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M-EQUIVALENCE OF MAPPINGS

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The paper investigates the relation of M-equivalence of mappings. We present functors preserving this relation. A new method for constructing examples of M-equivalent mappings is given and, as a corollary, we obtain a list of properties of mappings which are not preserved by M-equivalence. Some characterizations of M-equivalence of spaces in terms of M-equivalence of mappings are presented. A complete classification of A-equivalent mappings having right inverse up to A-equivalent spaces is given.

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Исследуется отношение M-эквивалентности отображений. Представлены функторы, сохраняющие это отношение. Приводится новый метод построения примеров M-эквивалентных отображений. Как следствие, получен набор свойств отображений, не сохраняющихся при M-эквивалентности. Представлены некоторые методы для описания M- эквивалентности пространств в терминах M-эквивалентности отображений. С точностью до A-эквивалентных пространств получена полная классификация A-эквивалентных отображений, имеющих правое обратное.

1. Introduction. All spaces are assumed to be Tychonoff. The notion of M-equivalent mappings was introduced by O.Okunev in [5]. He provided the first method for constructing such mappings and, as a corollary, some properties which are not preserved by the relation of M-equivalence of mappings. We refer to [7] for examples of categories and functors in topological algebra and basic results on their isomorphical classification. In Section 2 we give some basic properties preserved by the M-equivalence relation. Section 3 contains basic constructions preserving the M-equivalence relation. In Section 4 a powerful method for constructing examples of M-equivalent mappings is presented. We also give a list of properties which are not preserved by M-equivalence. In Section 5 we give a few methods to characterize of M-equivalence of spaces in terms of M-equivalence of mappings. In Section 6 we investigate the A-equivalence of the mappings having right inverse. The main result of the section is a complete A-classification of the mappings having right inverse up to A-equivalent spaces. As a corollary, we give some constructions preserving the A-equivalence of such mappings. The terminology is taken from [5] and [2].

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2. On M-invariant properties of mappings.

Definition 2.1. [5, p.160]. We call two mappings $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ M-equivalent if there exist topological isomorphisms $i\colon F(X_1) \to F(X_2)$ and $j\colon F(Y_1) \to F(Y_2)$ such that $j \circ f^* = g^* \circ i$ where $f^*\colon F(X_1) \to F(Y_1)$ and $g^*\colon F(X_2) \to F(Y_2)$ are homomorphisms extending f and g (we denote $f \overset{M}{\sim} g$).

Replacing in the last definition the functor of free topological group by the functor of free abelian topological group, free locally convex space and free locally convex space in the weak topology we obtain the definition of A-, L- and l-equivalent mappings.

Proposition 2.2. $f \overset{M}{\sim} g \Longrightarrow f \overset{A}{\sim} g \Longrightarrow f \overset{L}{\sim} g.$

Proof. For each topological isomorphism $i\colon F(X_1) \to F(X_2)$ there exists a topological isomorphism $i_A\colon A(X_1) \to A(X_2)$ such that $i_A \circ p_1 = p_2 \circ i$ where $p_i\colon F(X_i) \to A(X_i)$ are homomorphisms extending the identity map of X_i. We call such i_A the abelization of i.

Let $f \overset{M}{\sim} g$ and $i\colon F(X_1) \to F(X_2), j\colon F(Y_1) \to F(Y_2)$ be topological isomorphisms such that $j \circ f^* = g^* \circ i$. Denote by $f^*_A\colon A(X_1) \to A(Y_1)$ and $g^*_A\colon A(X_2) \to A(Y_2)$ the extensions of f and g to continuous homomorphisms; by i_A, j_A the abelizations of the topological isomorphisms i and j. Obviously, from $j \circ f^* = g^* \circ i$ it follows that $j_A \circ f^*_A = g^*_A \circ i_A$. Hence $f \overset{A}{\sim} g$. In the same manner we can prove that $f \overset{A}{\sim} g \Longrightarrow f \overset{L}{\sim} g$. □

Recall that two mappings $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ are said to be *homeomorphic* if there exist homeomorphisms $i\colon X_1 \to X_2$, $j\colon Y_1 \to Y_2$ such that $f \circ i = j \circ g$.

Proposition 2.3. [5, Proposition 1.8]. *A continuous surjection $p\colon X \to Y$ is R-quotient if and only if the homomorphism $p^*\colon F(X) \to F(Y)$ extending p is open.*

Corollary 2.4. [5, Corollary 1.9]. *Let $f \overset{M}{\sim} g$ be continuous surjections. If f is R-quotient then so is g.*

One can easily check that any mapping A-equivalent to a surjection is again a surjection.

Definition 2.5. Let $f\colon X \to Y$ be a mapping. Define on X an equivalence relation $\sim$ by putting $a \sim b \Longleftrightarrow f(a) = f(b)$. From each class of equivalence we take an arbitrary point and form the set H. The injectivity of the mapping f is the cardinal number $i(f) = \mid X \backslash H \mid$.

Proposition 2.6. *Let $f \overset{A}{\sim} g$. Then $i(f) = i(g)$.*

Proof. This follows from the fact that $\ker f^*$ is algebraically the free abelian group with set of generators having the cardinality $i(f)$. □

The following facts are obvious.

Corollary 2.7. *Any mapping A-equivalent to a condensation is again a condensation.*

Corollary 2.8. *Any mapping A-equivalent to a homeomorphism is a homeomorphism.*

3. Constructions preserving M-equivalence.

3.1. Sum of mappings. The sum of Tychonoff spaces is again Tychonoff [2, Theorem 2.2.7].

Proposition 3.1. *If $f_s \overset{M}{\sim} g_s$ for each $s \in S$ then $\underset{s\in S}{\oplus} f_s \overset{M}{\sim} \underset{s\in S}{\oplus} g_s$.*

Proof. Let $f_s\colon X_s \to \overline{X_s}$, $g_s\colon Y_s \to \overline{Y_s}$ be continuous mappings and $i_s\colon F(X_s) \to F(Y_s)$, $j_s\colon F(\overline{X_s}) \to F(\overline{Y_s})$ be topological isomorphisms.

First of all note that if we have $i_s\colon F(X_s) \to F(Y_s)$, $j_s\colon F(\overline{X}_s) \to F(\overline{Y}_s))$ then we can define topological isomorphisms $i\colon F(\oplus_{s\in S} X_s) \to F(\oplus_{s\in S} Y_s)$, $j\colon F(\oplus_{s\in S}\overline{X}_s) \to F(\oplus_{s\in S}\overline{Y}_s)$ We can do it in the manner of [1, Proposition 8.8].

One can easily check that $j \circ (\oplus_{s\in S} f_s)^* = (\oplus_{s\in S} g_s)^* \circ i$.

Thus $\oplus_{s\in S} f_s$ is M-equivalent to $\oplus_{s\in S} g_s$. □

3.2. Product of mappings. Product of Tychonoff spaces is again Tychonoff [2, Theorem 2.3.11].

There exist Tychonoff spaces X, Y_1, Y_2 such that $Y_1 \overset{M}{\sim} Y_2$ while $X \times Y_1 \overset{M}{\not\sim} X \times Y_2$ (see [6, Corollary 2.3]). Then, obviously, $\mathrm{id}_{Y_1} \overset{M}{\sim} \mathrm{id}_{Y_2}$ while $\mathrm{id}_{Y_1} \times \mathrm{id}_X \overset{M}{\not\sim} \mathrm{id}_{Y_2} \times \mathrm{id}_X$. This shows that the products of M-equivalent maps are not necessary M-equivalent.

We say that a triple (X, Y, Z) satisfies condition $(*)$ if either Z is locally compact or $(X \oplus Y) \times Z$ is a k-space. We say that a quadruple (X_1, X_2, Y_1, Y_2) satisfies condition $(*)$ if the triples (X_1, X_2, Y_1) and (Y_1, Y_2, X_2) satisfy condition $(*)$.

Proposition 3.2. *Let $f_i\colon X_i \to Z_i$, $g_i\colon Y_i \to T_i$, $f_i \overset{M}{\sim} g_i$, $i \in \{1, 2\}$, and the quadruples (X_1, X_2, Y_1, Y_2) and (Z_1, Z_2, T_1, T_2) satisfy condition $(*)$. Then $f_1 \times f_2 \overset{M}{\sim} g_1 \times g_2$. The same is valid for A- and L-equivalence.*

Proof. Let $h_i\colon F(X_i) \to F(Y_i)$ be topological isomorphisms. In virtue of [6, Proposition 1.2] we can construct isomorphisms $h_1'\colon F(X_1 \times Y_1) \to F(X_2 \times Y_1)$ and $h_2'\colon F(X_2 \times Y_1) \to F(X_2 \times Y_2)$, in the same way having isomorphisms $v_i\colon F(Z_i) \to F(T_i)$ we can construct isomorphisms $v_1'\colon F(Z_1 \times T_1) \to F(Z_2 \times T_1)$ and $v_2'\colon F(Z_2 \times T_1) \to F(Z_2 \times T_2)$.

One can get M-equivalence of $f_1 \times f_2$ and $g_1 \times g_2$ from the diagram:

$$\begin{array}{ccc}
F(X_1 \times X_2) & \xrightarrow{h_1' \circ h_2'} & F(Y_1 \times Y_2) \\
{\scriptstyle (f_1\times f_2)^*} \downarrow & & \downarrow {\scriptstyle (g_1\times g_2)^*} \\
F(Z_1 \times Z_2) & \xrightarrow{v_1' \circ v_2'} & F(P_1 \times P_2)
\end{array}$$

□

Example 3.3. *If $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ are M-equivalent and $(X_1 \oplus Y_1)^n \oplus (X_2 \oplus Y_2)^n$ is a k-space, then $f^n \overset{M}{\sim} g^n$.*

Proof. Follows from [6, Corollary 1.5]. We only note that in the proof of the corollary we actually need $X^j \times Y^{n-j}$ to be a k-space for each $0 \le j \le n$. The latter is equivalent to the fact that $(X \oplus Y)^n$ is a k-space. □

3.3. G-symmetric power functor. Let G be a subgroup of the n-symmetric group S_n. Recall that SP^n_G denotes the *G-symmetric power functor* defined as follows. For a space X the space $SP^n_G X$ is the orbit space of the n-th power X^n by the action of G defined as follows $(x_1, x_2, ...x_n) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, ...x_{\sigma(n)})$, where $\sigma \in G$. The orbit containing $(x_1, x_2, ...x_n)$ is denoted by $[x_1, x_2, ...x_n]_G$. The set $\{x_1, x_2, ...x_n\}$ is called the support of an element $[x_1, x_2, ...x_n]_G$ and is denoted by $\mathrm{supp}([x_1, x_2, ...x_n]_G)$.

Proposition 3.4. *If $X \overset{A}{\sim} Y$ and $(X \oplus Y)^n$ is a k-space then $SP^n_G X \overset{A}{\sim} SP^n_G Y$.*

Proof. Having a topological isomorphism $i\colon A(X) \to A(Y)$ we can "extend" it to a topological isomorphism $i_n\colon A(X^n) \to A(Y^n)$. Denote by $s_X\colon X^n \to SP^n_G X, s_Y\colon Y^n \to SP^n_G Y$ the quotient mappings, $s^*_X\colon A(X^n) \to A(SP^n_G X), s^*_Y\colon A(Y^n) \to A(SP^n_G Y)$ their homomorphic extensions. One can easy check that there exists a unique topological isomorphism $i_{SP^n_G}\colon A(SP^n_G X) \to A(SP^n_G Y)$ such that $s^*_Y \circ i_n = i_{SP^n_G} \circ s^*_X$.

$\square$

For a map $f\colon X \to Y$ the map $SP^n_G f\colon SP^n_G X \to SP^n_G Y$ is defined as follows

$$SP^n_G f[x_1, x_2, ...x_n]_G = [f(x_1), f(x_2), ...f(x_n)]_G.$$

Proposition 3.5. *If $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ are A-equivalent and $(X_1 \oplus Y_1)^n \oplus (X_2 \oplus Y_2)^n$ is a k-space then $SP^n_G f \overset{A}{\sim} SP^n_G g$.*

Proof. For every topological isomorphism $i\colon A(X) \to A(Y)$ let

$$i_{SP^n_G}\colon A(SP^n_G(X_1)) \to A(SP^n_G(X_2))$$

be a topological isomorphism as in Proposition 3.4. One can easily check that from $j \circ f^* = g^* \circ i$ it follows that $j_{SP^n_G} \circ (SP^n_G f)^* = (SP^n_G g)^* \circ i_{SP^n_G}$. Hence $SP^n_G f \overset{A}{\sim} SP^n_G g$.

$\square$

3.4. Spaces of quasicomponents. Denote by $Q(X)$ the space of quasicomponents of a Tychonoff space X, by q_X we denote the quotient map $X \to Q(X)$ [4, page 159].

For the mapping $f\colon X \to Y$ we denote by $Qf\colon Q(X) \to Q(Y)$ the mapping for which $q_Y \circ f = Qf \circ q_X$.

Proposition 3.6. *Let $X \overset{M}{\sim} Y$, such that $Q(X)$ and $Q(Y)$ are Tychonoff. Then the mappings q_X and q_Y are M-equivalent. In particular, the spaces $Q(X)$ and $Q(Y)$ are M-equivalent.*

Proof. Let $i\colon F(X) \to F(Y)$ be a topological isomorphism, $q_X\colon X \to Q(X), q_Y\colon Y \to Q(Y)$ be the quotient maps and $Q_X\colon FM(X) \to F(Q(X)), FY\colon F(Y) \to F(Q(Y))$ their homomorphic extensions. Let us show that there exist a continuous map f such that $Q_Y \circ (i|_X = f \circ q_X$. If $z \in Q(X)$ and $q_X(x_1) = z$ then denote by $f(x) = Q_Y(i(x_1))$. Let us show that such f is well-defined. Suppose that we have another x_2 with $q_X(x_2) = z$. Then $x_1 \in Q_{x_2}$. The space $Q(Y)$ is totally disconnected [4, pages 159,161], hence [3] $F(Q(Y))$ is totally disconnected. Thus $Q_Y(i(Q_{x_2}))$ is single. Since for any continuous f we have $f(Q_x) \subseteq Q_{f(x)}$, we can conclude that $Q_Y(i(x_1)) = Q_Y(i(x_2))$ and therefore f is well-defined. The continuity of f follows from the continuity of i and Q_Y and from the fact that q_X is quotient.

In the same manner we can define $g\colon Q(Y) \to F(Q(X))$.

Let us extend f, g to continuous homomorphisms f^* and g^*.

Then from the diagram

$$\begin{array}{ccccc}
F(X) & \xrightarrow{i} & F(Y) & \xrightarrow{i^{-1}} & F(X) \\
Q_X \downarrow & & \downarrow Q_Y & & \downarrow Q_X \\
F(Q(X)) & \xrightarrow{f^*} & F(Q(Y)) & \xrightarrow{g^*} & F(Q(X))
\end{array}$$

we have that $f^* \circ g^* = 1_{F(Q(X))}$. Similarly we can prove that $g^* \circ f^* = 1_{F(Q(Y))}$. The above shows that the map $f^*\colon F(Q(X)) \to F(Q(Y))$ is a topological isomorphism. $\square$

Proposition 3.7. *Let $f \overset{M}{\sim} g$. Then $Qf \overset{M}{\sim} Qg$.*

Proof. For every topological isomorphism $i\colon F(X) \to F(Y)$ a topological isomorphism $i_Q\colon F(Q(X)) \to F(Q(Y))$ is constructed in Proposition 3.6. One can easy check that from $j \circ f^* = g^* \circ i$ follow $j_Q \circ (Qf)^* = (Qg)^* \circ i_Q$. Hence $Qf \overset{M}{\sim} Qg$. □

The same is valid for A-equivalence relation.

3.5. Suspension, cone, open cone. Denote by Σ the suspension, by C the cone, by O the open cone [13]. For a continuous mapping $f\colon X \to Y$ let us denote by $\Sigma f\colon \Sigma X \to \Sigma Y$ its continuous extension.

Definition 3.8. We say that a topological isomorphism $i\colon F(X) \to F(Y)$ is *special* if the composition $h \circ i$ is constant on X, where $h\colon F(Y) \to \mathbb{Z}$ is a homomorphism extending the mapping equal 1 on Y.

Proposition 3.9. *Let $f\colon X_1 \to Y_1, g\colon X_2 \to Y_2$ be M-equivalent mappings, then there exist special isomorphisms $i_1\colon F(X_1) \to F(X_2)$ and $j_1\colon F(Y_1) \to F(Y_2)$ such that $j_1 \circ f^* = g^* \circ i_1$.*

Proof. Let $f \overset{M}{\sim} g$ and $i\colon F(X_1) \to F(X_2), j\colon F(Y_1) \to F(Y_2)$ be topological isomorphisms such that $j \circ f^* = g^* \circ i$. Using [6, Lemma 3.5] we can construct a topological automorphism $u\colon F(X) \to F(X)$ such that $i_1 = i \circ u$ is a special isomorphism. One can easily check that the map $f_1 = u \circ f|_X$ is topologically equivalent to f and there exists a topological isomorphism $v\colon F(Y) \to F(Y)$ such that $v \circ f_1^* = f^* \circ u$. Denote by $j_1 = j \circ v$. Then $j_1 \circ f_1^* = g^* \circ i_1$. Since i_1 is special, so is j_1. □

Proposition 3.10. *If $f \overset{M}{\sim} g$ then : $\Sigma f \overset{M}{\sim} \Sigma g$, $Cf \overset{M}{\sim} Cg$, $Of \overset{M}{\sim} Og$. The same is true for the relations of A- and L-equivalence.*

Proof. First we apply Proposition 3.9. For every special topological isomorphism $i\colon F(X_1) \to F(X_2)$ we can define topological isomorphisms: $i_\Sigma\colon F(\Sigma X_1) \to F(\Sigma X_2), i_C\colon F(CX_1) \to F(CX_2)$, $i_O\colon F(OX_1) \to F(OX_2)$ see [6, Proposition 4.4, 4.5]. One can easily check that from $j \circ f^* = g^* \circ i$ it follows $j_\Sigma \circ (\Sigma f)^* = (\Sigma g)^* \circ i_\Sigma$, $j_C \circ (Cf)^* = (Cg)^* \circ i_C$, $j_O \circ (Of)^* = (Og)^* \circ i_O$. Hence $\Sigma f \overset{M}{\sim} \Sigma g$, $Cf \overset{M}{\sim} Cg$, $Of \overset{M}{\sim} Og$. □

3.6. Dieudonné completion. For a topological space X, denote by μX the Dieudonné completion of X (see [2, section 8.5.13]). For a continuous mapping f denote by $\mu f\colon \mu X \to \mu Y$ its unique extension. The space admitting uniformity is Tychonoff, hence μX is Tychonoff for each X. It was proved in [11] that the functors $A \circ \mu$ and $R \circ A$ (where A is the functor of free abelian topological groups, R is the functor of Raikov completion) are naturally equivalent. From this fact it follows that $f \overset{A}{\sim} g$ follows $\mu f \overset{A}{\sim} \mu g$. The same proposition for M-equivalence follows from the results of [10].

4. Construction of examples of M-equivalent spaces. Two retractions r_1, r_2 of a space X are called *parallel* if $r_1 \circ r_2 = r_1$ and $r_2 \circ r_1 = r_2$.

Proposition 4.1. [5, Theorem 2.2]. *Assume that K_1 and K_2 are parallel retracts of a space X, $Y_1 = X/K_1$ and $Y_2 = X/K_2$ are R-quotient spaces and $p_1\colon X \to Y_1$ and $p_2\colon X \to Y_2$ are the natural mappings. Then the mappings p_1 and p_2 are M-equivalent. In particular, the spaces Y_1 and Y_2 are M-equivalent.*

Okunev's construction was generalized in [9].

Proposition 4.2. *Let X be a Tychonoff space and r_1 and r_2 its retractions onto the same retract K. Then $r_1 \overset{M}{\sim} r_2$.*

Proof. Obviously $r_1 \circ r_2 = r_2$ and $r_2 \circ r_1 = r_1$.

Consider the continuous mapping $i(x)\colon X \to F(X)$ defined by the formula $i(x) = r_1(x)x^{-1}r_2(x)$. Extend $i(x)$ to a continuous homomorphism $I(x)\colon F(X) \to F(X)$. Then

$$I \circ i(x) = r_1[r_1(x)x^{-1}r_2(x)] \times [r_1(x)x^{-1}r_2(x)]^{-1} \times r_2[r_1(x)x^{-1}r_2(x)] =$$

$$= r_1 \circ r_1(x) \times r_1(x)^{-1} \times r_1 \circ r_2(x) \times r_2(x)^{-1} \times x \times r_1(x)^{-1} \times r_2 \circ r_1(x) \times r_2(x)^{-1} \times r_2 \circ r_2(x) = x$$

Hence $I \circ I = 1_{F(X)}$

$$r_2 \circ i = r_2 \circ r_1(x) \times r_2(x)^{-1} \times r_2 \circ r_2(x) = r_1(x) \times r_2(x)^{-1} \times r_2(x) = r_1(x)$$
$$r_1 \circ i = r_1 \circ r_1(x) \times r_1(x)^{-1} \times r_1 \circ r_2(x) = r_1(x) \times r_1(x)^{-1} \times r_2(x) = r_2(x)$$

From this fact we can conclude that $r_1 \overset{M}{\sim} r_2$. □

Corollary 4.3. *Since any two parallel retractions are homeomorphic mappings, any two retractions onto parallel retracts are M-equivalent.*

A map $f\colon X \longrightarrow Y$ is called *finite-to-one (compact, pseudocompact)* if any $f^{-1}(y)$ is finite (compact, pseudocompact). A closed compact map is called *perfect.*

Example 4.4. *Let $X = \{1, 2, 3, ..., n, ...\}$ be the space of positive integers with usual order. Consider the mappings $f = \max(x, y)$, $g = \min(x, y)$. Obviously $f^{-1}(n) = (\{1...n\}, n) \cup (n, \{1..n\})$ is finite for all n and $g^{-1}(n) = (\{n, n + 1, ...\}, n) \cup (n, \{n, n + 1, ...)$ is infinite for all n.*

Corollary 4.5. *The following properties are not preserved by the relation of M-equivalence within the class of clopen mappings :*

1) *perfectness;*
2) *compactness;*
3) *pseudocompactness;*
4) *finite-to-one property.*

A map $f\colon X \longrightarrow Y$ is called *monotone, (easy, zero-dimensional, discrete)* [2, p.526,538] if any $f^{-1}(y)$ is connected (respectively hereditary disconnected, zero-dimensional, discrete). A map $f\colon X \longrightarrow Y$ is called functionally open(closed) if the preimage of every functionally open(closed) subset in Y is functionally open(closed) in X. A map which is functionally closed and functionally open is called *functionally clopen.* We say that $\dim(f) \leq n$ if $\dim(f^{-1}(y)) \leq n$ for any y. We say that $\mathrm{card}(f) \leq n$ if $\mathrm{card}(f^{-1}(y)) \leq n$ for any y.

Example 4.6. *Let $X = \mathbb{R}$. Then the mappings $f(x) = |x|$ and $g(x) = x^+ = (x + f(x))/2$ are retractions from $\mathbb{R}$ to $\mathbb{R}^+ = [0, \infty)$ so $f \overset{M}{\sim} g$.*

Corollary 4.7. *The following properties of maps are not preserved by the M-equivalence relation within the class of closed quotient retractions: monotonicity, easyness, dimension, zero-dimensionality, discreteness, cardinality, functional openness, functional clopenness.*

A map f is called a *local homeomorphism* if for any $x \in X$ there exist its neighbourhood $U(x)$ such that $f|_{U(x)}$ is a homeomorphism of $U(x)$ onto an open subspace of Y.

Corollary 4.8. *Let $X = [-2,-1] \cup \{0\} \cup [1,2]$ then consider the restrictions of the above defined mappings f and g. The restriction of f is a local homeomorphism, while the restriction of g is not. Both f and g are clopen mappings.*

5. On certain classes of M-equivalent mappings. Let X be a Tychonoff space. We denote by e_X the quotient mapping of X to the one-point space e, by id_X the identity of X, by D_X the condensation from a discrete space $D_{|X|}$ of cardinality $|X|$ onto X, by q_X the quotient mapping from the space X to the space $Q(X)$ of the quasicomponents of the space X, by μ_X the embedding $X \longrightarrow \mu X$.

Let $M[X]$ denote the class of Tychonoff spaces Y such that $X \overset{M}{\sim} Y$ and by $M[f]$ the class of continuous mappings g between Tychonoff spaces such that $f \overset{M}{\sim} g$.

Proposition 5.1. *For arbitrary Tychonoff X*

a) $M[e_X] = \{e_Y \mid Y \in M[X]\}$;

b) $M[\mathrm{id}_X] = \{\mathrm{id}_Y \mid Y \in M[X]\}$;

c) $M[D_X] = \{D_Y \mid Y \in M[X]\}$;

d) $M[q_X] = \{q_Y \mid Y \in M[X]\}$;

e) $M[\mu_X] = \{\mu_Y \mid Y \in M[X]\}$.

Similar statements hold for A-equivalent spaces and mappings.

Proof. d) The inclusion $M[q_X] \supseteq \{q_Y \mid Y \in M[X]\}$ follows from Proposition 3.6.

Let us prove the inclusion $M[q_X] \subseteq \{q_Y \mid Y \in M[X]\}$. Let $f\colon Y \to Z$ be such that $f \overset{M}{\sim} q_X$. Then $Y \in M[X]$ and $Q(f) \overset{M}{\sim} Q(q_X)$ by Proposition 3.7. Since $Q(q_X)$ is a homeomorphism, so is $Q(f)$. Hence $f \cong q_Y$.

e) The inclusion $M[\mu_X] \supseteq \{\mu_Y \mid Y \in M[X]\}$ follows from the fact that the extension μ_X is the embedding of the free topological group $F(X)$ into its Weil completion. Let us prove the inclusion $M[\mu_X] \subseteq \{\mu_Y \mid Y \in M[X]\}$. Let $f\colon Y \to Z$ be such that $f \overset{M}{\sim} \mu_X$. Then $Y \in M[X]$, Z is Dieudonné complete and $\mu(f) \overset{M}{\sim} \mu(\mu_X)$. Since $\mu(\mu_X)$ is a homeomorphism, so is $\mu(f)$. Thus $f \cong \mu_Y$. ☐

Proposition 5.2. *Let X and Y be A-equivalent spaces, $a \in X$, $b \in Y$ be arbitrary points. Then there exists a special topological isomorphism $h\colon A_M(X) \to A_M(Y)$ such that $h(a) = b$.*

Proof. By [6, Proposition] there exists a special isomorphism $j\colon A_M(X) \to A_M(Y)$. Let $A = \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n$ be $j^{-1}(b)$. Since j is special, $\sum_{i=1}^n \lambda_i = 1$. Consider the mappings $f, g\colon X \to A_M(X)$ defined as follows: $f(x) = x + A - a$, $g(x) = x - A + a$. Let $f^*, g^*\colon A_M(X) \to A_M(X)$ be their homomorphic extensions. Then

$$f^* \circ g^*(x) = f^*(x - A + a) = (x + A - a) - [\lambda_1(x_1 + A - a) + \lambda_2(x_2 + A - a) + \cdots +$$
$$+\lambda_n(x_n + A - a)] + (a + A - a) = x + A - a - [\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n] -$$
$$-\left(\sum_{i=1}^n \lambda_i\right) \times (A - a) + A = x + A - a - A - 1 \times (A - a) + A = x.$$

Thus $f^* \circ g^* = g^* \circ f^* = 1_{A_M(X)}$. Therefore f^* is a special isomorphism and $f^*(a) = A$. Then $h = j \circ f^*$ is a topological isomorphism and $h(a) = j \circ f^*(a) = j(A) = b$. ☐

Let $\{X_s\}_{s\in S}$ be a family of spaces with base point $x_s \in X_s$ for each $s \in S$. Let $\vee_{s\in S}(X_s, x_s) = (\oplus_{s\in S} X_s)/(\oplus_{s\in S} x_s)$ be the bouquet of this family.

Corollary 5.3. *Let $X_s \overset{A}{\sim} Y_s$ for each $s \in S$. Then $\vee_{s\in S} X_s \overset{A}{\sim} \vee_{s\in S} Y_s$.*

In the same manner we can prove

Proposition 5.4. *Let X and Y be L-equivalent spaces, $a \in X$, $b \in Y$ be arbitrary points. Then there exists a special linear homeomorphism $h\colon L(X) \to L(Y)$ such that $h(a) = b$.*

Denote by $t_{(X,x_0)}$ the embedding of the one point space e into X such that $t_{(X,x_0)}(e) = x_0$. Consider the continuous mapping $i\colon X \to F(X)$ such that $i(x) = a \times x^{-1} \times b$. Denote by $I\colon F(X) \to F(X)$ its extension. Then $I \circ t_{(X,a)} = t_{(X,b)}$. Hence $t_{(X,x_0)}$ does not depend, up to M-equivalence, on the base point x_0 so we will write shortly t_X.

Proposition 5.5. *For arbitrary Tychonoff space X, $A[t_X] = \{t_Y \mid Y \in A[X]\}$.*

6. On A-equivalence of the mappings having right inverse.

Proposition 6.1. *Let $r\colon X \to K$ be a retraction. Then r is M-equivalent to the R-quotient mapping $p\colon (X/K) \vee_e K \to K$.*

Proof. Considering Okunev's construction [5] we can construct a topological isomorphism $i\colon X \to X/K \vee_e K_1$, where K_1 is a homeomorphic copy of K (we fix a homeomorphism $h\colon K \to K_1$).

Applying this construction we came to the conclusion that the point $a \in X$ is mapped onto the point $i(a) = p_1(a) \times e^{-1} \times r_1(a)$, where $p_1\colon X \to X/K$ is the R-quotient mapping, $e = p_1(K)$, and r_1 is the composition of the retraction $r\colon X \to K$ and the homeomorphism $h\colon K \to K_1$. Denote by $p^*\colon F(X/K \vee_e K) \to F(K)$ the extension of p to a group homomorphism.

Then $p^* \circ i = p^* \circ p_1(a) \times p^* \circ (e)^{-1} \times p^* \circ r_1(a) = e \times e^{-1} \times r_1(a) = r_1(a)$. □

In the same manner one can prove the following proposition.

Proposition 6.2. *Let $r\colon X \to K$ be a retraction, $p\colon X \to X/K$ the R-quotient mapping. Then p is M-equivalent to the R-quotient mapping $q\colon X/K \vee_e K \to X/K$.*

Proposition 6.3. *For two retractions $r_i\colon X_i \to K_i$ the following are equivalent:*

I) $r_1 \overset{A}{\sim} r_2$;

II) The R-quotient maps $q_i\colon X_i \to X_i/K_i$ are A-equivalent;

III) $K_1 \overset{A}{\sim} K_2$ and $X_1/K_1 \overset{A}{\sim} X_2/K_2$.

Proof. (I $\Longrightarrow$ III) $X_1 \overset{A}{\sim} X_2$ follows from the definition of A-equivalent mappings. By Proposition 6.1 the retractions r_i are A-equivalent to R-quotients $p_i\colon X_i/K_i \vee_e K_i \to K_i$. Since $\ker(p_i^*) = A_0(X_i/K_i)$ and $A_M((X_i/K_i) = A_0((X_i/K_i) \times \mathbb{Z}$, we have $A_M((X_1/K_1) \simeq A_M((X_2/K_2)$.

(II $\Longrightarrow$ III) $X_1/K_1 \overset{A}{\sim} X_2/K_2$ follows from the definition of A-equivalent mappings. By Proposition 6.2, the R-quotients $q_i\colon X_i \to X_i/K_i$ are A-equivalent to quotients $p_i\colon X_i/K_i \vee_e$

$K_i \to X_i/K_i$. Since $\ker(p_i^*) = A_0(K_i)$ and $A_M(K_i) = A_0(K_i) \times \mathbb{Z}$, we have $A_M(K_1) \simeq A_M(K_2)$.

(III $\Longrightarrow$ I) By Proposition 6.1 it suffices to prove that the R-quotients $f_i\colon X_i/K_i \vee_{e_i} K_i \to K_i$ are A-equivalent. By Proposition 5.2 there exist topological isomorphisms $u\colon A(X_1/K_1) \to A(X_2/K_2)$ and $v\colon A(K_1) \to A(K_2)$ such that $u(e_1) = e_2$ and $v(e_1) = e_2$. Consider the mapping $s_1\colon X_1/K_1 \vee_{e_1} K_1 \to A(X_2/K_2 \vee_{e_2} K_2)$ by putting $s_1(x) = u(x)$ if $x \in X_1/K_1$ and $s_1(x) = v(x)$ if $x \in K_1$. In the same manner we can define $s_2\colon X_2/K_2 \vee_{e_2} K_2 \to A(X_1/K_1 \vee_{e_1} K_1)$. Then the extensions s_i^* of s_i are inverse continuous homomorphisms. So s_1 is topological isomorphism. Denote by

$$A_0(X_1/K_1) = \left\{ W \in A(X_1) : W = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n, x_i \in X_1/K_1, \sum_{i=1}^{n} \varepsilon_i = 0 \right\}.$$

Obviously $\ker(r_i^*) = A_0(X_i/K_i)$ and $s_1(A_0(X_1/K_1)) = (A_0(X_2/K_2))$, therefore by [5, Theorem 1.10] we have $r_1 \overset{A}{\sim} r_2$.

(III $\Longrightarrow$ II) Using Proposition 6.2 we can prove the implication similary to the previous one. $\square$

We call two retractions r_1 and r_2 of a space X are *orthogonal* [8] if the mappings $r_1 \circ r_2$ and $r_2 \circ r_1$ are constant.

Corollary 6.4. *Two orthogonal retractions $r_i\colon X \to K_i$, $i \in \{1, 2\}$ are A-equivalent iff $K_1 \overset{A}{\sim} K_2$.*

Proof. Since $K_1 \overset{A}{\sim} K_2$, by [8, Propositions 3.2,3.7.] we have $X/K_1 \overset{A}{\sim} X/K_2$. by Proposition 6.3 we see that $r_1 \overset{A}{\sim} r_2$. $\square$

Proposition 6.5. *Let $X \overset{M}{\sim} Y$ and a triple (X, Y, Z) satisfies condition (*). Consider the projection mappings $p_X\colon X \times Z \to X$, $p_Y\colon Y \times Z \to Y$, $f_X\colon X \times Z \to Z$, $f_Y\colon Y \times Z \to Z$. Then $p_X \overset{M}{\sim} p_Y, f_X \overset{M}{\sim} f_Y$.*

Proposition 6.6. *Let $f_1 \overset{A}{\sim} f_2, g_1 \overset{A}{\sim} g_2$ be the mappings that have right inverse. Then*

$$g_1 \circ f_1 \overset{A}{\sim} g_2 \circ f_2.$$

Proof. Let $f_i\colon X_i \to Y_i, g_i\colon Y_i \to Z_i$, $i \in \{1, 2\}$. By Proposition 6.1, $g_1 \circ f_1 \overset{A}{\sim} g_2 \circ f_2$ iff $X/Z_1 \overset{A}{\sim} X/Z_2$ and $Z_1 \overset{A}{\sim} Z_2$.

Let us show that $X/Z^+ \overset{A}{\sim} X/Y \oplus Y/Z$. The space X/Z contains a retract homeomorphic to Y/Z. Hence by [5, Theorem 2.4] $X/Z^+ \overset{A}{\sim} X/Y \oplus Y/Z$.

Since $f_1 \overset{A}{\sim} f_2$ implies $X_1/Y_1 \overset{A}{\sim} X_2/Y_2$, $g_1 \overset{A}{\sim} g_2$ implies $Y_1/Z_1 \overset{A}{\sim} Y_2/Z_2$, we have $(X_1/Z_1)^+ \overset{A}{\sim} (X_2/Z_2)^+$ so from [8, Proposition 3.7] we can conclude that $X_1/Z_1 \overset{A}{\sim} X_2/Z_2$. Hence $g_1 \circ f_1 \overset{A}{\sim} g_2 \circ f_2$. $\square$

Denote by $f \nabla g$ the *sum combination* [2, page 126] of the mappings f and g.

Proposition 6.7. *Let $f_1 \overset{A}{\sim} f_2, g_1 \overset{A}{\sim} g_2$ be mappings such that f_i has right inverse. Then*

$$f_1 \nabla g_1 \overset{A}{\sim} f_2 \nabla g_2.$$

Proof. Let $f_i\colon X_i \to Z_i, g_i\colon Y_i \to Z_i$, $i \in \{1,2\}$. Since f_i has right inverse, $f_i \nabla g_i$ also has right inverse. Then $f_1 \nabla g_1 \overset{A}{\sim} f_2 \nabla g_2$ if and only if $Y_1 \overset{A}{\sim} Y_2$ and $X_1/Z_1 \oplus Y_1 \overset{A}{\sim} X_2/Z_2 \oplus Y_2$.

Since $f_1 \overset{A}{\sim} f_2$ implies $X_1/Z_1 \overset{A}{\sim} X_2/Z_2$ and $g_1 \overset{A}{\sim} g_2$ implies $Y_1 \overset{A}{\sim} Y_2$, we obtain $X_1/Z_1 \oplus Y_1 \overset{A}{\sim} X_2/Z_2 \oplus Y_2$. $\square$

REFERENCES

1. Guran I. Y., Zarichnyi M. M. Elements of theory of topological groups, Inst. System Stud. Education, Kyiv, 1991.
2. Энгелькинг Р. Общая топология, М.: Мир, 1986.
3. Yunusov A. S. *On quasicomponent of free topological groups*, Mat. Issledovania **74** (1983), Kishinev, 163–165. (Russian)
4. Куратовский К. Топология, М.: Мир. Т.1. 1966, Т.2. 1969.
5. Okunev O. G. *A method for consructing examples of M-equivalent spaces*, Topology Appl **36** (1990), 157–171; Correction, Topology Appl **49** (1993), 191–192.
6. Okunev O. G. *M-equivalence of products*, Trudy Mosc. Math. Obsch. **56** (1995), 192–205.
7. Pestov V. G. *Universal arrows to forgetful functors from categories of topological algebra*, Bull. Austr. Math. Soc. **48** (1993), 209–249.
8. Pyrch N. M. *Orthogonal retractions and the relations of M-equivalence*, Matematychni Studii, **20**, No. 2 (2003), 151–161.
9. Pyrch N., Zarichnyi M. *On a generalization of Okunev's construction*, Algebraic structures and their applications, Proceedings of the Institute of Math.: Kiev, 2002, 346–350.
10. Sipacheva O. V. *Free topological groups of spaces and their subspaces*, Topology Appl., **101** (2000), 181–212.
11. Tkachenko M. G. *On the competeness of free Abelian topological groups*, Dokl. Akad. Nauk SSSR **269** (1983), 299–303. (Russian)
12. Ткачук В. В. *Об одном методе построения M-эквивалентных пространств*, Усп. матем. наук, **38** (1983), №6, 127–128.
13. Wagner C. H. Symmetric, cyclic and permutation products of manifolds, Dissertationes mathematicae, CLXXXII, Warszawa, 1980.

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