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A CHARACTERIZATION OF THE MENGER AND HUREWICZ PROPERTIES OF SUBSPACES OF THE REAL LINE

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We characterize the covering properties of Menger, Hurewicz, and two other selection principles of subsets of the real line in terms of their continuous images in the Baire space $\mathbb{N}^\omega$, and thus answer the corresponding question of B. Tsaban in positive.

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В данной работе мы характеризируем свойства Менгера, Гуревича и два других селекционных принципа подпространств прямой в терминах их непрерывных образов при отображениях в пространство $\mathbb{N}^\omega$, и таким образом даем положительный ответ на соответствующий вопрос Б. Цабана.

The properties of Menger and Hurewicz, which are the basic and oldest selection principles, take their origin in papers [2] and [1]. Both of them appeared as cover counterparts of σ-compactness. Recall from [9], that a topological space X has the Menger (resp. Hurewicz) property, if for every sequence $(u_n)_{n\in\omega}$ of open covers of X there exists a sequence $(v_n)_{n\in\omega}$ such that every v_n is a finite subset of u_n and the family $\{\bigcup v_n : n\in\omega\}$ is a cover (resp. γ-cover) of X, where an indexed family $\{A_n : n\in\omega\}$ is a γ-cover of X if for every $x\in X$ the set $\{n\in\omega : x\notin A_n\}$ is finite. It is easy to see that every σ-compact space X has the Hurewicz property ($\equiv$ is Hurewicz) and every Hurewicz space has the Menger property ($\equiv$ is Menger).

One of the main results of [2] is the characterization of the above two properties of a space X in terms of continuous images of X under maps $f\colon X\to\mathbb{R}^\omega$. It involves the *eventual dominance* preorder $\le^*$ on $\mathbb{R}^\omega$ defined as follows: $(x_n)_{n\in\omega}\le^*(y_n)_{n\in\omega}$ if and only if the set $\{n\in\omega : x_n>y_n\}$ is finite.

Theorem 1. (Hurewicz [2]) *Let X be a metrizable separable space. Then*

(1) *X has the Menger property if and only if $f(X)$ is not cofinal with respect to $\le^*$ for every continuous function $f\colon X\to\mathbb{R}^\omega$;*

(2) *X has the Hurewicz property if and only if $f(X)$ is bounded with respect to $\le^*$ for every continuous function $f\colon X\to\mathbb{R}^\omega$.*

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It is observed that for a zero-dimensional space X the same characterization in terms of continuous images in $\mathbb{N}^\omega$ holds.

Theorem 2. (Reclaw [7]) *Let X be a zero-dimensional metrizable separable space. Then*

(1) *X has the Menger property if and only if $f(X)$ is not cofinal with respect to $\leq^*$ for every continuous function $f\colon X \to \mathbb{N}^\omega$;*

(2) *X has the Hurewicz property if and only if $f(X)$ is bounded with respect to $\leq^*$ for every continuous function $f\colon X \to \mathbb{N}^\omega$.*

In addition to the properties of Menger and Hurewicz, two other selection principles were recently characterized in spirit of Theorem 2. Their definitions involve some special types of covers introduced in [14] and [5] respectively. A family $\{A_n : n \in \omega\}$ is

- a τ^*-*cover* of X, if for every $x \in X$ there exists an infinite subset I_x of $\{n \in \omega : x \in A_n\}$ such that for all $x_1, x_2 \in X$ either $|I_{x_1} \setminus I_{x_2}| < \infty$ or $|I_{x_2} \setminus I_{x_1}| < \infty$;
- an ω-*cover* of X, if for every finite subset K of X there exists $n \in \omega$ with $K \subset A_n$.

A topological space is said to have the property $\bigcup_{\text{fin}}(\mathcal{O}, T^*)$ ($\bigcup_{\text{fin}}(\mathcal{O}, \Omega)$), if for every sequence $(u_n)_{n\in\omega}$ of open covers of X there exists a sequence $(v_n)_{n\in\omega}$, where v_n is a finite subset of u_n, such that $\{\bigcup v_n : n \in \omega\}$ is a τ^*- (ω-)cover of X.

Theorem 3. ([14, Theorem 7.8], [11, Theorem 2.1]) *Let X be a zero-dimensional metrizable separable space. Then*

(1) *X has the property $\bigcup_{\text{fin}}(\mathcal{O}, T^*)$ if and only if $f(X)$ satisfies the weak excluded middle property for every continuous function $f\colon X \to \mathbb{N}^\omega$;*

(2) *X has the property $\bigcup_{\text{fin}}(\mathcal{O}, \Omega)$ if and only if $f(X)$ is not finitely dominating for every continuous function $f\colon X \to \mathbb{N}^\omega$.*

The reader is refered to papers [14] and [11] for corresponding definitions. Different forms of Theorem 2 are frequently used in literature, see [13] and [12]. It is well known, that every zero-dimensional metrizable separable space X is homeomorphic to a subspace of $\mathbb{N}^\omega$, and thus to a subspace of the space of irrational numbers, see [8]. The following question was asked by B. Tsaban in private communication: *Is the characterization in terms of images in $\mathbb{N}^\omega$ true for all subspaces of $\mathbb{R}$?*

The following theorem, which is the main result of this paper, answers this question in positive.

Theorem 4. *Let X be a subspace of the real line. Then X has the Menger (resp. Hurewicz, $\bigcup_{\text{fin}}(\mathcal{O}, T^*)$, $\bigcup_{\text{fin}}(\mathcal{O}, \Omega)$) property if and only if for every continuos function $f\colon X \to \mathbb{N}^\omega$ the image $f(X)$ is not cofinal (resp. is bounded, has the weak excluded middle property, is not finitely dominating) with respect to $\leq^*$.*

In the proof of Theorem 4 we shall use some properties of *set-valued maps*. Following [4] by a set-valued map from a set X into Y we understand a map $\Phi\colon X \to \mathcal{P}(Y) \setminus \{\varnothing\}$, where $\mathcal{P}(Y)$ denotes the family of all subsets of Y. Recall, that a set-valued map Φ from a topological space X into a topological space Y is called

- *compact-valued*, if $\Phi(x)$ is compact for every $x \in X$;
- *upper semicontinuous*, if for every open subset V of Y the set $\Phi^{-1}_{\subset}(V) = \{x \in X : \Phi(x) \subset V\}$ is open in X.

For a subset A of X and a set-valued map $\Phi : X \to \mathcal{P}(Y)$ we denote by $\Phi(A)$ the set $\bigcup_{x\in A}\Phi(X)$.

From now on we fix some property P among the Menger, Hurewicz, $\bigcup_{\text{fin}}(\mathcal{O},T^*)$, and $\bigcup_{\text{fin}}(\mathcal{O},\Omega)$ properties, and denote by ξ and S its counterparts among the four types of covers and among the properties of subsets of $\mathbb{N}^\omega$ according to Theorems 2 and 3. For example, if P is the property $\bigcup_{\text{fin}}(\mathcal{O},T^*)$, then ξ-covers coincide with τ^*-covers, and S stands for the weak excluded middle property.

Lemma 1. (1) *Let X be a topological space with the property P and C be a closed subset of X. Then C has the propety P.*

(2) *Let $\Phi\colon X \to Y$ be a compact-valued upper semicontinuous map between topological spaces X and Y such that $\Phi(X) = Y$. Then Y has the property P provided so does X. In particular, every continuous image of a space with the property P has this property as well.*

(3) *Let X be a topological space. Then the union $Y \cup Z$ of a subspace Y with the property P and a σ-compact subspace Z of X has the property P.*

Proof. 1. This simple statement probably belongs to folklore and its proof is left to the reader.

2. Let us fix an arbitrary sequence $(w_n)_{n\in\omega}$ of open covers of Y. For every $n \in \omega$ consider the family $u_n = \{\Phi^{-1}_{\subset}(\cup v) : v \subset w_n, |v| < \infty\}$. Since Φ is upper semicontinuous and compact-valued, each u_n is an open cover of X. The property P of X implies the existence of a sequence $(c_n)_{n\in\omega}$, where each c_n is a finite subset of u_n, such that $\{\bigcup c_n : n \in \omega\}$ is a ξ-cover of X. From the above it follows that for every $n \in \omega$ we can find a finite subset v_n of w_n with $\Phi(\bigcup c_n) \subset \bigcup v_n$. Therefore for every $y \in Y$ and $x \in X$ such that $y \in \Phi(x)$ we have

$$\left\{n \in \omega : y \in \bigcup v_n\right\} \supset \left\{n \in \omega : x \in \bigcup c_n\right\},$$

consequently $\{\bigcup v_n : n \in \omega\}$ is a ξ-cover of Y, and thus Y has the property P.

3. A trivial verifivation is left to the reader. □

Lemma 2. *Let X be a subspace of $\mathbb{R}$. Then there exists a zero-dimensional metrizable space X^* such that X^* is a continuous image of X and X^* has the property P if and only if so does X.*

Proof. First of all, denote by $\mathcal{E}$ the family of all (connected) components of X containing more than one element. Then $\mathcal{E}$ may be written in the form $\mathcal{E} = \mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}$, where $\mathcal{A} = \{(a^0_\alpha, a^1_\alpha) : \alpha \in A\}$, $\mathcal{B} = \{(b^0_\beta, b^1_\beta] : \beta \in B\}$, $\mathcal{C} = \{[c^0_\xi, c^1_\xi) : \xi \in C\}$, and $\mathcal{D} = \{[d^0_\zeta, d^1_\zeta] : \zeta \in D\}$, where A, B, C and D are at most countable sets. For every $\alpha \in A$ fix some $a_\alpha \in (a^0_\alpha, a^1_\alpha)$ and consider the map $f\colon X \to \mathbb{R}$,

$$f(x) = \begin{cases} a_\alpha, & \text{if } x \in (a^0_\alpha, a^1_\alpha) \\ b^1_\beta, & \text{if } x \in (b^0_\beta, b^1_\beta] \\ c^0_\xi, & \text{if } x \in [c^0_\xi, c^1_\xi) \\ x, & \text{otherwise.} \end{cases}$$

It is clear that f is continuous. Moreover, $f(X) \subset X$ and $X \setminus f(X)$ is open in $\mathbb{R}$, consequently $f(X)$ is closed subset of X such that the complement $X \setminus f(X)$ is σ-compact, and thus Lemma 1 implies X has the property P if and only if so is $X_1 = f(X)$. The space X_1 obviously has the following property: each connected component of X_1 is compact. Moreover, every component of X_1 coincides with the quasicomponent containing it, see [4, Ch.5, §46(V)] for corresponding definitions. Indeed, let G be a countable dense subset of $\mathbb{R} \setminus X$. Then the family $\mathcal{G} = \{(a,b) \cap X_1 : a,b \in G\}$ consists of clopen subsets of X_1 and every component of X_1 coincides with an itersection of all elements of $\mathcal{G}$ containing it. Following [4], we denote by $Q(X_1)$ the space of quasicomponets of X_1 endowed with the topology τ generated by a base $\{U^+ : U \text{ is clopen in } \mathbb{R}\}$, where $U^+ = \{K \in Q(X_1) : K \subset U\}$. Let $g\colon X_1 \to Q(X_1)$ be a map assigning to a point $x \in X_1$ its quasicomponent. It follows from [4, Ch.5, §46(Va) Th.1] that g is continuous. In addition, $Q(X_1)$ is regular and zero-dimensional by [4, Ch.5, §46(Va), Th.2]. Next, we shall show that g^{-1}, considered as a set-valued map from $Q(X_1)$ into X_1, is compact-valued upper semicontinuous. For this aim fix arbitrary $K \in Q(X_1)$ and an open subset U of X_1 with $K \subset W$. Let us write K in the form $K = [c,d]$ and find $\varepsilon > 0$ such that $(c-\varepsilon, d+\varepsilon) \subset U$. Since K is a component of X_1, there are $a \in G \cap (c-\varepsilon, c)$ and $b \in G \cap (d, d+\varepsilon)$. Now, it is clear that $g^{-1}(W) \subset U$, where $W = ((a,b) \cap X_1)^+$. Applying Lemma 1 once again, we conclude that $Q(X_1)$ has the property P if and only if so does X_1. The above argument gives us that the family $\{((a,b)\cap X_1)^+ : a,b \in G\}$ is a countable base of the topology τ, consequently $Q(X_1)$ is metrizable separable being second-countable and regular space, see [3, Ch.4 Th.17]. And finally, $X^* = Q(X_1)$ safisfies the requirements of this lemma. □

Proof of Theorem 4. Let X be a subspace of the real line. If X fails to have the property P, then Lemma 2 yields a zero-dimensional metrizable separable space X^* and a surjective continuous function $f\colon X \to X^*$ such that X^* fails to have the property P. Applying Theorems 2 and 3, we can find a continuous map $g\colon X^* \to \mathbb{N}^\omega$ such that $g(X^*)$ fails to have the property S. Then $(g \circ f)(X) = g(X^*)$ does not have the property S as well.

Now, assume that X has the property P. In this case it sufficies to note that the zero-dimensionality of X was not used in the proofs of "only if" parts of Theorems 2 and 3, see [7], [14] and [11]. □

In our proof of Lemma 2 we used a simple structure of connected subspaces of $\mathbb{R}$. Since the family of connected subspaes of $\mathbb{R}^2$ is much more farious, Theorem 4 fails for subspaces of the plane.

Example. There exists a connected subspace X of $\mathbb{R}^2$ which fails to be Menger. Consequently every continuous image $f(X)$ under a map $f\colon X \to \mathbb{N}^\omega$ contains only one point. To construct such a space, we denote by $\mathbb{I} \subset \mathbb{R}$ and $\mathbb{Q} \subset \mathbb{R}$ the sets of all irrational and rational numbers, respectively. Let

$$X = \mathbb{R} \times \{0\} \cup \mathbb{Q} \times [0,1) \cup \mathbb{I} \times \{1\} \subset \mathbb{R}^2.$$

Then the space X is obviously connected and fails to be Menger as a space containing a closed copy of irrationals, see [6]. □

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