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MAXIMUM K_{r+1} -FREE GRAPHS WHICH ARE NOT r -PARTITE*

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Turán's theorem states that the maximum size of a K_{r+1} -free graph G of order n is attained by a complete r -partite graph. Here we determine the maximum size of G on the additional restriction that G is not r -partite. Also, we present a new proof of the result of Andrásfai, Erdős, and Gallai on the maximum size of an order- n graph whose shortest odd cycle has given length $2l + 1$. The extremal graphs are characterized for all feasible values of parameters.

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Теорема Турана утверждает, что максимальный размер K_{r+1} -свободного графа порядка n достигается полным r -дольным графом. Здесь мы определяем максимальный размер G при дополнительном ограничении, что G не является r -дольным. Мы также приводим новое доказательство результата Андрашфай, Эрдеша и Галлаи о максимальном размере графа порядка n , у которого кратчайший нечетный цикл имеет заданную длину $2l + 1$. Охарактеризованы максимальные графы для подходящих значений параметра.

1. Introduction. Let K_m denote the complete graph on m vertices. The fundamental theorem of Turán [6] states that $\text{ex}(n, K_{r+1})$, the maximum size of a K_{r+1} -free graph G of order n , is attained by the complete r -partite graph with parts whose sizes differ by at most 1. This result is one of the cornerstones of extremal graph theory. For future use, let the *Turán graph* $T_r(n)$ be the corresponding extremal graph and let

$$t_r(n) = e(T_r(n)) = \text{ex}(n, K_{r+1}).$$

It is interesting to know what happens if we additionally require that G is not r -partite, that is, we consider the graphs in

$$\mathcal{G}_{n,r} = \{G : v(G) = n, G \not\supset K_{r+1}, \chi(G) > r\}.$$

The classical paper of Andrásfai, Erdős and Sós [1] determines $\max\{\delta(G) : G \in \mathcal{G}_{n,r}\}$, the largest minimum degree of $G \in \mathcal{G}_{n,r}$. Erdős and Simonovits [4] studied the more general problem of maximizing $\delta(G)$ over

$$\mathcal{H}_{n,r,s} = \{G : v(G) = n, G \not\supset K_{r+1}, \chi(G) > s\}.$$

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(Note that $\mathcal{G}_{n,r} = \mathcal{H}_{n,r,r}$.)

In this paper we investigate

$$p_r(n) = \max\{e(G) : G \in \mathcal{G}_{n,r}\}.$$

This problem is completely settled in Section 2. Namely, we will prove the following result.

Theorem 1. *Let $n \geq r + 3$ and $r \geq 2$. If $r > \frac{n-1}{2}$, then $p_r(n) = t_r(n) - 2$. If $r \leq \frac{n-1}{2}$, then*

$$p_r(n) = t_r(n) - \left\lfloor \frac{n}{r} \right\rfloor + 1. \quad (1)$$

Moreover, the extremal graphs are characterized by Theorem 4 and Lemma 5.

For convenience, the bound (1) on $p_r(n)$ is stated relative to the Turán function $t_r(n) = \text{ex}(n, K_{r+1})$ so that it is immediately obvious how many edges we lose by imposing the restriction $\chi(G) > r$. Note also that $\mathcal{G}_{n,r} = \emptyset$ for $n \leq r + 2$ or for $r = 1$. (In order to see the former claim, note that the complement of any K_{r+1} -free graph G of order $r + 2$ must contain either two disjoint edges or a triangle, so $\chi(G) \leq r$.)

It would be nice to compute $\max\{e(G) : G \in \mathcal{H}_{n,r,s}\}$. It is easy to see that this function equals $t_r(n) - O(n)$ if r, s are fixed. However, the exact computation seems to be a hard task in general.

For $r = 2$, we have canonical examples of graphs of chromatic number larger than r : odd cycles. Häggkvist [5] studied the problem which odd cycles must be present in a non-bipartite graph of given minimum degree. Andrásfai and, independently, Erdős and Gallai (see Erdős [3, Lemma 1]) computed $b_l(n)$, the maximum size of a non-bipartite graph of order n whose shortest odd cycle has length $2l + 1$. Unfortunately, this result is not well known, appearing only as an auxiliary lemma in [3]. In fact, we were unaware of it until we rather accidentally came across [3] while revising the current paper. Since no attempt to characterize the extremal graphs was made in [3] and our proof is different (albeit longer), we decided to keep it. Here is the precise statement of the result.

Theorem 2. *Let $l \geq 2$ and $n \geq 2l + 1$. Then*

$$b_l(n) = \left\lfloor \frac{(n - 2l + 3)^2}{4} \right\rfloor + 2l - 3. \quad (2)$$

All extremal graphs are described by the construction at the beginning of Section 3.

One of the extremal graphs for the $b_l(n)$ -problem is obtained by taking C_{2l+1} and appropriately cloning two adjacent vertices, but there are other constructions.

It is easy to see that the right-hand side of (2) strictly decreasing as a function of l for fixed n , $2 \leq l \leq \frac{n-1}{2}$. Hence, $b_l(n)$ is also equal to the maximum size of a non-bipartite graph G of order n without any odd cycle of length less than $2l + 1$.

Thus the problems of determining $p_r(n)$ and $b_l(n)$ overlap in a special case: $b_2(n) = p_2(n)$. A more remarkable relation is that our proofs of Theorems 1 and 2 are based on the same idea of Erdős [2].

2. Determining $p_r(n)$. A construction giving a lower bound on $p_r(n)$ can be obtained as follows. Let $n \geq r + 3$. Choose integers

$$1 \leq n_1 \leq \dots \leq n_r \text{ such that } \sum_{i=1}^r n_i = n - 1 \text{ and } n_{r-1} \geq 2, \quad (3)$$

and pairwise disjoint sets N_1, \dots, N_r , where $|N_i| = n_i$. Let s and t be the two smallest indices i (in either order) for which $n_i > 1$. (Thus $|s - t| = 1$.) Let $S = [r] \setminus \{s, t\}$. Choose any subset $A \subset N_s$ which is *proper* (that is, $A \neq \emptyset$ and $A \neq N_s$). Choose $y \in N_t$. To $K_r(N_1, \dots, N_r)$, the complete r -partite graph on $N_1 \cup \dots \cup N_r$, add a vertex x connected to everything in $(\cup_{i \in S} N_i) \cup (\{y\} \cup A)$ but remove all edges between y and A . Let us call the obtained graph $G = G(\mathbf{n})$, where $\mathbf{n} = (n_1, \dots, n_r)$. Of course, the isomorphism class of G depends on the choice of $|A|$ (and the choice of s, t if $n_s \neq n_t$) but the size $e(G)$ does not:

$$e(G) = \sigma_2(\mathbf{n}) + \sigma_1(\mathbf{n}) - n_s - n_t + 1, \quad (4)$$

where $\sigma_2(\mathbf{n}) = \sum_{i < j} n_i n_j$ and $\sigma_1(\mathbf{n}) = \sum_i n_i$. More generally, for an arbitrary (not necessarily increasing) sequence \mathbf{n} with at least two entries larger than 1, let $G(\mathbf{n})$ be obtained by properly ordering \mathbf{n} and then taking the above construction.

It is easy to see that $K_{r+1} \not\subset G$. Indeed, if some $(r+1)$ -set $K \subset V(G)$ spanned a complete graph, then $x, y \in K$ (because $G - x$ and $G - y$ are r -partite); however, $\Gamma(x) \cap \Gamma(y) = \cup_{i \in S} N_i$ is $(r-2)$ -partite, a contradiction. (Here, $\Gamma(x)$ denotes the neighborhood of x .)

Also, we have $\chi(G) > r$. Indeed, suppose on the contrary that we can color G with r colors. Choosing arbitrary $y_s \in N_s \setminus A$, $y_t \in N_t$, and $y_i \in N_i$, $i \in S$, we obtain a copy of K_r , so the colors of these vertices do not depend on their choices. But then x and each vertex of A see the same set of $r-1$ colors among its neighbors. Hence the set $A \cup \{x\}$ is monochromatic, which is a contradiction as x is connected to A .

Let us turn to proving upper bounds on $p_r(n)$. As we have already observed, $\mathcal{G}_{n,r} = \emptyset$ for $n \leq r+2$. Therefore we restrict our attention to $n \geq r+3$. First we prove the required upper bound in the following special case.

Lemma 3. *Let $r \geq 2$ and $n \geq r+3$. Let $G \in \mathcal{G}_{n,r}$ be such that for some vertex y we have $\chi(G - y) = r$. Then $e(G) \leq e(G(\mathbf{n}))$ for some \mathbf{n} satisfying (3).*

Proof. Take an r -coloring of $G - y$. Let $\{y_1\}, \dots, \{y_l\}, N_1, \dots, N_{r-l}$ be the color classes, of which l have size one. Let n_1, \dots, n_{r-l} , all at least 2, be the sizes of N_1, \dots, N_{r-l} , respectively. As $n \geq r+3$, we have $l < r$.

Let $Y = \{y, y_1, \dots, y_l\}$. Note that Y spans the complete subgraph for otherwise G is r -colorable. Let

$$M_i = \{x \in N_i : \Gamma(x) \supset Y\}$$

and $m_i = |M_i|$. By reordering, let us assume that $m_1 \leq \dots \leq m_{r-l}$.

We claim that each M_i is non-empty. Otherwise, for every $x \in N_i$ choose $f(x) \in Y$ such that $\{x, f(x)\} \notin E(G)$. The $l+1$ sets $\{z\} \cup \{x \in N_i : f(x) = z\}$, $z \in Y$, are independent and partition $Y \cup N_i$. Together with the $r-l-1$ remaining parts N_j , $j \neq i$, this gives an r -coloring of G , contradicting our assumption.

Thus each $m_i \geq 1$. Moreover, $l \leq r-2$ for otherwise $Y \cup \{x\}$ for some $x \in M_1$ spans a copy of K_{r+1} .

Let \bar{e}_{ij} be the number of edges missing between N_i and N_j . Potentially, Y creates $\prod_{i=1}^{r-l} m_i$ copies of K_{r+1} . A missing edge between N_i and N_j destroys at most $\frac{1}{m_i m_j} \prod_{h=1}^{r-l} m_h$ such copies. Hence,

$$\prod_{h=1}^{r-l} m_h \leq \sum_{i < j} \frac{\bar{e}_{ij}}{m_i m_j} \prod_{h=1}^{r-l} m_h \leq \frac{\sum_{i < j} \bar{e}_{ij}}{m_1 m_2} \prod_{h=1}^{r-l} m_h. \quad (5)$$

We see that $\sum_{i < j} \bar{e}_{ij}$, the total number of edges missing between the N_i 's, is at least $m_1 m_2$. Thus

$$e(G) \leq \sigma_2(1^{(l)}, n_1, \dots, n_{r-l}) - m_1 m_2 + l + \sum_{i=1}^{r-l} m_i, \quad (6)$$

where $1^{(l)}$ means the number 1 repeated l times. A simple optimization shows that it is best to take $m_i = n_i$ for $i \geq 3$, and $m_1 = 1$ (recall $m_1 \leq m_2$), which gives us $e(G) \leq e(G(1^{(l)}, n_1, \dots, n_{r-l}))$, as required. \square

Now we are ready to prove our main result whose proof relies on Lemma 3.

Theorem 4. *Let $r \geq 2$ and $n \geq r + 3$. Then $p_r(n)$ equals the maximum of $e(G(\mathbf{n}))$ over all integers satisfying (3). Moreover, all extremal graphs are described by our construction.*

Proof. Our argument is built upon the ideas from Erdős' proof [2] of Turán's theorem, where it is shown that the degree sequence of a K_{r+1} -free graph can be majorized by that of an r -partite graph.

Let $G \in \mathcal{G}_{n,r}$ have the maximum size. We prove the theorem by induction on r . We do not give a separate proof for the base case $r = 2$: the inductive step, when specialized to $r = 2$, gives a self-contained proof. We prove the desired bound first and then analyze the cases of equality.

Let $V = V(G)$. Choose $x \in V$ with its degree being equal to the maximal degree of G , that is, $d(x) = \Delta(G)$. Let $D = \Gamma(x)$. Let the graph H be obtained from G by removing all edges inside $C = V \setminus D$ and adding all edges between D and C .

We claim that $H \not\supset K_{r+1}$. Indeed, suppose otherwise. The vertex set K of this K_{r+1} must intersect C because $G[D]$ and $H[D]$ are the same. As C is an independent set in H , we have $|K \cap C| = 1$. By the symmetry of C , we can assume that $K \cap C = \{x\}$. But then K spans a complete graph in G , a contradiction.

Note that for every $y \in V$ we have $d_G(y) \leq d_H(y)$: if $y \in C$, this follows from $d_G(y) \leq \Delta(G) = d_H(y)$; if $y \in D$, then $\Gamma_G(y) \subset \Gamma_H(y)$.

If $H[D]$ is not $(r-1)$ -partite, then by the induction assumption we have $e(H[D]) \leq e(G(\mathbf{n}))$ for some $\mathbf{n} = (n_1, \dots, n_{r-1})$. Let the r -vector \mathbf{m} be obtained from \mathbf{n} by inserting the number $|C|$. We have

$$e(G) \leq e(H) \leq (n - |C|)|C| + e(G(\mathbf{n})) \leq e(G(\mathbf{m})),$$

proving the required upper bound.

Hence, we can assume that $H[D]$ is $(r-1)$ -partite: $D = \cup_{i=1}^{r-1} D_i$. (For $r = 2$ we get this conclusion for free: $D = \Gamma(x)$ is an independent set because $G \not\supset K_3$.) Let $d_i = |D_i|$, $d = |D|$, and $c = |C|$. We can assume that each d_i is at least 2 for otherwise the required upper bound follows by Lemma 3. Also, $c \geq 2$ for otherwise $G = H$ is r -partite.

Call a part D_i *good* if there is $y_i \in D_i$ which is connected in G to everything in $V \setminus D_i$. We claim that all, but at most one, parts are good. (We assume here that $r \geq 3$ as the claim is vacuously true for $r = 2$.) Suppose on the contrary that, for example, D_1 and D_2 are bad. Let the r -vector \mathbf{d} be made of the numbers d_1, \dots, d_{r-1}, c . We have

$$e(G) \leq \sigma_2(\mathbf{d}) - \frac{d_1 + d_2}{2},$$

which strictly beats the desired bound. Indeed, assuming $d_1 \leq d_2$, we have

$$\sigma_2(\mathbf{d}) - \frac{d_1 + d_2}{2} \leq \sigma_2(\mathbf{d}) - d_1 \leq e(G(d_1, d_2 - 1, d_3, \dots, d_{r-1}, c)) - 1.$$

Note that the last sequence has at least two elements which are at least 2 (namely, d_1 and c), so it still satisfies (3). This upper bound on $e(G)$ contradicts the maximality of G .

We also obtain a contradiction by assuming that all parts are good: if $G[C]$ is empty, then $\chi(G) \leq r$; otherwise $G \supset K_{r+1}$. So, let D_1 be the unique bad part. If each vertex of D_1 misses at least 2 neighbors in C , then

$$e(G) \leq \sigma_2(\mathbf{d}) - 2d_1 < \sigma_2(\mathbf{d}) - d_1,$$

which is too small. Hence, there is $y_1 \in D_1$ such that $C \setminus \Gamma(y_1)$ consists of a single vertex z . Choose $y_i \in D_i$, $i \in [2, r-1]$, which witnesses the fact that D_i is good. Then $\{y_1, \dots, y_{r-1}\}$ is an $(r-1)$ -clique which is connected (in G) to everything in $C \setminus \{z\}$. Hence, this set is independent, which implies that $\chi(G - z) \leq r$. Now, we can apply Lemma 3 again. The upper bound is proved.

Let us characterize the cases of equality. We go over the proof of the upper bound, using the same notation.

If $G[D]$ is not $(r-1)$ -partite, then by induction $G[D] \cong G(\mathbf{m})$ for some \mathbf{m} . Moreover, each vertex $y \in D$ is connected in G to everything in C : otherwise $d_G(y) < d_H(y)$ and $e(G) < e(H)$, a contradiction to the maximality of G . It follows that $G[C]$ is the empty graph and G is as desired.

Suppose that $G[D]$ is $(r-1)$ -partite. Our proof shows that there is a vertex y , either $y \in C$ or $y \in D$, such that $G - y$ is $(r-1)$ -partite. Let the parts of $V' = V \setminus \{y\}$ be $\{y_1\}, \dots, \{y_l\}, N_1, \dots, N_{r-l}$. (Now we use the notation of Lemma 3.) Of all possible choices of y and an $(r-1)$ -partition of V' , take one which minimizes l .

We must have $m_i = n_i$ for $i \geq 3$ and $m_1 = 1$. As we have equality in (5), all missing edges in $G[N_i, N_j]$ lie between M_1 and $M_2 \cup \dots \cup M_k$, where $m_2 = \dots = m_k$. In fact, all missing edges lie inside just *one* $G[M_1, M_i]$ for otherwise starting with $Y \cup M_1$ we can greedily add $z_i \in M_i$, consecutively for $i = 2, \dots, r-l$, to get a K_{r+1} -subgraph.

The case $m_2 = n_2$ is impossible: otherwise we can move the vertex in M_1 into N_2 to obtain another legitimate $(r-1)$ -partition of $V \setminus \{y\}$ with the new M_1 being empty, which is a contradiction as we already know.

If some y_i is not connected to some $u \in V \setminus Y$, then u belongs to N_j with $n_j = 2$. (Otherwise, moving u to the part $\{y_i\}$ we obtain a new legitimate r -partition of $V \setminus \{y\}$ with a smaller l .) Then either $j = 1$ or $j = 2$ (because $M_i = N_i$ for $i \geq 3$). As we have equality in (6), every vertex of $V \setminus Y$ has at most one non-neighbor in Y .

It is impossible that some two vertices $w, z \in Y$ have degree less than $n-1$ each. Otherwise, it follows from the above information that $n_1 = n_2 = 2$, the graph $G[N_1 \cup N_2 \cup \{w, z\}]$ contains a perfect matching in its complement and so is 3-colorable, which implies that $\chi(G) \leq r$, a contradiction.

Thus, all missing edges in $G[Y, V \setminus Y]$ are between $(N_1 \setminus M_1) \cup (N_2 \setminus M_2)$ and some $z \in Y$, which is exactly what we want. Now we know all the edges of G and in order to maximize $e(G)$ we must have that n_1, n_2 are the two smallest numbers among the n_i 's, which is precisely what our construction says.

The theorem is proved. □

It remains to describe which sequences \mathbf{n} give us $p_r(n)$.

Lemma 5. Let $n \geq r + 3 \geq 5$. If $r > \frac{n-1}{2}$, then $\mathbf{n} = (1^{(2r-n+1)}, 2^{(n-r-1)})$ is the (unique) sequence satisfying (3) and maximizing (4). If $r \leq \frac{n-1}{2}$, then the optimal sequences are precisely those \mathbf{n} in (3) which satisfy all of the following inequalities:

$$n_1 \geq 2, \quad (7)$$

$$n_2 \leq n_1 + 1, \quad (8)$$

$$n_r \leq n_1 + 2, \quad (9)$$

$$n_r \leq n_3 + 1. \quad (10)$$

Proof. Let \mathbf{n} be an optimal sequence and $s = \min\{i \in [r] : n_i \geq 2\}$. If $s \geq 2$, then decreasing n_r by 1 and increasing n_{s-1} by 1, we increase $e(G(\mathbf{n}))$ by $n_r + n_{s+1} - 4 \geq n_r - 2$. By the maximality of \mathbf{n} we conclude that $n_r = 2$, $r > \frac{n-1}{2}$, and \mathbf{n} is precisely the required (unique) sequence.

So, suppose that $n_1 \geq 2$ and thus $r \leq \frac{n-1}{2}$. If we move 1 from n_r to n_1 , then $e(G(\mathbf{n}))$ increases by at least $n_r - n_1 - 2$, implying (9). The remaining inequalities are proved likewise. We have shown that any optimal sequence has the stated form.

Finally, let us consider, for $r \leq \frac{n-1}{2}$, the set \mathcal{N} of sequences satisfying (3) and (7)–(10). It is a routine to see that $1 \leq |\mathcal{N}| \leq 3$ and $e(G(\mathbf{n}))$ is constant on \mathcal{N} . It follows that \mathcal{N} is precisely the set of optimal sequences. \square

Remark. Note that, depending on n and r , there are from one to three different sequences satisfying (3) and (7)–(10).

Proof of Theorem 1. Let $r \leq \frac{n-1}{2}$, for example. It is easy to see that among all sequences \mathbf{n} satisfying (3) and (7)–(10), we can choose one satisfying $n_r \leq n_1 + 1$. This choice of parts is the same as the choice for $T_r(n-1)$. Now, to construct the Turán graph $T_r(n)$ we add one more vertex to the smallest part, that is, $n_2 + \dots + n_r$ extra edges. Comparing this with the construction of $G(\mathbf{n})$ and using the fact that $n_2 = \lfloor n/r \rfloor$, we obtain (1). \square

3. Determining $b_l(n)$. Recall that $b_l(n)$ is the maximum size of a graph of order n containing a cycle of length $2l+1$ but no odd cycle of strictly smaller length. Trivially, $b_1(n) = \binom{n}{2}$ so let us assume that $l \geq 2$.

A lower bound on $b_l(n)$ is given by the following construction. Take the Turán graph $T_2(n-2l+3)$; let its parts be X and Y , $|X| - |Y| \in \{-1, 0, 1\}$. Let $x \in X$ and let A be an arbitrary proper subset of Y , that is, $A \neq \emptyset$ and $A \neq Y$. Add a set L of $2l-3$ vertices spanning a path; let its end-vertices be u and v . Remove all edges between x and A but add edges $\{x, u\}$ and $\{v, y\}$ for all $y \in A$.

As A is a proper subset of Y , the constructed graph G contains a $(2l+1)$ -cycle. On the other hand, the removal of any vertex of L makes G bipartite. Hence, any odd cycle C must traverse all vertices of L . But we need at least 4 more vertices in order to connect u to v . This implies that C has at least $2l+1$ vertices, as required.

It is easy to see that the size of G is given by formula (2) of Theorem 2. We claim that this construction is optimal.

Proof of Theorem 2. We have to prove the upper bound on $b_l(n)$. For $l = 2$, the theorem follows from the results of Section 2 because all $p_2(n)$ -extremal graphs contain a 5-cycle. So, let us assume that $l \geq 3$. If $n = 2l+1$, then $b_l(n) = n$ with C_{2l+1} being the only available graph. So, assume that $n > 2l+1$.

Let G be a graph attaining $b_l(n)$. Let $V = V(G)$.

Let $x \in V$ be a vertex of maximum degree and $Y = \Gamma(x)$ be its neighborhood. We have $|Y| \geq 3$. Let y be a vertex of Y of the largest degree; let $X = \Gamma(y)$. As $K_3 \not\subset G$, the sets X and Y are independent and $X \cap Y = \emptyset$. Let $C \subset G$ be a $(2l+1)$ -cycle visiting vertices $v_1, v_2, \dots, v_{2l+1}$ in this order. Define

$$\begin{aligned} L &= V(C) \setminus (X \cup Y), \\ F &= G[X \cup Y], \\ R &= V \setminus (X \cup Y \cup L), \\ r &= |R|, \\ H &= G[\overline{R}] = G[X \cup Y \cup L]. \end{aligned}$$

We claim that $|L| \geq 2l - 3$. Let us suppose the contrary. As F is bipartite, we can find distinct $a, b \in X \cup Y$ which are connected by a path P whose interior points lie inside L such that the length of P is odd if and only if a, b lie in the same part. This gives us the required contradiction. For example, if a, b lie in different parts, then we can connect them by a path $P' \subset F$ of length one or three (use x, y if $\{a, b\} \not\subset E(G)$). But then $P \cup P'$ is an odd cycle of length at most $(2l - 4) + 3$, a contradiction.

Suppose first that $|L| = 2l - 3$, that is, C intersects $X \cup Y$ in 4 vertices. The properties of $x, y \in F$ imply, after a moment's thought, that C has precisely two vertices in each part of F and, moreover, these 4 vertices are consecutive vertices of C , say, $v_1, v_3 \in X$ and $v_2, v_4 \in Y$.

Clearly, C is an induced cycle for otherwise we can find a shorter odd cycle. Also, it is routine to see that the vertices v_6, \dots, v_{2l} of C cannot send any edges to $X \cup Y$ for otherwise we can find a shorter odd cycle by using the vertices x, y . Likewise, $\Gamma(v_5) \cap X = \Gamma(v_{2l+1}) \cap Y = \emptyset$ for otherwise we can find a 5-cycle, contradicting our assumption $l \geq 3$. Let $A = \Gamma(v_{2l+1}) \cap X \neq \emptyset$, $a = |A|$, $B = \Gamma(v_5) \cap Y \neq \emptyset$, and $b = |B|$.

We already know the H -degrees of all vertices in L : $(b+1, 2, 2, \dots, 2, a+1)$. Please also notice that we cannot have any edges between A and B for it would create a shorter odd cycle. By the choice of x , the H -degree of each vertex in $X \setminus A$ (resp. A) is at most $|Y|$ (resp. $|Y| - b + 1$). Likewise, each H -degree in $Y \setminus B$ (resp. B) is at most $|X|$ (resp. $|X| - a + 1$). And, of course, every vertex has degree at most $\Delta(G) = |Y|$ in G .

This shows that

$$\begin{aligned} 2e(G) &\leq 2e(H) + 2 \sum_{z \in R} d_G(z) \leq (b+1) + (a+1) + (|L| - 2) \times 2 \\ &\quad + (|X| - a)|Y| + a(|Y| - b + 1) + (|Y| - b)|X| + b(|X| - a + 1) + 2r|Y| \\ &= 2(|X| + r) \cdot |Y| + a + b - ab + |L| - 1. \end{aligned}$$

A simple optimization, in view of $|X| + |Y| + r + |L| = n$ and the inequalities $a \geq 1$ and $b \geq 1$, shows that we have to take $\min(a, b) = 1$ and that the sets Y and $X \cup R$ must be nearly equal. This gives

$$e(G) \leq \left\lfloor \frac{(n - |L|)^2}{4} \right\rfloor + |L|. \quad (11)$$

Recall that $|L| = 2l - 3$, obtaining precisely the required upper bound.

In order to characterize the extremal graphs (at least in this case) let us show that $R = \emptyset$. Suppose on the contrary that R is non-empty. Then there are no edges connecting R to $(X \setminus A) \cup (Y \setminus B)$ because each such edge is counted three times in the bound on $2e(G)$.

Similarly, R spans no edge in G but every $z \in R$ has degree $\Delta(G)$. As $\min(a, b) = 1$, it follows that $\max(a, b) = |Y| - 1$ and each vertex $z \in R$ is connected to everything in $A \cup B$. But then $G \supset C_5$, a contradiction. Now, it easily follows from $R = \emptyset$ that G is given by our construction.

Hence, let us assume that for any choice of a $(2l+1)$ -cycle C we have $|L| \geq 2l-2$. Suppose first that there is C intersecting both X and Y . Because the diameter of the bipartite graph F is 3, we can assume by symmetry that $v_1 \in X$ and then that $v_2 \in Y$ or $v_4 \in Y$. (We do not exclude the possibility that $|V(C) \cap (X \cup Y)| = 3$.)

Let us consider the case $v_2 \in Y$. By threatening to create a $(2l+1)$ -cycle having 4 vertices in $X \cup Y$ or a shorter odd cycle via x, y , one can show that there is no edge between $X \cup Y$ and $\{v_4, \dots, v_{2l}\}$. Thus v_3, v_{2l+1} are the only vertices of L that are connected to $X \cup Y$. Repeating the argument leading to (11), but with respect to $X' = X \cup \{v_3\}$, $Y' = Y \cup \{v_{2l+1}\}$, and $L' = L \setminus \{v_{2l+1}, v_1, v_2, v_3\}$, we obtain (11) (with L' instead of L). However, this time the inequality is strict, because when we passed from X, Y to X', Y' at least one of these sets strictly increased its size. This is the desired contradiction.

If $v_4 \in Y$, then one can argue that v_2, v_3 are the only vertices of L that can be connected to $(X \cup Y) \setminus \{v_1, v_4\}$ and the same contradictory bound arises.

The case when C intersects, say X , in two vertices u, v reduces to the above case: replace the even uv -path along C by two edges $\{u, y\}$ and $\{y, v\}$, obtaining an odd cycle of length at most $2l+1$ which intersects both X and Y .

Therefore, it remains to consider only the case when every $(2l+1)$ -cycle C intersects $X \cup Y$ in at most one vertex. Suppose first that $v_1 \in X$. Then v_2, v_{2l+1} cannot be connected to Y . (Otherwise we get K_3 or C_5 .) Likewise, v_3, v_{2l} cannot be connected to X . Moreover, $\Gamma(v_3) \cap \Gamma(v_{2l}) = \emptyset$. Also, none of v_4, \dots, v_{2l-1} can be connected to $X \cup Y$ for otherwise we would get a shorter odd cycle or a cycle intersecting both X and Y , a contradiction. Thus

$$\sum_{v \in L} d_H(v) \leq 2(|X| + 1) + (|Y| + 4) + (2l - 4) \times 2 = 2|X| + |Y| + 4l - 2.$$

Similarly to what we did before (and using $|X| = n - |Y| - 2l - r$), we obtain

$$\begin{aligned} 2e(G) &\leq 2|X| \cdot |Y| + (2|X| + |Y| + 4l - 2) + 2r|Y| \\ &= |Y| (2n - 2|Y| - 4l - 1) + 2n - 2r - 2. \end{aligned}$$

The optimal choice here is $r = 0$ and $y = \lfloor \frac{n}{2} \rfloor - l$. By considering odd/even $n \geq |L| + 4 = 2l + 4$ separately, one can see that the obtained bound is strictly smaller than (2), a contradiction. Similarly, we show that $v_1 \in Y$ is also impossible.

Finally, assume that every $(2l+1)$ -cycle is disjoint from $X \cup Y$. Suppose first that v_1 is connected to, for example, X . The routine analysis shows that, in order to prevent a shorter odd cycle or a C_{2l+1} intersecting $X \cup Y$, there are no edges between the following pairs of sets: $(X, \{v_2, v_4, v_{2l-1}, v_{2l+1}\})$, $(Y, \{v_1, v_3, v_{2l}\})$ and $(X \cup Y, \{v_5, \dots, v_{2l-2}\})$. Moreover, the X -neighborhoods of v_1, v_3, v_{2l} are disjoint as well as the Y -neighborhoods of $v_2, v_4, v_{2l-1}, v_{2l+1}$. We conclude that

$$\begin{aligned} 2e(G) &\leq 2|X| \cdot |Y| + |X| + |Y| + (2l + 1) \times 2 + 2r|Y| \\ &= 2|Y| (n - |Y| - 2l - 1) + n - r + 2l + 1. \end{aligned} \tag{12}$$

Optimizing, we take $|Y| = \lfloor \frac{n-1}{2} \rfloor - l$ and $r = 0$. One can see that the obtained bound is strictly less than (2) for $n \geq |L| + 4 \geq 2l + 5$.

If there are no edges between C and $X \cup Y$, then we get a strictly better bound than (12). This finishes the proof of the theorem. \square

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