We denote $n_0(t, f) = n(t, 1/f) - n(t, f)$, where $n(t, f) = \sum_{b_\nu \in A(t)} 1$ and $\{b_\nu\}$ are the poles of f in A and for $R \in [1, R_0)$

$$m(R, f) = m(R, \infty) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\theta})|d\theta, \quad m_0(R, f) = m(R, f) + m\left(\frac{1}{R}, f\right),$$

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$$N(R,f)=\int\limits_1^R \frac{n(t,f)}{t}dt.$$

The function

$$T_0(R,f)=m_0(R,f)-2m(1,f)+N(R,f),\quad 1\le R<R_0,$$

we call *the Nevanlinna characteristic* of f in annulus. The function $T_0(R,f)$ is nonnegative, continuous, nondecreasing and convex with respect to $\log R$. These facts have been proved in [19].

If f is a holomorphic function on A we denote $M(r,f)=\max\limits_{|z|=r}|f(z)|$ and

$$M_0(R,f)=\log^+ M(R,f)+\log^+ M(1/R,f)-2m(1,f),\quad 1\le R<R_0.$$

3. Main results. Let $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$.

Theorem 1. *Let $f(z)$ be a meromorphic function in A and $1<R<R_0\le\infty$. Let $\{a_\mu\}$ be the zeroes and $\{b_\nu\}$ be the poles of f in A. If $f(z)\ne 0,\infty$ $(z\in\mathbb{T})$ then for $z=re^{i\varphi}$ $(1/R<r<R)$ we have*

$$\begin{gathered}\log|f(z)|=\\ =\frac{1}{2\pi}\int\limits_0^{2\pi}\log|f(Re^{i\theta})|\mathcal{P}(r,R,\theta-\varphi)d\theta+\frac{1}{2\pi}\int\limits_0^{2\pi}\log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|\mathcal{P}(1,rR,\theta-\varphi)d\theta-\\ -\frac{1}{4\pi}\int\limits_0^{2\pi}\log|f(e^{i\theta})|\left(\mathcal{P}(r,R^2,\theta-\varphi)+\mathcal{P}(1,rR^2,\theta-\varphi)\right)d\theta-\\ -\frac{1}{2\pi}\int\limits_0^{2\pi}\left(\frac{\partial\log|f(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1}\log\left|\frac{R^2-e^{-i\theta}z}{\frac{1}{R^2}-e^{-i\theta}z}\right|d\theta-\sum_{a_\mu\in A^R}G_R(a_\mu,z)-\\ -\sum_{a_\mu\in A_{1/R}}G_{1/R}(a_\mu,z)+\sum_{b_\nu\in A^R}G_R(b_\nu,z)+\sum_{b_\nu\in A_{1/R}}G_{1/R}(b_\nu,z)+2k(f)\log R,\end{gathered}\tag{1}$$

where $k(f)=\frac{1}{2\pi i}\int_{|z|=1}\frac{f'(z)}{f(z)}dz\in\mathbb{Z}$, $G_d(\xi,z)=\log\left|\frac{d^2-\bar{\xi}z}{d(\xi-z)}\right|$, *and* $\mathcal{P}(x,X,T-t)=$ $=\dfrac{X^2-x^2}{X^2-2Xx\cos(T-t)+x^2}=\mathrm{Re}\,\dfrac{Xe^{iT}+xe^{it}}{Xe^{iT}+xe^{it}}$ *is the Poisson kernel.*

Theorem 2. *Let $f(z)$ be a holomorphic function in A and $1\le r<R<R_0\le\infty$. Then $M_0(r,f)$ possess the following properties: 1) $M_0(r,f)\ge T_0(r,f)\ge 0$;*
2) $M_0(r,f)$ is convex with respect to $\log r$; 3) $M_0(r,f)$ is nondecreasing;
4) If f has no zeroes on $\mathbb{T}$ then

$$\begin{gathered}M_0(r,f)\le\left(\frac{R+r}{R-r}+\frac{Rr+1}{Rr-1}\right)T_0(R,f)+2\left(\frac{R+r}{R-r}+\frac{Rr+1}{Rr-1}-1\right)m(1,f)+\\ +\frac{2(R^4-1)r}{(R^2-r)(R^2r-1)}\,m(1,1/f)+C(f)\log r,\quad C(f)=2\sup_{\theta\in[0,2\pi]}\left|\left(\frac{\partial\log|f(\rho e^{i\theta})|}{\partial\rho}\right)\right|\Bigg|_{\rho=1}.\end{gathered}\tag{2}$$

Corollary. *Let $f(z)$ be a holomorphic function in $\mathbb{C}\backslash\{0\}$, $f(z) \neq 0$ when $|z| = 1$. Then*

$$M_0(r,f) \leq 6T_0(2r,f) + \alpha(f)\log r + \beta(f),$$

where

$$\alpha(f) = 2 \sup_{\theta\in[0,2\pi]} \left|\left(\frac{\partial \log|f(\rho e^{i\theta})|}{\partial\rho}\right)\right|\Bigg|_{\rho=1}, \quad \beta(f) = \frac{5}{\pi}\int_0^{2\pi} \log^+|f(e^{i\theta})|d\theta + \frac{5}{3\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(e^{i\theta})|}d\theta.$$

This is a mere consequence of (2), when $R = 2r$. To prove Theorem 1 we need the following definitions and propositions from [18] and [21].

A path γ is a piecewise continuously differentiable map $\gamma\colon [0,1] \to \mathbb{C}$. Let $\gamma_i\colon [0,1] \to A$, $i \in \{1,2\}$ be closed paths. γ_1 and γ_2 are called *homotopic* in A if there exists continuous function $H(s,t)\colon [0,1]\times[0,1] \to A$ such that: 1) $H(s,0) = \gamma_1(s)$; 2) $H(s,1) = \gamma_2(s)$; 3)$\forall\, t \in [0,1] : H(0,t) = H(1,t)$.

Lemma A [18]. *For any $a \in \mathbb{C}$ and for any closed path $\gamma\colon [0,1] \to \mathbb{C}\backslash\{a\}$*

$$\mathrm{Ind}_\gamma(a) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = n \in \mathbb{Z}.$$

Corollary. *For any closed path γ in A, $\gamma(0) = 1$, there exists $n \in \mathbb{Z}$ such that γ is homotopic in A to $\gamma_0^n\colon [0,1] \to A(R)$, $\gamma_0^n(t) = e^{2\pi int}$, $t \in [0,1]$.*

Proof. By Lemma A

$$\mathrm{Ind}_\gamma(0) = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(s)}{\gamma(s)}ds = n \in \mathbb{Z}.$$

It is easy to verify that

$$H(s,t) = (t + (1-t)|\gamma(s)|)\exp\left\{ i(2\pi nst + (1-t)\,\mathrm{Im}\int_0^s \frac{\gamma'(\tau)}{\gamma(\tau)}d\tau)\right\}$$

is the homotopy between γ and γ_0^n. □

Lemma 1. *Let f be a holomorphic function in A, $f(z) \neq 0,\infty$ $(z \in A)$. Then for each closed path γ in A, $\gamma(0) = 1$ there exists $k \in \mathbb{Z}$ such that for $g(z) = z^{-k}f(z)$ we have*

$$\int_\gamma \frac{g'(z)}{g(z)}dz = 0.$$

Proof. Denote $\Gamma(t) = f(\gamma_0^1(t))$, $t \in [0,1]$. Then $\frac{1}{2\pi i}\int_{\gamma_0^1}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_\Gamma \frac{d\xi}{\xi} = k \in \mathbb{Z}$ and

$$\frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz = \frac{1}{2\pi i}\left(\int_\gamma \frac{f'(z)}{f(z)}dz - k\int_\gamma \frac{dz}{z}\right).$$

By the corollary of Lemma A there exists $n \in \mathbb{Z}$ such that γ is homotopic to γ_0^n. Since integrals over homotopic paths are equal, we see that

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{g'(z)}{g(z)}dz = \frac{1}{2\pi i}\int\limits_{\gamma_0^n}\frac{f'(z)}{f(z)}dz - \frac{k}{2\pi i}\int\limits_{\gamma_0^n}\frac{dz}{z}dz = nk - nk = 0.$$

□

The proved Lemma 1 allows us to determine the logarithm of $g(z) = z^{-k}f(z)$, where f satisfies the conditions of that lemma. Indeed, let $z_0 = 1$ and let $\log g(z_0)$ be determined. Put

$$\log g(z) = \log g(z_0) + \int\limits_{z_0}^{z}\frac{g'(\zeta)}{g(\zeta)}d\zeta,$$

where the integral is taken along a path joined z_0 and z in the domain A.

Proof of Theorem 1. **Case 1.** Assume that $f(z)$ has neither poles nor zeros in $A(R)$. Let $F(z) = \log g(z)$, $g(z) = z^{-k}f(z)$, where $k = k(f)$. Consider $z = re^{i\varphi}$, $\frac{1}{R} < r < R$.
By the Cauchy integral formula

$$\begin{aligned} F(z) &= \frac{1}{2\pi i}\left(\int\limits_{|\xi|=R}\frac{F(\xi)}{\xi - z}d\xi - \int\limits_{|\xi|=1/R}\frac{F(\xi)}{\xi - z}d\xi\right) = \\ &= \frac{1}{2\pi}\left(\int\limits_0^{2\pi}F(Re^{i\theta})\frac{Re^{i\theta}}{Re^{i\theta} - re^{i\varphi}}d\theta - \int\limits_0^{2\pi}F(\frac{1}{R}e^{i\theta})\frac{e^{i\theta}}{e^{i\theta} - Rre^{i\varphi}}d\theta\right). \end{aligned} \tag{3}$$

Since $z_1 = \frac{R^2}{\overline{z}} = \frac{R^2}{r}e^{i\varphi} \notin A^R$ and $z_2 = \frac{1}{R^2\overline{z}} = \frac{1}{R^2 r}e^{i\varphi} \notin A_{1/R}$, we see that

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\left(\int\limits_{|\xi|=R}\frac{F(\xi)}{\xi - z_1}d\xi - \int\limits_{|\xi|=1}\frac{F(\xi)}{\xi - z_1}d\xi\right) = \\ &= \frac{1}{2\pi}\left(\int\limits_0^{2\pi}F(Re^{i\theta})\frac{re^{i\theta}}{re^{i\theta} - Re^{i\varphi}}d\theta - \int\limits_0^{2\pi}F(e^{i\theta})\frac{re^{i\theta}}{re^{i\theta} - R^2e^{i\varphi}}d\theta\right) \end{aligned} \tag{4}$$

and

$$\begin{aligned} 0 &= \frac{1}{2\pi i}\left(\int\limits_{|\xi|=1}\frac{F(\xi)}{\xi - z_2}d\xi - \int\limits_{|\xi|=1/R}\frac{F(\xi)}{\xi - z_2}d\xi\right) = \\ &= \frac{1}{2\pi}\left(\int\limits_0^{2\pi}F(e^{i\theta})\frac{R^2re^{i\theta}}{R^2re^{i\theta} - e^{i\varphi}}d\theta - \int\limits_0^{2\pi}F\left(\frac{1}{R}e^{i\theta}\right)\frac{Rre^{i\theta}}{Rre^{i\theta} - e^{i\varphi}}d\theta\right). \end{aligned} \tag{5}$$

After subtracting (4) and (5) from (3) we have

$$F(z)=\frac{1}{2\pi}\int\limits_0^{2\pi}F(Re^{i\theta})\mathcal{P}(r,R,\theta-\varphi)d\theta+\frac{1}{2\pi}\int\limits_0^{2\pi}F\left(\frac{1}{R}e^{i\theta}\right)\mathcal{P}(1,rR,\theta-\varphi)d\theta+$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi}F(e^{i\theta})\left(\frac{re^{i\theta}}{re^{i\theta}-R^2e^{i\varphi}}-\frac{R^2re^{i\theta}}{R^2re^{i\theta}-e^{i\varphi}}\right)d\theta. \tag{6}$$

Denote

$$B(e^{i\theta},z)=\frac{re^{i\theta}}{re^{i\theta}-R^2e^{i\varphi}}-\frac{R^2re^{i\theta}}{R^2re^{i\theta}-e^{i\varphi}}\ ,\qquad \Phi(\xi,z)=\log\left|\frac{R^2-\xi\overline{z}}{\frac{1}{R^2}-\xi\overline{z}}\right|.$$

It is a matter of direct computation to verify that

$$\left(\frac{\partial\Phi}{\partial\rho}(\rho e^{i\theta},re^{i\varphi})\right)\Big|_{\rho=1}=\operatorname{Re}B(e^{i\theta},z)=-\frac{1}{2}\left(\mathcal{P}(r,R^2,\theta-\varphi)+\mathcal{P}(1,rR^2,\theta-\varphi)\right), \tag{7}$$

$$\frac{\partial\Phi}{\partial\theta}(e^{i\theta},z)=-\operatorname{Im}B(e^{i\theta},z). \tag{8}$$

After taking real parts in (6) we have

$$\log|g(z)|=\frac{1}{2\pi}\int\limits_0^{2\pi}\log|g(Re^{i\theta})|\mathcal{P}(r,R,\theta-\varphi)d\theta+$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi}\log|g(e^{i\theta})|\operatorname{Re}B(e^{i\theta},z)d\theta-\frac{1}{2\pi}\int\limits_0^{2\pi}\operatorname{Im}\log g(e^{i\theta})\operatorname{Im}B(e^{i\theta},z)d\theta+$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi}\log\left|g(\frac{1}{R}e^{i\theta})\right|\mathcal{P}(1,rR,\theta-\varphi)d\theta.$$

If relation (8) is combined with the Cauchy-Riemann equations one obtains

$$-\int\limits_0^{2\pi}\operatorname{Im}\log g(e^{i\theta})\operatorname{Im}B(e^{i\theta},z)d\theta=\int\limits_0^{2\pi}\arg\ g(e^{i\theta})\frac{\partial\Phi}{\partial\theta}(e^{i\theta},z)d\theta=$$
$$=\Phi(1,z)\left(\arg\ g(e^{2\pi i})-\arg\ g(e^{0i})\right)-\int\limits_0^{2\pi}\frac{d\ \arg\ g(e^{i\theta})}{d\theta}\Phi(e^{i\theta},z)d\theta=$$
$$=-\int\limits_0^{2\pi}\left(\frac{\partial\log|g(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1}\Phi(e^{i\theta},z)d\theta.$$

Thus we have

$$\log|g(z)| = \frac{1}{2\pi}\int\limits_0^{2\pi} \log|g(Re^{i\theta})|\mathcal{P}(r,R,\theta-\varphi)d\theta +$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi} \log|g(e^{i\theta})| \operatorname{Re} B(e^{i\theta},z)d\theta - \frac{1}{2\pi}\int\limits_0^{2\pi} \left(\frac{\partial \log|g(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1} \Phi(e^{i\theta},z)d\theta +$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|g\left(\frac{1}{R}e^{i\theta}\right)\right| \mathcal{P}(1,rR,\theta-\varphi)d\theta.$$

Observe that $\int_0^{2\pi} \mathcal{P}(x,X,\theta-\varphi)d\theta = 2\pi$ for all $X > 0$, $0 \le x < X$, $\varphi \in \mathbb{R}$ and $\frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{i\theta},z)d\theta = \log\left|\frac{R^2}{\bar{z}}\right| = 2\log R - \log r$. We therefore get the required relation for $f(z)$

$$\log|f(z)| = \frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(Re^{i\theta})|\mathcal{P}(r,R,\theta-\varphi)d\theta +$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(e^{i\theta})| \operatorname{Re} B(e^{i\theta},z)d\theta - \frac{1}{2\pi}\int\limits_0^{2\pi} \left(\frac{\partial \log|f(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1} \Phi(e^{i\theta},z)d\theta + \tag{9}$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right| \mathcal{P}(1,rR,\theta-\varphi)d\theta + 2k\log R.$$

Case 2. f has zeroes and no poles in $A(R)$. Put

$$p(z) = f(z)\prod_{a_\mu \in A^R}\left\{\frac{R^2-\bar{a}_\mu z}{R(z-a_\mu)}\right\}\prod_{a_\mu \in A_{1/R}}\left\{\frac{\frac{1}{R^2}-\bar{a}_\mu z}{\frac{1}{R}(z-a_\mu)}\right\}.$$

It is easy to see that $p(z)$ is a holomorphic function with neither poles nor zeroes in $A(R)$, $k(p) = k(f)$. Since $\left|\frac{R(z-a)}{R^2-\bar{a}z}\right| = 1$ for $|z| = R$, then after applying formula (9) to $p(z)$ we obtain :

$$\log|f(z)| + \sum_{a_\mu \in A^R} G_R(a_\mu,z) + \sum_{a_\mu \in A_{1/R}} G_{1/R}(a_\mu,z) =$$
$$= \frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(Re^{i\theta})|\mathcal{P}(r,R,\theta-\varphi)d\theta + \frac{1}{2\pi}\sum_{a_\mu \in A_{1/R}}\int_0^{2\pi} G_{1/R}(a_\mu,Re^{i\theta})\mathcal{P}(r,R,\theta-\varphi)d\theta +$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(e^{i\theta})| \operatorname{Re} B(e^{i\theta},z)d\theta + \frac{1}{2\pi}\sum_{a_\mu \in A^R}\int\limits_0^{2\pi} G_R(a_\mu,e^{i\theta}) \operatorname{Re} B(e^{i\theta},z)d\theta +$$
$$+\frac{1}{2\pi}\sum_{a_\mu \in A_{1/R}}\int\limits_0^{2\pi} G_{1/R}(a_\mu,e^{i\theta}) \operatorname{Re} B(e^{i\theta},z)d\theta - \frac{1}{2\pi}\int\limits_0^{2\pi} \left(\frac{\partial \log|f(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1} \Phi(e^{i\theta},z)d\theta - \tag{10}$$

$$-\frac{1}{2\pi}\sum_{a_\mu\in A^R}\int\limits_0^{2\pi}\left(\frac{\partial}{\partial\rho}G_R(a_\mu,\rho e^{i\theta})\right)\Big|_{\rho=1}\Phi(e^{i\theta},z)d\theta-$$

$$-\frac{1}{2\pi}\sum_{a_\mu\in A_{1/R}}\int\limits_0^{2\pi}\left(\frac{\partial}{\partial\rho}G_{1/R}(a_\mu,\rho e^{i\theta})\right)\Big|_{\rho=1}\Phi(e^{i\theta},z)d\theta+$$

$$+\frac{1}{2\pi}\int\limits_0^{2\pi}\log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|\mathcal{P}(1,rR,\theta-\varphi)d\theta+$$

$$+\frac{1}{2\pi}\sum_{a_\mu\in A^R}\int_0^{2\pi}G_R(a_\mu,\frac{1}{R}e^{i\theta})\mathcal{P}(1,rR,\theta-\varphi)d\theta+2k(f)\log R.$$

Apply the classical Poisson-Jensen formula to calculate the following integrals

$$I_1=-\frac{1}{2\pi}\int_0^{2\pi}G_{1/R}(a,Re^{i\theta})\mathcal{P}(r,R,\theta-\varphi)d\theta=\log\left|\frac{R(R^2-\overline{a}z)}{R^4a-z}\right|,$$

$$I_2=-\frac{1}{2\pi}\int_0^{2\pi}G_R\left(a,\frac{1}{R}e^{i\theta}\right)\mathcal{P}(1,rR,\theta-\varphi)d\theta=\log\left|\frac{R(1-R^2a\overline{z})}{R^4\overline{z}-\overline{a}}\right|.$$

If we recall (7), we see that

$$\frac{1}{2\pi}\int\limits_0^{2\pi}G_R(a,e^{i\theta})\operatorname{Re}B(e^{i\theta},z)d\theta=-\frac{1}{2}\left(\frac{1}{2\pi}\int\limits_0^{2\pi}G_R(a,e^{i\theta})\mathcal{P}(r,R^2,\theta-\varphi)d\theta+\right.$$
$$\left.+\frac{1}{2\pi}\int\limits_0^{2\pi}G_R(a,e^{i\theta})\mathcal{P}(1,rR^2,\theta-\varphi)d\theta\right)=\frac{1}{2}\left(\log\left|\frac{R(z-R^2a)}{R^4-\overline{a}z}\right|+\log\left|\frac{R(1-R^2a\overline{z})}{R^4\overline{z}-\overline{a}}\right|\right).$$

By the analogy

$$\frac{1}{2\pi}\int\limits_0^{2\pi}G_{1/R}(a,e^{i\theta})\operatorname{Re}B(e^{i\theta},z)d\theta=-\frac{1}{2}\left(\frac{1}{2\pi}\int\limits_0^{2\pi}G_{1/R}(a,e^{i\theta})\mathcal{P}(r,R^2,\theta-\varphi)d\theta+\right.$$
$$\left.+\frac{1}{2\pi}\int\limits_0^{2\pi}G_{1/R}(a,e^{i\theta})\mathcal{P}(1,rR^2,\theta-\varphi)d\theta\right)=\frac{1}{2}\left(\log\left|\frac{R(R^2-\overline{a}z)}{R^4a-z}\right|+\log\left|\frac{R(R^2\overline{z}-\overline{a})}{R^4a\overline{z}-1}\right|\right).$$

It is easy to verify that for $a=|a|e^{i\alpha}$

$$\left(\frac{\partial}{\partial\rho}G_R(a,\rho e^{i\theta})\right)\Big|_{\rho=1}=\frac{1}{2}\left(\mathcal{P}(1,|a|,\theta-\alpha)-\mathcal{P}(|a|,R^2,\theta-\alpha)\right),$$

$$\left(\frac{\partial}{\partial\rho}G_{1/R}(a,\rho e^{i\theta})\right)\Big|_{\rho=1}=\frac{1}{2}\left(\mathcal{P}(1,R^2|a|,\theta-\alpha)-\mathcal{P}(|a|,1,\theta-\alpha)\right).$$

If we use these equalities we obtain that for $a=|a|e^{i\alpha}$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial\rho}G_R(a,\rho e^{i\theta})\right)\bigg|_{\rho=1}\Phi(e^{i\theta},z)d\theta=$$

$$=\frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi}\Phi(e^{i\theta},z)\mathcal{P}(1,|a|,\theta-\alpha)d\theta-\frac{1}{2\pi}\int_0^{2\pi}\Phi(e^{i\theta},z)\mathcal{P}(|a|,R^2,\theta-\alpha)d\theta\right)=$$

$$=\frac{1}{2}\left(\log\left|\frac{R^2(R^2\overline{a}-\overline{z})}{R^2\overline{a}z-1}\right|-\log\left|\frac{R^2(R^4-a\overline{z})}{R^4z-a}\right|\right).$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial\rho}G_{1/R}(a,\rho e^{i\theta})\right)\bigg|_{\rho=1}\Phi(e^{i\theta},z)d\theta=$$

$$=\frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi}\Phi(e^{i\theta},z)\mathcal{P}(1,R^2|a|,\theta-\alpha)d\theta-\frac{1}{2\pi}\int_0^{2\pi}\Phi(e^{i\theta},z)\mathcal{P}(|a|,1,\theta-\alpha)d\theta\right)=$$

$$=\frac{1}{2}\left(\log\left|\frac{R^2(R^4\overline{a}-\overline{z})}{R^4a\overline{z}-1}\right|-\log\left|\frac{R^2(R^2-\overline{a}z)}{R^2z-a}\right|\right).$$

So that (10) yields

$$\log|f(z)|=\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\mathcal{P}(r,R,\theta-\varphi)d\theta-$$
$$-\frac{1}{4\pi}\int_0^{2\pi}\log|f(e^{i\theta})|\left(\mathcal{P}(r,R^2,\theta-\varphi)+\mathcal{P}(1,rR^2,\theta-\varphi)\right)d\theta+$$
$$+\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|\mathcal{P}(1,rR,\theta-\varphi)d\theta-\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial\log|f(\rho e^{i\theta})|}{\partial\rho}\right)\bigg|_{\rho=1}\Phi(e^{i\theta},z)d\theta-$$
$$-\sum_{a_\mu\in A^R}G_R(a_\mu,z)-\sum_{a_\mu\in A_{1/R}}G_{1/R}(a_\mu,z)+2k(f)\log R. \tag{11}$$

Case 3. f has zeroes and poles in $A(R)$. The result is a mere consequence of case 2 in view of the possibility to represent f as follows $f=f_0\frac{1}{f_\infty}$, where f_0 and f_∞ are the meromorphic functions with no poles in $A(R)$. □

Now we can prove the following analog of Jensen formula for annuli by means of Theorem 1.

Theorem 3. *Let f be a nonidentical zero meromorphic function on $A=\{z:\frac{1}{R_0}<|z|<R_0\}$, $1<R_0\le+\infty$. Then*

$$\int_1^R\frac{n_0(t,f)}{t}dt=\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|d\theta+\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|d\theta-\frac{1}{\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta \tag{12}$$

for every R such that $1<R<R_0$.

Proof. We prove that theorem in the case when f has no poles, for simplicity. First assume that f has no zeroes in $\mathbb{T}$. Put $z = e^{i\varphi}$ and integrate (1) with respect to φ. Then

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\log|f(Re^{i\theta})| + \log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|\right)\mathcal{P}(1,R,\theta-\varphi)d\theta d\varphi =$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\log|f(Re^{i\theta})| + \log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|\right)\int_0^{2\pi}\mathcal{P}(1,R,\theta-\varphi)d\varphi d\theta =$$
$$= \int_0^{2\pi}\log|f(Re^{i\theta})|d\theta + \int_0^{2\pi}\log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|d\theta.$$

The similar computation gives

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|\mathcal{P}(1,R^2,\theta-\varphi)d\theta d\varphi = \int_0^{2\pi}\log|f(e^{i\theta})|d\theta$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\partial\log|f(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1}\Phi(e^{i\theta},e^{i\varphi})d\theta d\varphi = 2\log R\int_0^{2\pi}\left(\frac{\partial\log|f(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1}d\theta.$$

Consider the function $g(z) = z^{-k}f(z)$, where $k = \frac{1}{2\pi i}\int_{|z|=1}\frac{f'(z)}{f(z)}dz$. According to Lemma 1 we can determine a branch of logarithm of $g(z)$ in some neighborhood of $\mathbb{T}$. Then in view of the Cauchy-Riemann equations we have

$$\int_0^{2\pi}\left(\frac{\partial\log|g(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1}d\theta = Var_{\gamma_0^1}\arg g(z) = \frac{1}{i}\int_{\gamma_0^1}\frac{g'(z)}{g(z)}dz = 0.$$

So, we state that

$$\int_0^{2\pi}\left(\frac{\partial\log|f(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1}d\theta = 2\pi k. \tag{13}$$

Since $\frac{1}{2\pi}\int_0^{2\pi}\log|a-e^{i\theta}|d\theta = \log^+|a|$ then for $|a| \in (1,R)$

$$\int_0^{2\pi}G_R(a,e^{i\theta})d\varphi = 2\pi\log\frac{R}{|a|}$$

and for $|a| \in (1/R,1)$

$$\int_0^{2\pi}G_{1/R}(a,e^{i\theta})d\varphi = 2\pi\log R|a|.$$

Further

$$2\pi\sum_{1<|a_\mu|<R}\log\frac{R}{|a_\mu|}+2\pi\sum_{\frac{1}{R}<|a_\mu|<1}\log R|a_\mu|=2\pi\int_1^R\frac{n(t,1/f)}{t}dt$$

and we obtain the statement of the theorem.
When f has zeroes in $\mathbb{T}$ apply the statement of the previous case to

$$q(z)=f(z)\prod_{a_\mu\in\mathbb{T}}\left\{\frac{\frac{1}{R^2}-\bar{a}_\mu z}{\frac{1}{R}(z-a_\mu)}\right\}$$

This gives (12) and completes the proof.

□

Proof of Theorem 2. 1) The proof is trivial.
2) The Hadamard three circles theorem implies that $\log^+ M(r,f)$ is convex with respect to $\log r$, $\frac{1}{R}<r<R$. Hence, by the definition of convexity with respect to $\log r$, $\log^+ M(1/r,f)$ is convex with respect to $\log r$ and the assertion follows immediately.
3) Let $1<r_1<r_2$. By property 2)

$$\log^+ M(r_1,f)+\log^+ M(1/r_1,f)<$$
$$<\frac{\log r_2-\log r_1}{\log r_2-\log(1/r_2)}(\log^+ M(1/r_2,f)+\log^+ M(r_2,f))+$$
$$+\frac{\log r_1-\log(1/r_2)}{\log r_2-\log(1/r_2)}(\log^+ M(r_2,f)+\log^+ M(1/r_2,f))=\log^+ M(r_2,f)+\log^+ M(1/r_2,f).$$

4) Let $z=re^{i\theta}\in A(R)$. According to Theorem 1 and in view of (13) we have

$$\log|f(z)|=\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\mathcal{P}(r,R,\theta-\varphi)d\theta-$$
$$-\frac{1}{4\pi}\int_0^{2\pi}\log|f(e^{i\theta})|\left(\mathcal{P}(r,R^2,\theta-\varphi)+\mathcal{P}(1,rR^2,\theta-\varphi)\right)d\theta+$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|\mathcal{P}(1,rR,\theta-\varphi)d\theta+$$
$$+\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial\log|f(\rho e^{i\theta})|}{\partial\rho}\right)\Bigg|_{\rho=1}\log\left|\frac{1-R^2e^{i\theta}\overline{z}}{R^2-e^{i\theta}\overline{z}}\right|d\theta-\sum_{a_\mu\in A^R}G_R(a_\mu,z)-\sum_{a_\mu\in A_{1/R}}G_{1/R}(a_\mu,z). \tag{14}$$

Since the sums of the right hand of (14) are negative and

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\log\left|\frac{1-R^2e^{i\theta}\overline{z}}{R^2-e^{i\theta}\overline{z}}\right|\right|d\theta=|\log|z||,$$

we see that

$$\log^+|f(z)| \le \frac{R+r}{R-r}\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+|f(Re^{i\theta})|d\theta + \left(\frac{R^2+r}{R^2-r}+\frac{R^2r+1}{R^2r-1}\right)\frac{1}{4\pi}\int\limits_0^{2\pi}\log^+\frac{1}{|f(e^{i\theta})|}d\theta +$$
$$+\frac{Rr+1}{Rr-1}\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|d\theta + \sup_{\theta\in[0,2\pi]}\left|\left(\frac{\partial\log|f(\rho e^{i\theta})|}{\partial\rho}\right)\right|\Bigg|_{\rho=1}|\log r|.$$

This gives for $r \ge 1$

$$\log^+ M(r,f) + \log^+ M(1/r,f) \le$$
$$\le \left(\frac{R+r}{R-r}+\frac{Rr+1}{Rr-1}\right)\left(\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+|f(Re^{i\theta})|d\theta + \frac{1}{2\pi}\int\limits_0^{2\pi}\log^+\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|d\theta\right) +$$
$$+\left(\frac{R^2+r}{R^2-r}+\frac{R^2r+1}{R^2r-1}\right)\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+\frac{1}{|f(e^{i\theta})|}d\theta + 2\sup_{\theta\in[0,2\pi]}\left|\left(\frac{\partial\log|f(\rho e^{i\theta})|}{\partial\rho}\right)\right|_{\rho=1}\Bigg|\log r.$$

After simple transformation we obtain the required inequality.

□

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