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A. Ya. Khrystiyanyn, A. A. Kondratyuk

ON THE NEVANLINNA THEORY FOR MEROMORPHIC FUNCTIONS ON ANNULI. II

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In this paper we prove analogs of the Lemma on Logarithmic Derivative and the Second Fundamental Theorem of the Nevanlinna theory for meromorphic functions on annuli.

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Доказываются аналоги леммы о логарифмической производной и второй основной теоремы теории распределения значений Неванлинны для функций, мероморфных в плоском круговом кольце.

In this part of our paper we use the definitions, notations and results of the first part [1].

1. Main results.

Theorem 1. (Lemma on the logarithmic derivative) *Let f be a nonconstant meromorphic function on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 \le +\infty$, and let $\lambda \ge 0$. Then*

(i) in the case $R_0 = +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O(\log(RT_0(R, f))) \tag{1}$$

for $R \in (1; +\infty)$ except for the set Δ_R such that

$$\int\limits_{\Delta_R} R^{\lambda-1} dR < +\infty; \tag{2}$$

(ii) in the case $R_0 < +\infty$,

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right) \tag{3}$$

for $R \in (1; R_0)$ except for the set Δ'_R such that

$$\int\limits_{\Delta'_R} \frac{dR}{(R_0 - R)^{\lambda+1}} < +\infty. \tag{4}$$

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Theorem 2. *Let f be a nonconstant meromorphic function on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 \le +\infty$. Let $a_1, a_2, ..., a_p$ be p distinct finite complex numbers and $\lambda \ge 0$. Then*

$$m_0(R, f) + \sum_{\nu=1}^{p} m_0\left(R, \frac{1}{f - a_\nu}\right) \le 2T_0(R, f) - N_0^{(1)}(R, f) + S(R, f), \tag{5}$$

where

$$N_0^{(1)}(R, f) = N_0\left(R, \frac{1}{f'}\right) + 2N_0(R, f) - N_0(R, f'), \tag{6}$$

and

1) *in the case $R_0 = +\infty$,*

$$S(R, f) = O(\log(RT_0(R, f))) \tag{7}$$

for $R \in (1; +\infty)$ except for the set Δ_R such that $\int_{\Delta_R} R^{\lambda-1} dR < +\infty$;

2) *if $R_0 < +\infty$ then*

$$S(R, f) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right) \tag{8}$$

for $R \in (1; R_0)$ except for the set Δ_R' such that $\int_{\Delta_R'} \frac{dR}{(R_0-R)^{\lambda+1}} < +\infty$.

Definition 1. Let f be a nonconstant meromorphic function on $A = \{z : 0 < |z| < +\infty\}$ and $a \in \mathbf{C}$. The value

$$\delta_0(a) = \varliminf_{R\to+\infty} \frac{m_0\left(R, \frac{1}{f-a}\right)}{T_0(R, f)}$$

is called the *deficiency* or *defect* of the function f for the value a,

$$\delta_0(\infty) = \varliminf_{R\to+\infty} \frac{m_0(R, f)}{T_0(R, f)}.$$

Theorem 3. *Let f be a nonconstant meromorphic function on $A = \{z : 0 < |z| < \infty\}$. Let $\{a_\nu\}$ be any finite collection of complex numbers possibly including ∞. Then,*

$$\sum_{\nu=1}^{q} \delta_0(a_\nu) \le 2.$$

Theorems 1 – 3 are generalizations of respective results of the Nevanlinna value distribution theory [2] for meromorphic functions on $\{z : |z| < R_0\}$, $R_0 \le +\infty$.

2. Auxiliary results. Let $\overset{\circ}{T}_0(R, f)$ be the spherical characteristic of the function f. Then [1]

$$\overset{\circ}{T}_0(R, f) = N_0\left(R, \frac{1}{f - a}\right) + \overset{\circ}{m}_0(R, a), \quad R > 1.$$

Since $0 \le k(w, a) \le 1$, we have

$$N_0\left(R, \frac{1}{f - a}\right) \le \overset{\circ}{T}_0(R, f) + \frac{1}{\pi}\int_0^{\pi} \log \frac{1}{k(f(e^{i\theta}), a)} d\theta.$$

If μ is a nonnegative unit mass distribution on the plane then

$$\int\limits_{|a|<+\infty} N_0\left(R, \frac{1}{f-a}\right) d\mu \leq \overset{\circ}{T}_0(R) + \frac{1}{2\pi} \int\limits_{|a|<+\infty} \left(\int\limits_0^{\pi} \log \frac{1}{k(f(e^{i\theta}), a)} d\theta \right) d\mu$$

or

$$\begin{aligned} \int\limits_{|a|<+\infty} N_1\left(R, \frac{1}{f-a}\right) d\mu + \int\limits_{|a|<+\infty} N_2\left(R, \frac{1}{f-a}\right) d\mu \leq \overset{\circ}{T}_0(R, f) + \\ + \frac{1}{2\pi} \int\limits_{|a|<+\infty} \left(\int\limits_0^{\pi} \log \frac{1}{k(f(e^{i\theta}), a)} d\theta \right) d\mu. \end{aligned} \tag{9}$$

We put

$$Q_i(R) = \int\limits_{|a|<+\infty} N_i\left(R, \frac{1}{f-a}\right) d\mu, \quad i \in \{1, 2\}.$$

Then (9) can be written as follows:

$$Q_1(R) + Q_2(R) \leq \overset{\circ}{T}_0(R, f) + \frac{1}{2\pi} \int\limits_{|a|<+\infty} \left(\int\limits_0^{\pi} \log \frac{1}{k(f(e^{i\theta}), a)} d\theta \right) d\mu. \tag{10}$$

Note that

$$\int\limits_{1/R}^{1} \frac{n_1\left(t, \frac{1}{f-a}\right)}{t} dt = \int\limits_1^R \frac{n_1\left(\frac{1}{\tau}, \frac{1}{f-a}\right)}{\tau} d\tau.$$

Denote

$$\Omega_1(t) = \int\limits_{|a|<+\infty} n_1\left(\frac{1}{t}, \frac{1}{f-a}\right) d\mu, \quad \Omega_2(t) = \int\limits_{|a|<+\infty} n_2\left(t, \frac{1}{f-a}\right) d\mu.$$

Then

$$Q_i(R) = \int\limits_1^R \frac{\Omega_i(t)}{t} dt, \quad i \in \{1, 2\}.$$

Let μ be absolutely continuous as a function of a set on the (a)-plane. Then μ has nonnegative finite density $\rho(a)$ almost everywhere and therefore the mass $\mu(e)$ of the measurable set e is equal to the Lebesgue integral

$$\mu(e) = \int\limits_e \rho(a) d\sigma_a$$

where $d\sigma_a$ means an element of the measure on the plane.

Then [2]

$$\Omega_1(R) = \int\limits_{|a|<+\infty} n_1\left(\frac{1}{R}, \frac{1}{f-a}\right) \rho(a) d\sigma_a = \iint\limits_{\frac{1}{R} \leq |z| \leq 1} |f'(z)|^2 \rho(f(z)) d\sigma_z,$$

where $d\sigma_z = r dr d\varphi$, $z = re^{i\varphi}$.

Similarly,

$$\Omega_2(R) = \int\limits_{|a|<+\infty} n_2\left(R, \frac{1}{f-a}\right)\rho(a)d\sigma_a = \iint\limits_{1\le|z|\le R} |f'(z)|^2\rho(f(z))d\sigma_z.$$

We also suppose that the function $\rho(a)$ is such that the derivatives $\Omega_1'(R)$ and $\Omega_2'(R)$ exist and are finite. Thus,

$$\Omega_1'(R) = -\frac{1}{R^2}\left(-\frac{1}{R}\int\limits_0^{2\pi}\left|f'\left(\frac{1}{R}e^{i\varphi}\right)\right|^2\rho\left(f\left(\frac{1}{R}e^{i\varphi}\right)\right)d\varphi\right) =$$

$$= \frac{1}{R^3}\int\limits_0^{2\pi}\left|f'\left(\frac{1}{R}e^{i\varphi}\right)\right|^2\rho(f(\frac{1}{R}e^{i\varphi}))d\varphi, \quad \Omega_2'(R) = R\int\limits_0^{2\pi}|f'(Re^{i\varphi})|^2\rho(f(Re^{i\varphi}))d\varphi.$$

Lemma 1. *Let f be a nonconstant meromorphic function on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 \le +\infty$. Let μ be an absolutely continuous mass distribution and $\rho(a)$ be its density. Then*

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+(|f'(Re^{i\theta})|^2\rho(f(Re^{i\theta})))d\theta \le \log^+\frac{\Omega_2'(R)}{2\pi R} + \log 2, \tag{11}$$

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+\left(\left|f'\left(\frac{1}{R}e^{i\theta}\right)\right|^2\rho\left(f\left(\frac{1}{R}e^{i\theta}\right)\right)\right)d\theta \le \log^+\frac{R^3\Omega_1'(R)}{2\pi} + \log 2 \tag{12}$$

for $R \in (1; R_0)$

The proof of Lemma 1 is similar to that of Lemma 1 from [2, p. 249].

Lemma 2. *Let f be a nonconstant meromorphic function on $\{z : 0 < |z| < +\infty\}$, and let $\lambda \ge 0$. Then*

(i)

$$\log\frac{R^3\Omega_1'(R)}{2\pi} < 4\log Q_1(R) + (3\lambda + 2)\log R$$

for $R \in (1; +\infty)$ except for the set $\Delta_R^{(1)}$ such that $\int_{\Delta_R^{(1)}} R^{\lambda-1}dR < +\infty$;

(ii)

$$\log\frac{\Omega_2'(R)}{2\pi R} < 4\log Q_2(R) + 3\lambda\log R$$

for $R \in (1; +\infty)$ except for the set $\Delta_R^{(2)}$ such that $\int_{\Delta_R^{(2)}} R^{\lambda-1}dR < +\infty$.

Proof. The proof of statement (ii) of Lemma 2 may be obtained by the same way as the proof of Lemma 2 ([2], p. 250).

The proof of statement (i) is also similar to that from [2]. Namely, choose $\Delta_R^{(1)}$ such that

$$\Omega_1(R) \ge R^\lambda(Q_1(R))^2 \quad \text{and} \quad \Omega_1'(R) \ge R^{\lambda-1}(\Omega_1(R))^2).$$

Then ([2], p.250) $\int_{\Delta_R^{(1)}} d\left(\frac{R^\lambda}{\lambda}\right) < +\infty$. Thus, we obtain that everywhere on the set $(1;+\infty)\backslash\Delta_R^{(1)}$

$$R^3\Omega_1'(R) < R^3R^{\lambda-1}(\Omega_1(R))^2 = R^{\lambda+2}(\Omega_1(R))^2 < R^{\lambda+2}R^{2\lambda}(Q_1(R))^4 = R^{3\lambda+2}(Q_1(R))^4.$$

Lemma 2 is proved. □

Lemma 3. *Let f be a nonconstant meromorphic function on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 < +\infty$, and let $\lambda \geq 0$. Then*

(i)

$$\log\frac{\Omega_2'(R)}{R} < 9\log^+ Q_2(R) + (4\lambda+4)\log\frac{1}{R_0 - R}$$

for $R \in (1; R_0)$ except for the set $\Delta_R^{(2)}$ such that $\int_{\Delta_R^{(2)}} \frac{dR}{(R_0-R)^{\lambda+1}} < +\infty$;

(ii)

$$\log R^3\Omega_1'(R) < 9\log^+ Q_1(R) + (4\lambda+4)\log\frac{1}{R_0 - R} + 3\log R_0$$

for $R \in (1; R_0)$ except for the set $\Delta_R^{(1)}$ such that $\int_{\Delta_R^{(1)}} \frac{dR}{(R_0-R)^{\lambda+1}} < +\infty$.

Proof. (*i*) Let $\Delta'_{R,(2)}$ be a set of R ($1 < R < R_0$) such that

$$\Omega_2(R) \geq \frac{Q_2(R)(1 + (\log Q_2(R))^2)}{(R_0 - R)^{\lambda+1}}.$$

Then

$$\int\limits_{\Delta'_{R,(2)}} \frac{dR}{(R_0-R)^{\lambda+1}} = \int\limits_{\Delta'_{R,(2)}} \frac{dR}{dQ_2(R)}\frac{dQ_2(R)}{(R_0-R)^{\lambda+1}} = \int\limits_{\Delta'_{R,(2)}} \frac{R}{\Omega_2(R)}\frac{dR}{(R_0-R)^{\lambda+1}} \leq$$

$$\leq R_0 \int\limits_{\Delta'_{R,(2)}} \frac{(R_0-R)^{\lambda+1}}{Q_2(R)(1+(\log Q_2(R))^2)}\frac{dQ_2(R)}{(R_0-R)^{\lambda+1}} = R_0 \int\limits_{\Delta'_{R,(2)}} \frac{d\log Q_2(R)}{(1+(\log Q_2(R))^2)} \leq \pi R_0.$$

Similarly, if $\Delta''_{R,(2)}$ is a set of R ($1 < R < R_0$) such that

$$\frac{\Omega_2'(R)}{R} \geq \frac{\Omega_2(R)(1 + (\log \Omega_2(R))^2)}{(R_0 - R)^{\lambda+1}}$$

then

$$\int\limits_{\Delta''_{R,(2)}} \frac{dR}{(R_0-R)^{\lambda+1}} = \int\limits_{\Delta''_{R,(2)}} \frac{dR}{d\Omega_2(R)}\frac{d\Omega_2(R)}{(R_0-R)^{\lambda+1}} = \int\limits_{\Delta''_{R,(2)}} \frac{R}{R\Omega_2'(R)}\frac{d\Omega_2(R)}{(R_0-R)^{\lambda+1}} \leq$$

$$\leq \int\limits_{\Delta''_{R,(2)}} \frac{(R_0-R)^{\lambda+1}}{\Omega_2(R)(1+(\log \Omega_2(R))^2)}\frac{d\Omega_2(R)}{(R_0-R)^{\lambda+1}} = \int\limits_{\Delta''_{R,(2)}} \frac{d\log \Omega_2(R)}{(1+(\log \Omega_2(R))^2)} \leq \pi.$$

Thus, we have

$$\int\limits_{\Delta_R^{(2)}} \frac{dR}{(R_0 - R)^{\lambda+1}} \leq (R_0 + 1)\pi,$$

where $\Delta_R^{(2)} = \Delta'_{R,(2)} \cup \Delta''_{R,(2)}$. So, on $(1; R_0) \backslash \Delta_R^{(2)}$,

$$\begin{gathered} \log \frac{\Omega_2'(R)}{R} < \log \frac{\Omega_2(R)(1 + (\log \Omega_2(R))^2)}{(R_0 - R)^{\lambda+1}} < \\ < \log \left(\frac{Q_2(R)(1 + (\log Q_2(R))^2)}{(R_0 - R)^{2\lambda+2}} (1 + (\log \Omega_2(R))^2) \right) = \log Q_2(R) + \\ + \log(1 + (\log Q_2(R))^2) + (2\lambda + 2) \log \frac{1}{R_0 - R} + \log(1 + (\log \Omega_2(R))^2) \leq \\ \leq \log^+ Q_2(R) + 2\log^+ Q_2(R) + (2\lambda + 2) \log \frac{1}{R_0 - R} + 2\log^+ \Omega_2(R). \end{gathered} \tag{13}$$

Since

$$\log^+ \Omega_2(R) \leq \log^+ \frac{Q_2(R)(1 + (\log Q_2(R))^2)}{(R_0 - R)^{\lambda+1}} \leq \log^+ Q_2(R) + 2\log^+ Q_2(R) + \log \frac{1}{(R_0 - R)^{\lambda+1}},$$

(13) can be rewritten as follows:

$$\log \frac{\Omega_2'(R)}{R} < 9\log^+ Q_2(R) + (4\lambda + 4) \log \frac{1}{R_0 - R}.$$

(ii) Let $\Delta'_{R,(1)}$ be a set of R $(1 < R < R_0)$ such that

$$\Omega_1(R) \geq \frac{Q_1(R)(1 + (\log Q_1(R))^2)}{(R_0 - R)^{\lambda+1}}.$$

Then similarly as in case (i) we have

$$\int\limits_{\Delta'_{R,(1)}} \frac{dR}{(R_0 - R)^{\lambda+1}} \leq \pi R_0.$$

Let $\Delta''_{R,(1)}$ be a set of R $(1 < R < R_0)$ such that

$$\Omega_1'(R) \geq \frac{\Omega_1(R)(1 + (\log \Omega_1(R))^2)}{(R_0 - R)^{\lambda+1}}.$$

Then

$$\int\limits_{\Delta''_{R,(1)}} \frac{dR}{(R_0 - R)^{\lambda+1}} = \int\limits_{\Delta''_{R,(1)}} \frac{dR}{d\Omega_1} \frac{d\Omega_1(R)}{(R_0 - R)^{\lambda+1}} \leq \int\limits_{\Delta''_{R,(1)}} \frac{d\Omega_1(R)}{\Omega_1(R)(1 + (\log \Omega_1(R))^2)} \leq \pi.$$

Thus

$$\int\limits_{\Delta_R^{(1)}} \frac{dR}{(R_0 - R)^{\lambda+1}} \leq (R_0 + 1)\pi,$$

where $\Delta_R^{(1)} = \Delta'_{R,(1)} \cup \Delta''_{R,(1)}$.

Let $R \in (1; R_0) \backslash \Delta_R^{(1)}$, then

$$\log(R^3 \Omega_1'(R)) = 3 \log R + \log \Omega_1'(R) < 3 \log R_0 + \log \Omega_1'(R).$$

Since

$$\log \Omega_1'(R) < \log \frac{\Omega_1(R)(1 + (\log \Omega_1(R))^2)}{(R_0 - R)^{\lambda+1}} <$$

$$< \log \left(\frac{Q_1(R)(1 + (\log Q_1(R))^2)}{(R_0 - R)^{2\lambda+2}} (1 + (\log \Omega_1(R))^2) \right),$$

we have

$$\log R^3 \Omega_1'(R) < 9 \log^+ \Omega_1(R) + (4\lambda + 4) \log \frac{1}{R_0 - R} + 3 \log R_0.$$

Lemma 3 is proved. $\square$

Choose the mass distribution μ with the density

$$\rho(a) = \frac{1}{2\pi^2} \frac{1}{|a|^2(1 + (\log |a|)^2)}.$$

Then

$$\int\limits_{|a|<+\infty} d\mu = \int\limits_{|a|<+\infty} \rho(a) d\sigma_a = \frac{1}{\pi} \int\limits_0^{+\infty} \frac{d|a|}{|a|(1 + (\log |a|)^2)} = 1$$

Let $w \in \overline{\mathbf{C}}$. At first we consider the expression

$$P(w) = \int\limits_{|a|<+\infty} \log \frac{1}{k(w, a)} d\mu.$$

Lemma 4. *$P(w)$ becomes infinite only whenever w is equal to* 0 *or to* ∞.

Proof. We have

$$P(w) = \log \sqrt{1 + |w|^2} + U(w),$$

where

$$U(w) = \int\limits_{|a|<+\infty} \log \frac{\sqrt{1 + |a|^2}}{|w - a|} d\mu.$$

It is easy to see that $-U(w)$ is a subharmonic in $\mathbf{C}$ function and depends only on $|w|$. According to the Jensen theorem for subharmonic functions ([3]) we can write

$$\frac{1}{2\pi} \int\limits_0^{2\pi} (-U(|w| e^{i\theta})) d\theta - \frac{1}{2\pi} \int\limits_0^{2\pi} (-U(|w_0| e^{i\theta})) d\theta = N(|w|, -U) - N(|w_0|, -U), \quad |w_0| < |w|, \tag{14}$$

where

$$N(|w|, -U) - N(|w_0|, -U) = \int\limits_{|w_0|}^{|w|} \frac{n(t)}{t} dt$$

and $n(t)=\mu(\{z:|z|\le t\})$. Thus,

$$-\frac{\partial U(|w|)}{\partial|w|}=\frac{n(|w|)}{|w|}=\frac{1}{|w|}\int\limits_{|a|\le|w|}d\mu=\frac{1}{|w|}\frac{1}{\pi}\int\limits_{|w|}\frac{d|a|}{|a|(1+(\log|a|))^2}.$$

After integrating we obtain

$$U(w)=\frac{1}{\pi}(\log\sqrt{1+(\log|w|)^2})-\log|w|\cdot\operatorname{arctg}\log|w|-\frac{\pi}{2}\log|w|)+U(1),$$

where

$$U(1)=\int\limits_{|a|<+\infty}\log\frac{\sqrt{1+|a|^2}}{|a-1|}d\mu=\frac{2}{\pi}\int\limits_0^{+\infty}\log\sqrt{1+e^{-2t}}\frac{dt}{1+t^2}$$

is finite. Thus $P(w)$ is finite for all complex w except for $w=0$ and $w=\infty$. This completes the proof of Lemma 4. $\square$

3. Proof of Theorem 1. We follow [2]. Find upper estimate for $m(R,\frac{f'}{f})$

$$m\left(R,\frac{f'}{f}\right)=\frac{1}{4\pi}\int\limits_0^{2\pi}\log^+\left|\frac{f'(Re^{i\theta})}{f(Re^{i\theta})}\right|^2d\theta\le\frac{1}{4\pi}\int\limits_0^{2\pi}\log^+\frac{|f'(Re^{i\theta})|^2}{|f(Re^{i\theta})|^2(1+(\log|f(Re^{i\theta})|)^2)}d\theta+$$
$$+\frac{1}{4\pi}\int\limits_0^{2\pi}\log\sqrt{1+(\log|f(Re^{i\theta})|)^2}d\theta. \tag{15}$$

Let $R_0=+\infty$. Using Lemmas 1 and 2 we can estimate the first integral in (15)

$$\frac{1}{4\pi}\int\limits_0^{2\pi}\log^+\frac{|f'(Re^{i\theta})|^2}{|f(Re^{i\theta})|^2(1+(\log|f(Re^{i\theta})|)^2)}d\theta\le\frac{1}{2}(4\log^+Q_2(R)+3\lambda\log R)\le$$
$$\le\frac{1}{2}(4\log^+(Q_1(R)+Q_2(R))+3\lambda\log R), \tag{16}$$

where $\lambda\ge0$. This inequality is valid for $R\in(1;+\infty)$ except for the set $\Delta_R^{(2)}$ from Lemma 2.

Similarly,

$$m\left(\frac{1}{R},\frac{f'}{f}\right)=\frac{1}{4\pi}\int\limits_0^{2\pi}\log^+\left|\frac{f'(\frac{1}{R}e^{i\theta})}{f(\frac{1}{R}e^{i\theta})}\right|^2d\theta\le\frac{1}{4\pi}\int\limits_0^{2\pi}\log^+\frac{|f'(\frac{1}{R}e^{i\theta})|^2}{|f(\frac{1}{R}e^{i\theta})|^2(1+(\log|f(\frac{1}{R}e^{i\theta})|)^2)}d\theta+$$
$$+\frac{1}{4\pi}\int\limits_0^{2\pi}\log\sqrt{1+\left(\log\left|f\left(\frac{1}{Re^{i\theta}}\right)\right|\right)^2}d\theta \tag{17}$$

and

$$\frac{1}{4\pi}\int\limits_0^{2\pi}\log^+\frac{|f'(\frac{1}{R}e^{i\theta})|^2}{|f(\frac{1}{R}e^{i\theta})|^2(1+(\log|f(\frac{1}{R}e^{i\theta})|)^2)}d\theta\le\frac{1}{2}(4\log^+Q_1(R)+(3\lambda+2)\log R)\le$$
$$\le\frac{1}{2}(4\log^+(Q_1(R)+Q_2(R))+(3\lambda+2)\log R), \tag{18}$$

where $\lambda \geq 0$. This inequality holds for $R \in (1; +\infty)$ except for the set $\Delta_R^{(1)}$ from Lemma 2.

The case when $R_0 < +\infty$ is similar. Inequalities (15) and (17) are still valid. Using Lemmas 1 and 3, we have

$$\frac{1}{4\pi}\int_0^{2\pi} \log^+ \frac{|f'(Re^{i\theta})|^2}{|f(Re^{i\theta})|^2(1+(\log|f(Re^{i\theta})|)^2)} d\theta \leq \\ \leq \frac{1}{2}\left(9\log^+(Q_1(R)+Q_2(R)) + (4\lambda+4)\log\frac{1}{R_0-R}\right), \tag{19}$$

where $\lambda \geq 0$. This inequality holds for $R \in (1; R_0)$ except for the set $\Delta_R^{(2)}$ from Lemma 3. And

$$\frac{1}{4\pi}\int_0^{2\pi} \log^+ \frac{|f'(\frac{1}{R}e^{i\theta})|^2}{|f(\frac{1}{R}e^{i\theta})|^2(1+(\log|f(\frac{1}{R}e^{i\theta})|)^2)} d\theta \leq \\ \leq \frac{1}{2}\left(9\log^+(Q_1(R)+Q_2(R)) + (4\lambda+4)\log\frac{1}{R_0-R} + 3\log R_0\right), \tag{20}$$

where $\lambda \geq 0$. This inequality holds for $R \in (1; R_0)$ except for the set $\Delta_R^{(1)}$ from Lemma 3.

In the case when the function f has neither zeroes nor poles on the circle $\{z : |z| = 1\}$, we obtain, using (10), (16) and (18), that the first integrals in (15) and (17) are $O(\log(R\overset{\circ}{T}_0(R,f)))$, for $R \in (1; +\infty)$ except for the set $\Delta_R^{(1)} \cup \Delta_R^{(2)}$. Similarly, if $R_0 < +\infty$ then the first integrals in (15) and (17) are $O\left(\log\frac{\overset{\circ}{T}_0(R,f)}{R_0-R}\right)$, for $R \in (1; R_0)$ except for the set $\Delta_R^{(1)} \cup \Delta_R^{(2)}$.

Now we can estimate the last integrals of (15) and (17):

$$\frac{1}{4\pi}\int_0^{2\pi} \log\sqrt{1+(\log|f(Re^{i\theta})|)^2} d\theta \leq \log\frac{1}{4\pi}\int_0^{2\pi}\sqrt{1+(\log|f(Re^{i\theta})|)^2} d\theta \leq \\ \leq \frac{1}{4\pi}\int_0^{2\pi}(1+|\log|f(Re^{i\theta})||)d\theta = \log\left(1+m(R,f)+m\left(R,\frac{1}{f}\right)\right). \tag{21}$$

Similarly,

$$\frac{1}{4\pi}\int_0^{2\pi} \log\sqrt{1+\left(\log\left|f\left(\frac{1}{R}e^{i\theta}\right)\right|\right)^2} d\theta \leq \log\left(1+m\left(\frac{1}{R},f\right)+m\left(\frac{1}{R},\frac{1}{f}\right)\right). \tag{22}$$

Since $T_0(R,f) = m(R,f) + m(\frac{1}{R},f) - 2m(1,f) + N_0(R,f)$ and $T_0(R,f) = T_0(R,\frac{1}{f})$, we have

$$\log\left(1+m(R,f)+m\left(R,\frac{1}{f}\right)\right) \leq \log(1+2T_0(R,f)+4m(1,f)) \leq \\ \leq \log^+ T_0(R,f) + O(1), \quad R \in (1; R_0), \tag{23}$$

$$\log\left(1+m\left(\frac{1}{R},f\right)+m\left(\frac{1}{R},\frac{1}{f}\right)\right) \leq \log(1+2T_0(R,f)+4m(1,f)) \leq \\ \leq \log^+ T_0(R,f) + O(1), \quad R \in (1; R_0). \tag{24}$$

Using (15)-(18), (21)-(24) and the relation $T_0(R,f) = \overset{o}{T}_0(R,f) + O(1)$, $R \in (1;+\infty)$ ([1]), we obtain in the case $R_0 = +\infty$

$$m\left(R, \frac{f'}{f}\right) = O(\log(RT_0(R,f))), \quad m\left(\frac{1}{R}, \frac{f'}{f}\right) = O(\log(RT_0(R,f))), \quad R \to +\infty \tag{25}$$

except for the sets $\Delta_R^{(1)}$ and $\Delta_R^{(2)}$ mentioned above.

When $R_0 < +\infty$ we obtain using (15), (17), (19), (20), (21)-(24) and the relation $T_0(R,f) = \overset{o}{T}_0(R,f) + O(1)$, $R \in (1;R_0)$, [1]:

$$m\left(R, \frac{f'}{f}\right) = O\left(\log \frac{T_0(R,f)}{R_0 - R}\right), \quad m\left(\frac{1}{R}, \frac{f'}{f}\right) = O\left(\log \frac{T_0(R,f)}{R_0 - R}\right), \quad R \to +\infty \tag{26}$$

except for the sets $\Delta_R^{(1)}$ and $\Delta_R^{(2)}$.

In the case when the function f has zeroes or poles on the circle $\{z : |z| = 1\}$ we consider the function

$$f_1(z) = f(z) \frac{\prod\limits_{j=1}^{s} (z - b_j)^{l_j}}{\prod\limits_{j=1}^{n} (z - a_j)^{k_j}}$$

where a_j are the zeroes, b_j are the poles of the function f on the $\{z : |z| = 1\}$ and k_j, l_j are their multiplicities respectively. Then the function f_1 has neither zeroes nor poles on $\{z : |z| = 1\}$. The functions $m(R,f)$ and $m(R,f_1)$ differ at most by the value $O(\log R)$ for $R \in (1;R_0)$. The same holds for the pairs of the functions $m(\frac{1}{R}, f)$, $m(\frac{1}{R}, f_1)$, and $T_0(R,f)$, $T_0(R,f_1)$ for $R \in (1;R_0)$. Therefore, (25) holds in this case too.

4. Proof of Theorem 2. Using the well-known inequality $\log^+ ab \le \log^+ a + \log^+ b$ we obtain

$$\begin{aligned} m(R,f') &= m\left(R, f\frac{f'}{f}\right) \le m(R,f) + m\left(R, \frac{f'}{f}\right), \\ m\left(\frac{1}{R}, f'\right) &= m\left(\frac{1}{R}, f\frac{f'}{f}\right) \le m\left(\frac{1}{R}, f\right) + m\left(\frac{1}{R}, \frac{f'}{f}\right). \end{aligned} \tag{27}$$

Let $a_1, a_2, ..., a_p$, $p \ge 2$, be a set of distinct finite complex numbers and $\delta = \min\{|a_n - a_k|, \ 1\}$, $n \ne k$. Then as in [2, p. 244-245] we have

$$\begin{aligned} m\left(R, \frac{1}{f'}\right) &> \sum_{\nu=1}^{p} m(R, \frac{1}{f - a_\nu}) - m\left(R, \sum_{\nu=1}^{p} \frac{f'}{f - a_\nu}\right) - p\log\frac{2p}{\delta} - \log 3, \\ m\left(\frac{1}{R}, \frac{1}{f'}\right) &> \sum_{\nu=1}^{p} m\left(\frac{1}{R}, \frac{1}{f - a_\nu}\right) - m\left(\frac{1}{R}, \sum_{\nu=1}^{p} \frac{f'}{f - a_\nu}\right) - p\log\frac{2p}{\delta} - \log 3. \end{aligned} \tag{28}$$

We obtain from (27) and (28)

$$m(R,f') + m\left(\frac{1}{R}, f'\right) \le m(R,f) + m\left(\frac{1}{R}, f\right) + m\left(R, \frac{f'}{f}\right) + m\left(\frac{1}{R}, \frac{f'}{f}\right), \tag{29}$$

$$
\begin{aligned}
m\left(R,\frac{1}{f'}\right)+m\left(\frac{1}{R},\frac{1}{f'}\right) > \sum_{\nu=1}^{p} m\left(R,\frac{1}{f-a_\nu}\right)+\sum_{\nu=1}^{p} m\left(\frac{1}{R},\frac{1}{f-a_\nu}\right)- \\
-m\left(R,\sum_{\nu=1}^{p}\frac{f'}{f-a_\nu}\right)-m\left(\frac{1}{R},\sum_{\nu=1}^{p}\frac{f'}{f-a_\nu}\right)-2p\log\frac{2p}{\delta}-2\log 3,
\end{aligned}
\tag{30}
$$

Add $-2m(1,f')+N_0(R,f')$ and $-2m(1,\frac{1}{f'})+N_0(R,\frac{1}{f'})$ to both sides of (29) and (30) respectively. Using the equality $T_0(R,f')=T_0(R,\frac{1}{f'})$, $1<R<R_0$ ([1]) we can make the conclusion that the characteristic $T_0(R,f')$ lays between two values

$$
\sum_{\nu=1}^{p} m\left(R,\frac{1}{f-a_\nu}\right)+\sum_{\nu=1}^{p} m\left(\frac{1}{R},\frac{1}{f-a_\nu}\right)-m\left(R,\sum_{\nu=1}^{p}\frac{f'}{f-a_\nu}\right)-m\left(\frac{1}{R},\sum_{\nu=1}^{p}\frac{f'}{f-a_\nu}\right)-
$$

$$
-2m\left(1,\frac{1}{f'}\right)+N_0\left(R,\frac{1}{f'}\right)-2p\log\frac{2p}{\delta}-2\log 3
$$

and

$$
m(R,f)+m\left(\frac{1}{R},f\right)+m\left(R,\frac{f'}{f}\right)+m\left(\frac{1}{R},\frac{f'}{f}\right)-2m(1,f')+N_0(R,f').
$$

In particular, the first value is less then the second one. Therefore,

$$
\begin{aligned}
\sum_{\nu=1}^{p} m\left(R,\frac{1}{f-a_\nu}\right)+\sum_{\nu=1}^{p} m\left(\frac{1}{R},\frac{1}{f-a_\nu}\right) < m(R,f)+m\left(\frac{1}{R},f\right)+m\left(R,\frac{f'}{f}\right)+ \\
+m\left(\frac{1}{R},\frac{f'}{f}\right)-2m(1,f')+N_0(R,f')+m\left(R,\sum_{\nu=1}^{p}\frac{f'}{f-a_\nu}\right)+m\left(\frac{1}{R},\sum_{\nu=1}^{p}\frac{f'}{f-a_\nu}\right)+ \\
+2m\left(1,\frac{1}{f'}\right)-N_0\left(R,\frac{1}{f'}\right)+2p\log\frac{2p}{\delta}+2\log 3.
\end{aligned}
\tag{31}
$$

Add $m(R,f)+m(\frac{1}{R},f)$ to both sides of inequality (31). Then

$$
m_0(R,f)+\sum_{\nu=1}^{p} m_0\left(R,\frac{1}{f-a_\nu}\right) < 2T_0(R,f)-N_0^{(1)}(R,f)+S(R,f), \tag{32}
$$

where

$$
N_0^{(1)}(R,f)=N_0\left(R,\frac{1}{f'}\right)+2N_0(R,f)-N_0(R,f') \tag{33}
$$

and

$$
\begin{aligned}
S(R,f)=m\left(R,\frac{f'}{f}\right)+m\left(\frac{1}{R},\frac{f'}{f}\right)+m\left(R,\sum_{\nu=1}^{p}\frac{f'}{f-a_\nu}\right)+m\left(\frac{1}{R},\sum_{\nu=1}^{p}\frac{f'}{f-a_\nu}\right)+ \\
+4m(1,f)-2m(1,f')+2m\left(1,\frac{1}{f'}\right)++2p\log\frac{2p}{\delta}+2\log 3.
\end{aligned}
\tag{34}
$$

Application of Theorem 1 completes the proof of Theorem 2.

5. Proof of Theorem 3. Theorem 3 is a consequence of Theorem 2, Theorem 4 from [1] and the respective result of Nevanlinna theory [2, p.270].

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Faculty of Mechanics and Mathematics
Lviv Ivan Franko National University
kond@franko.lviv.ua

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