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OPEN-MULTICOMMUTATIVITY OF SOME FUNCTORS RELATED TO THE FUNCTOR OF PROBABILITY MEASURES

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The property of a normal functor in the category of compact Hausdorff spaces to be open-multicommutative proposed by R. Kozhan and M. Zarichnyi is investigated. A number of normal functors related to the functor of probability measures and equipped with convex structure are considered here and it is proved that the functors cc, ccP, $G_{cc}P$ and $\lambda_{cc}P$ are open-multicommutative.

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Исследуется свойство открытой мультикоммутативности нормальных функторов в категории компактов, предложенное Р. Кожаном и М. Заричным. Здесь мы рассматриваем ряд нормальных функторов, связанных с функтором вероятностных мер и наделенным выпуклой структурой. Доказано, что функторы cc, ccP, $G_{cc}P$ и $\lambda_{cc}P$ являются открыто-мультикоммутативными.

1. Introduction. Classical objects in topology and functional analysis — spaces of probability measures — are widely used in economics and game theory during the last decades (see Lucas and Prescott (1971), Mas-Colell (1984), Jovanovic and Rosenthal (1988)). All these investigations deal with the notion of the set value correspondence map which assigns to every probability measures on the factors of the product of compacta the set of probability measures with these marginals,

$$\psi(\mu_1, ..., \mu_n) = \{\lambda \in P(X_1 \times ... \times X_n) \colon P\mathrm{pr}_i(\lambda) = \mu_i,\ i \in \{1, \dots, n\}\}.$$

The problem of continuity of this map can be equivalently redefined in terms of openness of the characteristic map of the following bicommutative diagram:

$$\begin{tikzcd} P(X \times Y) \arrow[r, "P\mathrm{pr}_1"'] \arrow[d, "P\mathrm{pr}_2"'] & P(X) \arrow[d, "P\mathbf{1}_*"] \\ P(Y) \arrow[r, "P\mathbf{1}_*"'] & P(\{*\}) \end{tikzcd}$$

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Due to the well known theorem of Ditor and Eifler (1972) the functor of probability measures is open. Shchepin (1981) showed that the openness property of a normal functor is closely related to the property of bicommutativity.

A common generalization of these two properties generates a new notion of open-multicommutativity of the normal function which was proposed by Kozhan and Zarichnyi (2004). In their paper they investigate the functor of probability measures and show that this functor is open-multicommutative.

In economic theory, in addition to the space of probability measures, different convex structures are commonly used, in particular, spaces of convex closed subsets of the space of probability measures have wide applications.

Here the property of normal functors to be open-multicommutative is investigated for functors cc, ccP, $G_{cc}P$ and $\lambda_{cc}P$.

2. Notations and definitions. Suppose that $\mathcal{G}$ is a finite partially ordered set and we also regard it as a finite directed graph. Denote by $\mathcal{VG}$ the class of all vertices of the graph $\mathcal{G}$ and by $\mathcal{EG}$ the set of its edges. A map $\mathcal{O}\colon \mathcal{G} \to \mathrm{Comp}$ is called a *diagram.*

Definition 2.1. The set of morphisms

$$(X \stackrel{g_A}{\to} \mathcal{O}(A))_{A\in\mathcal{VG}} \tag{2.1}$$

is said to be a *cone* over the diagram $\mathcal{O}$ if and only if for every objects $A, B \in \mathcal{VG}$ and for every edge $\varphi\colon A \to B$ in $\mathcal{EG}$ the diagram

$$\begin{tikzcd} & X \arrow[dl, "g_A"'] \arrow[dr, "g_B"] & \\ \mathcal{O}(A) \arrow[rr, "\mathcal{O}(\varphi)"'] & & \mathcal{O}(B) \end{tikzcd}$$

is commutative.

Definition 2.2. Cone (2.1) is called a *limit* of the diagram $\mathcal{O}$ if the following condition is satisfied: for each cone $C' = (X' \stackrel{g'_A}{\to} \mathcal{O}(A))_{A\in\mathcal{VG}}$ there exists a unique morphism $\chi_{C'}\colon X' \to X$ such that $g'_A = g_A \circ \chi_{C'}$ for every $A \in \mathcal{VG}$.

Further, we denote this cone by $\lim(\mathcal{O})$. The map $\chi_{C'}$ is called the *characteristic map* of C'.

Definition 2.3. Cone $C' = (X' \stackrel{g'_A}{\to} \mathcal{O}(A))_{A\in\mathcal{VG}}$ is called *open-multicommutative* if the characteristic map $\chi_{C'}$ is open and surjective.

Let F be a covariant functor in the category Comp. Define the diagram $F(\mathcal{O})\colon \mathcal{G} \to \mathrm{Comp}$ in the following way: for every $A \in \mathcal{VG}$ let $F(\mathcal{O})(A) = F(\mathcal{O}(A))$ and for every edge $\varphi \in \mathcal{EG}$ we set $F(\mathcal{O})(\varphi) = F(\mathcal{O}(\varphi))$.

Definition 2.4. The functor F is called *open-multicommutative* if it preserves the open-multicommutative cones, i.e. the cone

$$F(C') = (F(X') \stackrel{Fg'_A}{\to} F(\mathcal{O}(A)))_{A\in\mathcal{VG}}$$

over the diagram $F(\mathcal{O})$ is open-multicommutative.

Definition 2.5. The functor F is called *finite open-multicommutative* if it preserves the open-multicommutative cones which consist of finite spaces.

Further, we assume that the functor F is weakly normal.

We denote a limit of the diagram $F(\mathcal{O})$ by the cone $(Y_F \overset{\mathrm{pr}_A}{\to} F(\mathcal{O}(A)))_{A\in\mathcal{VG}}$.

Let us assume that X is a compact convex subspace of some locally convex space E. We consider the functor cc: Conv $\to$ Comp. This functor is defined as follows:

$$cc(X) = \{A \subset X\colon A \neq \varnothing \text{ is closed and convex}\} \subset \exp(X),$$

$cc(f)(A) = f(A)$ for any continuous affine map $f\colon X \to Y$ of compact convex subspaces of some locally convex spaces and $A \in cc(X)$.

Recall the well-known constructions of the inclusion hyperspace and superextension functors. We set

$$G(X) = \{\mathcal{A} \in \exp^2(X)\colon A \in \mathcal{A} \text{ and } A \subseteq B \in \exp(X) \Rightarrow B \in \mathcal{A}\},$$

$$\lambda(X) = \{\mathcal{A} \in G(X)\colon \mathcal{A} \text{ is a maximal linked system }\}.$$

The base of topology on $G(X)$ is formed by the sets of the form

$$(U_1^+ \cap \ldots \cap U_m^+) \cap (V_1^- \cap \ldots \cap V_n^-),$$

where

$$U_i^+ = \{\mathcal{A} \in G(X)\colon \text{ there exists } A \in \mathcal{A} \text{ such that } A \subset U_i\}$$
$$V_i^- = \{\mathcal{A} \in G(X)\colon \ A \cap V_i \neq \varnothing \text{ for every } A \in \mathcal{A}\},$$

and $U_1, \ldots, U_m, V_1, \ldots, V_n$ are open subsets in X. For properties of the functors G and λ reader is referred to Teleiko and Zarichnyi (1999).

In order to combine these constructions with convex structure we use defined above functors and the functor cc. In such a way we can define a space

$$G_{cc}(X) = \{\mathcal{A} \subset cc(X)\colon \ \mathcal{A} \text{ is closed and } A \in \mathcal{A} \text{ and } A \subseteq B \in cc(X) \Rightarrow B \in \mathcal{A}\}.$$

Let us define for every $X \in |\mathrm{Comp}|$ a map $iX\colon G_{cc}(X) \to G(X)$ in the following way: $iX(\mathcal{A}) = \{B \in \exp(X)\colon A \in \mathcal{A} \text{ and } A \subseteq B \Rightarrow B \in \mathcal{A}\}$.

We endow the space $G_{cc}(X)$ with the topology generated by the pre-base consisting of the sets

$$U_{cc}^+ = iX^{-1}(U^+) = \{\mathcal{A} \in G_{cc}(X)\colon \text{ there exists } A \in \mathcal{A} \text{ such that } A \subset U\}$$

and

$$V_{cc}^- = iX^{-1}(V^-) = \{\mathcal{A} \in G_{cc}(X)\colon \ A \cap V \neq \varnothing \text{ for every } A \in \mathcal{A}\},$$

where U is open subset in X. It is obviously that according to the defined above topology the map iX is an embedding.

For a map $f\colon X \to Y$ in Comp and $\mathcal{A} \in G_{cc}(X)$ let

$$G_{cc}f(\mathcal{A}) = \{B \in G_{cc}(Y)\colon \ f(A) \subset B,\ A \in \mathcal{A}\}.$$

Thus a covariant functor G_{cc}: Conv $\to$ Comp is defined. In the same way we can construct a functor λ_{cc}: Conv $\to$ Comp.

Due to the natural convex structure of the functor P one can compose the functors G_{cc} and λ_{cc} with the functor of probability measures and it gives us two new constructions of functors in the category Comp:

$$G_{cc}P = G_{cc} \circ P\colon \mathrm{Comp} \to \mathrm{Comp}$$

$$\lambda_{cc}P = \lambda_{cc} \circ P\colon \mathrm{Comp} \to \mathrm{Comp}.$$

3. Open-multicommutativity of the functor ccP and related functors.

Proposition 3.1. *A weakly normal bicommutative functor F is multicommutative.*

Proof. This Proposition is actually proved in Kozhan and Zarichnyi (2004) for the functor of probability measures. But in the proof they use only continuity, epimorhness and bicommutativity of functor P and that is why it can be applied for every normal bicommutative functor. ☐

The following proposition is also a generalization of the result of Kozhan and Zarichnyi (2004).

Proposition 3.2. *For a weakly normal open bicommutative functor F the following properties are equivalent:*
F is open-multicommutative;
F is finite open-multicommutative.

Proof. The proof is analogical to that of Theorem 4.1 in Kozhan and Zarichnyi (2004). The special properties of functor P are used only in the case of finite spaces $\mathcal{O}(A)$. The rest of the proof uses only facts that the functor P is open, bicommutative, continuous and epimorphic. Hence the scheme of the proof can be applied to every weakly normal open bicommutative functor. ☐

Proposition 3.3. *The functors G and λ are bicommutative.*

Proof. Let us consider an arbitrary bicommutative diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "h"] \\ Y \arrow[r, "j"'] & T \end{tikzcd} \tag{3.1}$$

Let $\mathcal{A} \in G(X)$ and $\mathcal{B} \in G(Y)$ such that $Gh(\mathcal{A}) = Gj(\mathcal{B}) = \mathcal{C} \in G(T)$. These equalities imply that for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we can find sets $A_B \in \mathcal{A}$ and $B_A \in \mathcal{B}$ such that $\overline{h(A)} = \overline{j(B_A)}$ as well as $\overline{j(B)} = \overline{h(A_B)}$. Since elements $A, A_B \in \exp(X)$ and $B, B_A \in \exp(Y)$, it follows from the bicommutativity of the hyperspace functor that there exist D_A and $D_B \in \exp(Z)$ such that $\overline{f(D_A)} = A$, $\overline{g(D_A)} = B_A$ and $\overline{f(D_B)} = A_B$, $\overline{g(D_B)} = B$. Denote

$$\mathcal{D} = \{D \in \exp(Z)\colon \exists A \in \mathcal{A} \text{ or } B \in \mathcal{B}\colon D_A \subset D \text{ or } D_B \subset D\}.$$

It is clear that $\mathcal{D} \in G(Z)$ and the following equalities are satisfied $Gf(\mathcal{D}) = \mathcal{A}$ and $Gg(\mathcal{D}) = \mathcal{B}$. This competes the proof of the bicommutativity of the functor G.

Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are maximal linked systems. As above, for elements $A \in \mathcal{A}$ and $B_A \in \mathcal{B}$ we can find a closed set $D_A \in \exp(Z)$ such that $\overline{f(D_A)} = A$ and $\overline{g(D_A)} = B_A$. Denote by D'_A the maximal of such sets with respect to the partial order $\subseteq$. The set D'_B is defined similarly. Let us show that

$$\mathcal{D} = \{D \in \exp(Z)\colon \exists A \in \mathcal{A} \text{ or } B \in \mathcal{B}\colon D'_A \subset D \text{ or } D'_B \subset D\}$$

is a linked system. Indeed, suppose that there exist two elements $D'_A, D'_{\tilde{A}} \in \mathcal{D}$ for some $A, \tilde{A} \in \mathcal{A}$ such that $D'_A \cap D'_{\tilde{A}} = \varnothing$. Denote $B = \overline{g(D'_A)}$ and $\tilde{B} = \overline{g(D'_{\tilde{A}})}$. Since $\mathcal{A}$ and $\mathcal{B}$ are linked systems $A \cap \tilde{A} \neq \varnothing$ and $B \cap \tilde{B} \neq \varnothing$. By the definition of D'_A and $D'_{\tilde{A}}$ we can find points $a \in A \cap \tilde{A}$ and $b \in B \cap \tilde{B}$ such that $h(a) = j(b)$. The bicommutativity of diagram (3.1) implies that there exists $d \in Z$ with $f(d) = a$ and $g(d) = b$. Consider two sets $D'_A \cup \{d\}$ and $D'_{\tilde{A}} \cup \{d\}$. The closure of images of both of them by the maps f and g are A, $\tilde{A}$, B and $\tilde{B}$ respectively but this contradicts to the maximality of the sets D'_A and $D'_{\tilde{A}}$.

Once $\mathcal{D}$ is a linked system, it can be extended to the maximal one $\tilde{\mathcal{D}}$. Let D be an element of $\tilde{\mathcal{D}}$. This means that it intersects each element of $\tilde{D}$ and this implies that $\overline{f(D)}$ and $\overline{g(D)}$ intersect all elements of $\mathcal{A}$ and $\mathcal{B}$ respectively and due to maximality of $\mathcal{A}$ and $\mathcal{B}$ we have $\overline{f(D)} \in \mathcal{A}$ and $\overline{g(D)} \in \mathcal{B}$. Thus, $Gf(\tilde{\mathcal{D}}) = \mathcal{A}$ and $Gg(\tilde{\mathcal{D}}) = \mathcal{B}$ which proves the proposition. □

Proposition 3.4. *The functors* exp, G *and* λ *are open-multicommutative.*

Proof. Let us check whether the conditions of Proposition 3.2 are satisfied. The functor exp is normal and bicommutative (see, e.g., Teleiko and Zarichnyi (1999)). Assume that every compactum $\mathcal{O}(A)$ is finite for each $A \in \mathcal{VG}$. Since every finite compactum $\mathcal{O}(A)$ is a discrete space, $\exp(\mathcal{O}(A))$ is also discrete, which follows from the properties of the Vietoris topology. It is known (see Kozhan and Zarichnyi (2004)) that $Y_{\exp} \subseteq \prod_{A \in \mathcal{VG}} \exp(\mathcal{O}(A))$, which is discrete as well as $\exp(X)$, is a subset of the discrete space $\exp(\prod_{A \in \mathcal{VG}} \mathcal{O}(A))$. Thus the characteristic map $\chi_{\exp}$ is the map of two discrete spaces and this necessarily implies that it is open. The spaces $G(X)$ and $\lambda(X)$ are subsets of the space $\exp^2(X)$ for every compactum X and therefore are also discrete if so is X . Then weak normality, openness and bicommutativity of functors G and λ (see Propositions 3.2 and 3.3) imply the open-multicommutativity of them. □

Lemma 3.5. *Let* $B \subset T \times T$ *and* $\mathrm{pr}_1(B) \subset C \subset T$. *If* C *is a convex set then this implies that* $\mathrm{pr}_1(\mathrm{conv}(B)) \subset C$. *If in addition we have that* $\mathrm{pr}_1(B) \supset C$ *then* $\mathrm{pr}_1(\mathrm{conv}(B)) = C$.

Proof. Consider an arbitrary point $x \in \mathrm{conv}(B) \setminus B$. There exist two points $x_1, x_2 \in B$ and $\alpha \in (0, 1)$ such that $x = \alpha x_1 + (1 - \alpha)x_2$. Since $\mathrm{pr}_1(x_1), \mathrm{pr}_1(x_2) \in C$ and C is convex, therefore $\mathrm{pr}_1(x) \in C$. The point x is arbitrarily chosen thus the first statement of the lemma is proved.

The second statement of the lemma is evident. □

Proposition 3.6. *The functor* $cc\colon \mathrm{Conv} \to \mathrm{Comp}$ *is open-multicommutative.*

Proof. Let us prove that the characteristic map $\chi_{cc}\colon cc(X) \to Y_{cc}$ is open. Consider an arbitrary point $B \in cc(X)$ and an arbitrary net $\{C_\alpha\}_{\alpha \in \Gamma} \subset Y_{cc}$ such that $\lim C_\alpha = C$ with $C = \chi_{cc}(B)$. Recall that for every compactum X with convex structure the space

$cc(X) \subset \exp(X)$. Since $B \in \exp(X)$ and $\chi_{\exp}$ is open map, this implies that there exists a net $\{B_\alpha\}_{\alpha\in\Gamma} \subset \exp(X)$ such that

$$\lim B_\alpha = B \text{ and } \chi_{\exp}(B_\alpha) = C_\alpha$$

for every $\alpha \in \Gamma$. Denote by D_α the convex hull of the set B_α. Since the function $\mathrm{conv}\colon$ $exp(T) \to cc(T)$ is continuous in the Vietoris topology for every compactum T with the convex structure, we have that

$$\lim D_\alpha = \lim \mathrm{conv}(B_\alpha) = \mathrm{conv}(\lim B_\alpha) = \mathrm{conv}(B) = B \in cc(X).$$

Each C_α is a convex set therefore $\mathrm{pr}_A(C_\alpha)$ is also a convex set for every $A \in \mathcal{VG}$ hence Lemma 3.5 implies that $\chi_{cc}(D_\alpha) = C_\alpha$ for every $\alpha \in \Gamma$. Thus, the map χ_{cc} is open. The surjectivity of the characteristic map is obvious since every element in $\prod_{A\in\mathcal{VG}} cc(\mathcal{O}(A))$ is also in $cc(\prod_{A\in\mathcal{VG}} \mathcal{O}(A))$. ☐

Proposition 3.7. *Let Q_1, Q_2, $Q_3 \subset \mathrm{Top}$ be categories and let functors $F_1\colon Q_1 \to Q_2$ and $F_2\colon Q_3 \to Q_1$ be open-multicommutative then the composition $F_1 \circ F_2\colon Q_3 \to Q_2$ is also open-multicommutative.*

Proof. The open-multicommutativity of the functor F_2 implies that a characteristic map $\chi_{F_2,\mathcal{O}}\colon F_2(X) \to Y_{F_2}$ is open and surjective. Since the functor F_1 is open, the composition $F_1\chi_{F_2,\mathcal{O}}\colon F_1 \circ F_2(X) \to F_1(Y_{F_2})$ is also open and surjective. Consider now a diagram $\mathcal{O}_1$ over a graph $\mathcal{G}$ such that for every $A \in \mathcal{VG}$ we have $\mathcal{O}_1(A) = F_2(\mathcal{O}(A))$ and for each $\varphi \in \mathcal{EG}$ we have $\mathcal{O}_1(\varphi) = F_2(\mathcal{O}_1(\varphi))$. The functor F_1 is open-multicommutative, so this implies that the characteristic map $\chi_{F_1,\mathcal{O}_1}\colon F_1(Y_{F_2}) \to Y_{F_1\circ F_2}$ is open and surjective. Thus, a map $\chi_{F_1\circ F_2,\mathcal{O}} = F_1\chi_{F_2,\mathcal{O}} \circ \chi_{F_1,\mathcal{O}_1}\colon F_1 \circ F_2(X) \to Y_{F_1\circ F_2}$ is open and surjective for any diagram $\mathcal{O}$ as the composition of two open and surjective maps. This implies that the functor $F_1 \circ F_2$ is open-multicommutative. ☐

Corollary 3.8. *The functor $ccP\colon \mathrm{Comp} \to \mathrm{Comp}$ is open-multicommutative.*

Proof. This follows from the open-multicommutativity of the functors P (see Kozhan and Zarichnyi (2004)), cc (see Proposition 3.6) and Proposition 3.7. ☐

Proposition 3.9. *The functors G_{cc} and λ_{cc} are open-multicommutative.*

Proof. Let us define a retraction for every

$$r_{cc}X\colon G(X) \to G_{cc}(X)$$

in the following way: for every $\mathcal{A} \in G(X)$

$$r_{cc}X(\mathcal{A}) = \{B \in \exp(X)\colon B = \mathrm{conv}(A),\ A \in \mathcal{A}\}.$$

It is easy to see that $r_{cc}X(\mathcal{A}) \in G_{cc}(X)$.

Let us show that the map $r_{cc}X$ is continuous for every $X \in \mathrm{Conv}$. To prove the continuity at a point $\mathcal{A}_0 \in G(X)$ it is sufficient to show that for every element of the pre-base U_{cc}^+ and U_{cc}^- that contains $\mathcal{B}_0 = r_{cc}X(\mathcal{A}_0)$ there exists a neighborhood $\mathbf{V} \subset G(X)$ of the point $\mathcal{A}_0$ such that

$$r_{cc}X(\mathbf{V}) \subset U_{cc}^+ \text{ (} U_{cc}^- \text{ respectively).}$$

First assume that $\mathbf{U} = U_{cc}^{-}$. Denote $\mathbf{V} = U^{-}$. Then for every $\mathcal{A} \in \mathbf{V}$ we have

$$\forall A \in \mathcal{A}\colon A \cap U \neq \varnothing \Rightarrow \operatorname{conv}(A) \cap U \neq \varnothing.$$

Since

$$\mathcal{B} = r_{cc}X(\mathcal{A}) = \{B \in \exp(X)\colon B = \operatorname{conv}(A),\ A \in \mathcal{A}\},$$

this implies that for any $B \in \mathcal{B}$ we have $B \cap U \neq \varnothing$ and therefore $\mathcal{B} \in U_{cc}^{-} = \mathbf{U}$.

Assume now that $\mathbf{U} = U_{cc}^{+}$. Since X is a subset of a locally convex space E, there exists a base of the topology in X consisting of convex sets. This implies that we can find an open convex set $V \subset U$. Denote $\mathbf{V} = V^{+}$. For every $\mathcal{A} \in \mathbf{V}$ there is a set $A_1 \in \mathcal{A}$ such that $A_1 \subset V$. The set V is convex, thus $\operatorname{conv}(A_1) \subset V \subset U$. This implies that for $\mathcal{B} = r_{cc}X(\mathcal{A})$ we can find $B_1 \in \mathcal{B}$ such that $B_1 \subset U$ and this means that $\mathcal{B} \in U_{cc}^{+} = \mathbf{U}$. Thus, the map r_{cc} is continuous.

Let us prove that the characteristic map $\chi_{G_{cc}}\colon G_{cc}(X) \to Y_{G_{cc}}$ is open. Consider an arbitrary point $\mathcal{B} \in G_{cc}(X)$ and an arbitrary net $\{\mathcal{C}_\alpha\}_{\alpha\in\Gamma} \subset Y_{G_{cc}}$ such that $\lim \mathcal{C}_\alpha = \mathcal{C}$, where $\mathcal{C} = \chi_{G_{cc}}(B)$. For every compactum X the space $G_{cc}(X) \subset G(X)$ and then $\mathcal{B} \in G(X)$. Since χ_G is an open map, this implies that there exists a sequence $\{\mathcal{B}_\alpha\}_{\alpha\in\Gamma} \subset G(X)$ such that

$$\lim \mathcal{B}_\alpha = \mathcal{B} \text{ and } \chi_G(\mathcal{B}_\alpha) = \mathcal{C}_\alpha$$

for every $\alpha \in \Gamma$.

Denote by $\mathcal{D}_\alpha$ the image of the set $\mathcal{B}_\alpha$ under the map r_{cc}. Since the map r_{cc} is continuous, we see that

$$\lim \mathcal{D}_\alpha = \lim r_{cc}(\mathcal{B}_\alpha) = r_{cc}(\lim \mathcal{B}_\alpha) = r_{cc}(\mathcal{B}) = \mathcal{B} \in G_{cc}(X).$$

Each $\mathcal{C}_i$ is in $\prod_{A\in\mathcal{VG}} G_{cc}(\mathcal{O}(A))$ and let $\mathcal{C}'_\alpha = \chi_{G_{cc}}(\mathcal{D}_\alpha)$ for every $\alpha \in \Gamma$. Let us prove that $\mathcal{C}_\alpha = \mathcal{C}'_\alpha$. For every $A \in \mathcal{VG}$ we have that

$$G_{cc}\mathrm{pr}_A(\mathcal{D}_\alpha) = \mathrm{pr}_A(\mathcal{C}'_\alpha) = \{C' \in cc(X)\colon\ \mathrm{pr}_A(D) \subset C',\ D \in \mathcal{D}_\alpha\}.$$

Lemma 3.5 proves that if for some $C \in cc(X)$ the statement $\mathrm{pr}_A(B) \subset C$ is satisfied for $B \in \mathcal{B}_\alpha$ then $\mathrm{pr}_A(D) = \mathrm{pr}_A(\operatorname{conv}(B)) \subset C$. This implies that $G_{cc}\mathrm{pr}_A(\mathcal{B}_\alpha) \subset G_{cc}\mathrm{pr}_A(\mathcal{D}_\alpha)$. On the other hand, if $\mathrm{pr}_A(D) \subset C'$ for some $D \in \mathcal{D}_\alpha$ then

$$\mathrm{pr}_A(B) \subset \mathrm{pr}_A(\operatorname{conv}(B)) = \mathrm{pr}_A(D) \subset C' \text{ for } B \in \mathcal{B}_\alpha.$$

This means that $G_{cc}\mathrm{pr}_A(\mathcal{B}_\alpha) \supset G_{cc}\mathrm{pr}_A(\mathcal{D}_\alpha)$. Since these two inclusions are satisfied for every $A \in \mathcal{VG}$, therefore

$$\prod_{A\in\mathcal{VG}} G_{cc}\mathrm{pr}_A(\mathcal{B}_i) = \prod_{A\in\mathcal{VG}} G_{cc}\mathrm{pr}_A(\mathcal{D}_i)$$

and this is equivalent to $\mathcal{C}_\alpha = \mathcal{C}'_\alpha$. This immediately implies that $\chi_{G_{cc}}(\mathcal{D}_\alpha) = \mathcal{C}_\alpha$ for every $\alpha \in \Gamma$. So, the characteristic map $\chi_{G_{cc}}$ is open and the functor G_{cc} is open-multicommutative. This map is also surjective since for every $\mathcal{C} \in Y_{G_{cc}}$ we have $\chi_{G_{cc}}(\mathcal{D}_C) = \mathcal{C}$, where

$$\mathcal{D}_C = \left\{ D \in cc(X)\colon\ D \supset \prod_{A\in\mathcal{VG}} C_A,\ C_A \in \mathrm{pr}_A(\mathcal{C}) \right\}.$$

Let us note that the restriction

$$r_{cc}|_{\lambda(X)}\colon \lambda(X) \to \lambda_{cc}(X)$$

is also a continuous retraction. Using this and the open-commutativity of the functor λ we can in the same way conclude that the functor λ_{cc} is also open-multicommutative. $\square$

Consider now the functors $G_{cc}P$ and $\lambda_{cc}P$ (see, e.g., Teleiko and Zarichnyi (1999) for the definition). Actually, $G_{cc}P(X) = G_{cc}(P(X))$ for every compactum X and this functor is the composition of two functors G_{cc} and P (the same situation is for the functor $\lambda_{cc}P$). The following result follows from Propositions 3.7 and 3.9.

Corollary 3.10. *The functors $G_{cc}P$ and $\lambda_{cc}P$ are open-multicommutative.*

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