УДК 517.956

R. V. Andrusyak, V. M. Kyrylych

GLOBAL SOLVABILITY OF HYPERBOLIC STEFAN PROBLEM

R. V. Andrusyak, V. M. Kyrylych. *Global solvability of hyperbolic Stefan problem*, Matematychni Studii, **23** (2005) 191–206.

Applying the method of contractive mappings, the local correct solvability of a free boundary problem with unseparated boundary conditions for a semilinear hyperbolic system of first order equations is established. Under additional assumptions on the monotonicity of initial data as well as the growth extent of both right-hand sides of the system and domain boundaries, sufficient conditions for the global solvability of the problem are formulated.

Р. В. Андрусяк, В. М. Кирилич. *Глобальная разрешимость гиперболической задачи Стефана* // Математичні Студiї. – 2005. – Т.23, №2. – C.191–206.

С помощью метода сжимающих отображений установлено локальную корректную разрешимость задачи со свободными границами с неразделенными граничными условиями для полулинейной гиперболической системы уравнений первого порядка. При выполнении дополнительных предположений о монотонности исходных данных, порядке роста правых частей системы и границы области, изложены достаточные условия глобальной разрешимости задачи.

The construction of a solution, global with respect to the argument t, of a mixed problem for a semilinear hyperbolic system of first-order equations with two independent variables was considered in [1]. A similar result for a quasilinear hyperbolic system was obtained in [2]. Some questions of the global solvability of mixed problems for such systems were also studied in [3].

Applying the methodology of the papers [2], [4], [5], we have established the theorem of local and global solvability of a free boundary problem for a semilinear hyperbolic system of first-order equations with nonlocal (unseparated) boundary conditions. In the case of fixed boundaries, the problems for quasilinear hyperbolic systems studied by I. Kmit in [5, chap.V] are close in statement to our problem.

Some variations of solution construction for a problem with nonlocal boundary conditions in the case of hyperbolic system with unknown boundaries (the hyperbolic Stefan problem) was examined in [6]. For the special kind of linear hyperbolic first-order system, the existence of a unique classical solution for a free boundary problem was proved for all $t > 0$ in [7].

Hyperbolic free boundary problems appear in many applied questions. In particular, questions of gas dynamics, heat conduction, phase transformation theory, polymer diffusion, seismicity, dynamics of biopopulation and some others lead to such problems (see also [4], [7]).

2000 *Mathematics Subject Classification:* 35L50.
Keywords: global solvability, hyperbolic Stefan problem, contractive mapping
doi:10.30970/ms.23.2.191-206

1. Statement of problem. To simplify the exposition, we introduce the notations: $u(x,t) = (u_1(x,t),\dots,u_n(x,t)) \in \mathbb{R}^n$; $a^u(t) = (a_1^u(t), a_2^u(t)) \in \mathbb{R}^2$; $u(a^u(t),t) = (u(a_1^u(t),t), u(a_2^u(t),t)) \in \mathbb{R}^{2n}$; $g(x) = (g_1(x),\dots,g_n(x)) \in \mathbb{R}^n$; $a^0 = (a_1^0, a_2^0) \in \mathbb{R}^2$; $g(a^0) = (g(a_1^0), g(a_2^0)) \in \mathbb{R}^{2n}$; $\zeta = (\zeta^1,\zeta^2) \in \mathbb{R}^2$; $\omega^k = (\omega_1^k,\dots,\omega_n^k) \in \mathbb{R}^n$; $\omega = (\omega^1,\omega^2) \in \mathbb{R}^{2n}$.

Consider the system of partial differential equations

$$\frac{\partial u_i}{\partial t} + \lambda_i(x,t)\frac{\partial u_i}{\partial x} = f_i(x,t,u), \quad i \in \{1,\dots,n\}, \tag{1}$$

where the functions $u_i(x,t)$ are defined in the domain $G_T^u = \{(x,t) \in \mathbb{R}^2 : 0 \le t \le T,\ a_1^u(t) \le x \le a_2^u(t)\}$, with boundaries $a_k^u(t), k \in \{1,2\}$, not set in advance and satisfying, in turn, the system of differential equations

$$\frac{da_k^u}{dt} = h_k\big(t, a^u(t), u(a^u(t),t)\big), \quad k \in \{1,2\}. \tag{2}$$

The unknown functions are subject to the initial conditions

$$a_k^u(0) = a_k^0, \quad a_1^0 < a_2^0,\ k \in \{1,2\}, \tag{3}$$

$$u_i(x,0) = g_i(x), \quad a_1^0 \le x \le a_2^0,\ i \in \{1,\dots,n\}. \tag{4}$$

Moreover, assuming the inequalities

$$\lambda_i(a_k^0,0) \ne h_k\big(0,a^0,g(a^0)\big), \quad i \in \{1,\dots,n\},\ k \in \{1,2\} \tag{5}$$

and introducing the notations
$I_1 = \big\{i : \lambda_i(a_1^0,0) > h_1(0,a^0,g(a^0))\big\}$, $\quad I_2 = \big\{i : \lambda_i(a_2^0,0) < h_2(0,a^0,g(a^0))\big\}$,
we impose boundary conditions as follows:

$$\sum_{k=1}^{2}\sum_{j=1}^{n} b_{ijk}(a^u(t),t)u_j(a_k^u(t),t) = H_i(a^u(t),t), \quad i \in \{1,\dots,N\}, \quad ,N = \sum_{k=1}^{2} \operatorname{card} I_k. \tag{6}$$

2. Generalized solution. Introduce a metric space S_T of the functions (u,a^u) such that $u_i \in C(G_T^u), i \in \{1,\dots,n\}$; $a_k^u \in C[0,T], k \in \{1,2\}$, $a_1^u(t) < a_2^u(t), 0 \le t \le T$, with the initial conditions (3),(4) satisfied and the metric defined by the formula

$$\rho(u,v) = \max\big\{\max_{k,t}|a_k^u(t) - a_k^v(t)|, \max_{i,x,t}|\overline{u}_i(x,t) - \overline{v}_i(x,t)|\big\}.$$

Here and subsequently for any $u_i : G_T^u \to \mathbb{R}$ ($g_i : [a_1^0,a_2^0] \to \mathbb{R}$) we use $\overline{u_i}$ ($\overline{g_i}$) to denote functions extending u_i (respectively g_i) to $\mathbb{R} \times [0,T]$ (respectively $\mathbb{R}$) by the rule $\overline{u_i}(x,t) = u_i(a_1^u(t),t)$, $x < a_1^u(t)$; $\overline{u_i}(x,t) = u_i(a_2^u(t),t)$, $x > a_2^u(t)$ (and respectively $\overline{g_i}(x) = g_i(a_1^0)$, $x < a_1^0$; $\overline{g_i}(x) = g_i(a_2^0)$, $x > a_2^0$).

Introduce some notations. Let $\varphi_i(\tau;x,t)$ be a solution, called a characteristic, of the Cauchy problem

$$\frac{d\xi}{d\tau} = \lambda_i(\xi,\tau), \quad \xi\Big|_{\tau=t} = x, \quad i \in \{1,\dots,n\},$$

and $\chi_i(x,t;u)$ be the minimal value τ at which the function $\varphi_i(\tau;x,t)$ reaches any boundary of G_T^u. For every $i \in \{1,\dots,n\}$ we define G_{Tg}^{ui}, G_{T1}^{ui}, and G_{T2}^{ui} to be the sets of points

$(x,t)\in G_T^u$ such that respectively $\chi_i(x,t;u)=0$, $\chi_i(x,t;u)>0$ with $\varphi_i(\chi_i(x,t;u);x,t)=a_1^u(\chi_i(x,t;u))$, and $\chi_i(x,t;u)>0$ with $\varphi_i(\chi_i(x,t;u);x,t)=a_2^u(\chi_i(x,t;u))$. Set $\mu_i^u(t)=u_i(a_1^u(t),t)$, $i\in I_1$; $\nu_i^u(t)=u_i(a_2^u(t),t)$, $i\in I_2$;

$$B(a^u(t),t)=\begin{pmatrix} b_{1i_11} & \dots & b_{1i_{p_1}1} & b_{1j_12} & \dots & b_{1j_{p_2}2}\\ b_{2i_11} & \dots & b_{2i_{p_1}1} & b_{2j_12} & \dots & b_{2j_{p_2}2}\\ \dots & .. & \dots & \dots & .. & \dots\\ b_{Ni_11} & \dots & b_{Ni_{p_1}1} & b_{Nj_12} & \dots & b_{Nj_{p_2}2}\end{pmatrix}$$

where i_r, j_s, $r\in\{1,\dots,p_1\}$, $s\in\{1,\dots,p_2\}$ are indices put in the order of increase and taken from the corresponding sets I_1,I_2, $p_k=\operatorname{card}I_k$, $k\in\{1,2\}$. Suppose also that

$$\det B(a^0,0)\neq 0. \tag{7}$$

If in (1) the functions u_i were continuously differentiable, and for $0\le t\le T$ the following inequalities held:

$$\lambda_i(a_k^u(t),t)\neq h_k\big(t,a^u(t),u(a^u(t),t)\big),\quad k\in\{1,2\},\ i\in\{1,\dots,n\}, \tag{8}$$
$$\det B(a^u(t),t)\neq 0, \tag{9}$$

then integrating (1) along characteristics would yield the system of functional integral equations

$$u_i(x,t)=\vartheta_i(x,t;u)+\int\limits_{\chi_i(x,t;u)}^{t} f_i(\varphi_i(\tau;x,t),\tau,u(\varphi_i(\tau;x,t),\tau))\,d\tau,\ i\in\{1,\dots,n\},\ (x,t)\in G_T^u, \tag{10}$$

where

$$\vartheta_i(x,t;u)=\begin{cases} g_i(\varphi_i(0;x,t)), & \text{if } (x,t)\in G_{Tg}^{ui},\\ \mu_i^u(\chi_i(x,t;u)), & \text{if } (x,t)\in G_{T1}^{ui},\\ \nu_i^u(\chi_i(x,t;u)), & \text{if } (x,t)\in G_{T2}^{ui};\end{cases}$$

and

$$\mu_i^u(t)=\frac{1}{\det B(a^u(t),t)}\sum_{j=1}^{N}B_{ji1}(a^u(t),t)\Big[H_j(a^u(t),t)-\sum_{k=1}^{2}\sum_{l\notin I_k}b_{jlk}(a^u(t),t)W_l^u\big(a_k^u(t),t\big)\Big], \tag{11}$$

$$\nu_i^u(t)=\frac{1}{\det B(a^u(t),t)}\sum_{j=1}^{N}B_{ji2}(a^u(t),t)\Big[H_j(a^u(t),t)-\sum_{k=1}^{2}\sum_{l\notin I_k}b_{jlk}(a^u(t),t)W_l^u\big(a_k^u(t),t\big)\Big]; \tag{12}$$

with

$$W_l^u\big(a_k^u(t),t\big)=\vartheta_l\big(a_k^u(t),t;u\big)+\int\limits_{\chi_l(a_k^u(t),t;u)}^{t} f_l\big(\varphi_l(\tau;a_k^u(t),t),\tau;u\big(\varphi_l(\tau;a_k^u(t),t),\tau\big)\big)\,d\tau, \tag{13}$$

$$k\in\{1,2\},\ l\notin I_k,\ 0\le t\le T.$$

Here B_{ijk}, $i\in\{1,\dots,N\}, k\in\{1,2\}, j\in I_k$, denote the algebraic cofactors to the corresponding elements b_{ijk} of the matrix B.

Similarly, integrating (2) from 0 to t, we arrive at the system of integral equations

$$a_k^u(t) = a_k^0 + \int\limits_0^t h_k\big(\tau, a^u(\tau), u(a^u(\tau), \tau)\big)\, d\tau, \quad k \in \{1,2\},\ 0 \le t \le T. \tag{14}$$

By a generalized (Lipschitz) solution of the problem (1)-(4),(6) we will mean the pair $(u, a^u) \in S_T$ of functions satisfying conditions (8),(9) and systems of equations (10),(14), with $u_i(x,t), i \in \{1,\dots,n\}$, being Lipschitz with respect to all their arguments. When no confusion can arise, we will use the symbol u in place of the pair (u, a^u). Remark that for any $\phi(x_1,\dots,x_n)$ the Lipschitz property with respect to all x_j with constant L is understood in the following form: $|\phi(x_1,\dots,x_n) - \phi(y_1,\dots,y_n)| \le L \max\limits_{1\le i\le n}\{|x_i - y_i|\}$.

3. Local solvability. Assume that $f_i(x,t,u)$, $i \in \{1,\dots,n\}$, are continuous in $\mathbb{R}\times[0,T]\times\mathbb{R}^n$ and possess the local Lipschitz property with respect to x and u; $h_k(t,\zeta,\omega), k \in \{1,2\}$, being defined in $[0,T]\times\mathbb{R}^{2+2n}$ are locally Lipschitz there with respect to all arguments; $\lambda_i(x,t), i \in \{1,\dots,n\}$, are continuous in $\mathbb{R}\times[0,T]$ as well as locally Lipschitz with respect to x; $g_i(x), i \in \{1,\dots,n\}$, are Lipschitz in $[a_1^0, a_2^0]$; and finally $b_{ijk}(\zeta,t)$, $H_i(\zeta,t)$, $i \in \{1,\dots,N\}$, $j \in \{1,\dots,n\}$, $k \in \{1,2\}$, being defined in $\mathbb{R}^2\times[0,T]$ possess the local Lipschitz property there with respect to all arguments.

In addition, assume satisfaction of zero-order fitting conditions

$$\sum_{j=1}^{n}\sum_{k=1}^{2} b_{ijk}(a^0, 0) g_j(a_k^0) = H_i(a^0, 0), \quad i \in \{1,\dots,N\}. \tag{15}$$

Theorem 1. *Under the above-mentioned assumptions on the continuity and the Lipschitzness of the given functions with the conditions (5),(7), and (15), there exists a unique generalized solution of the problem (1)-(4),(6) for $0 \le t \le \varepsilon_0$ when $\varepsilon_0 > 0$ is small enough.*

Proof. Denote by $S = S_{\varepsilon\alpha\beta p}$ a subset of S_ε, $\varepsilon \in (0,T]$, that consists of functions (u, a^u) satisfying the conditions:

I. the functions $t \mapsto a_k^u(t) - h_k\big(0, a^0, g(a^0)\big)t$, $k \in \{1,2\}$, $0 \le t \le \varepsilon$ are Lipschitz with a constant α;

II. $|u_i(x,t) - \overline{g}_i(x)| \le \beta$, $i \in \{1,\dots,n\}$, $(x,t) \in G_\varepsilon^u$;

III. the functions u_i are Lipschitz with respect to x with a constant p.

In the space S we define an operator A in the following way. Let $u \in S$, then $Au = (A_1 u, \dots, A_n u)$ where $A_i u\colon G_\varepsilon^{Au} \to \mathbb{R}$, with G_ε^{Au} being restricted on sides by the lines

$$x = a_k^{Au}(t) = a_k^0 + \int\limits_0^t h_k\big(\tau, a^u(\tau), u(a^u(\tau), \tau)\big)\, d\tau, \quad k \in \{1,2\},\ 0 \le t \le \varepsilon,$$

and the values $A_i u$ being fixed by the formula

$$(A_i u)(x,t) = \vartheta_i(x,t;\tilde{A}u) + \int\limits_{\chi_i(x,t;\tilde{A}u)}^{t} f_i\big(\varphi_i(\tau;x,t), \tau, (\tilde{A}u)(\varphi_i(\tau;x,t),\tau)\big)\, d\tau.$$

Here $\tilde{A}u = (\tilde{A}_1 u, \dots, \tilde{A}_n u)$ where $\tilde{A}_i u$ is the restriction of $\overline{u}_i$ to G_ε^{Au}. Notice that $\chi_i(x,t;\tilde{A}u) = \chi_i(x,t;Au)$ and $a_k^{\tilde{A}u}(t) = a_k^{Au}(t)$.

Fix $\varepsilon = \varepsilon_0$, $\alpha = \alpha_0$, $\beta = \beta_0$, ε_0 and β_0 being small enough for a_k^{Au}, $k \in \{1,2\}$ to satisfy condition I (the possibility of such a choice follows from the continuity of h_k and restrictions I,II on the introduced space S). Let $u \in S$, $(x,t) \in G_{\varepsilon_0}^{Au}$, then from the continuity of the corresponding functions it follows that $|f_i(x,t,\tilde{A}u)| \le F$, $|\lambda_i(x,t)| \le \Lambda, i \in \{1,\dots,n\}$; $|h_k\big(t, a^u(t), u(a^u(t),t)\big)| \le H, \in \{1,2\}$, ($F$,$\Lambda$, and H being some constants which are known to exist); and by local Lipschitzness we conclude that $f_i(x,t,\tilde{A}u), i \in \{1,\dots,n\}$ with respect to x, u, $\lambda_i(x,t), i \in \{1,\dots,n\}$ with respect to x, and $h_k\big(t, a^u(t), u(a^u(t),t)\big), k \in \{1,2\}$ with respect to all arguments t,ζ and ω are Lipschitz with some constants f_0,λ_0 and h_0 respectively. Now reduce ε_0 and β_0 so that

$$|h_k\big(t, a^u(t), u(a^u(t),t)\big) - \lambda_i(a_k^{Au}(t),t)| \ge \gamma > 0,\ k \in \{1,2\},\ i \in \{1,\dots,n\},\ 0 \le t \le \varepsilon_0,$$
$$|\det B(a^{Au}(t),t)| \ge \kappa > 0,\ 0 \le t \le \varepsilon_0 \tag{16}$$

(γ and κ being some constants which are known to exist). The operator A is thus well defined.

Similarly, for ε_0, α_0, β_0 set above and provided $u \in S, 0 \le t \le \varepsilon_0$, we deduce that $|b_{ijk}(a^{Au}(t),t)| \le b$, $|H_i(a^{Au}(t),t)| \le \tilde{H}$, $i \in \{1,\dots,N\}, j \in \{1,\dots,n\}, k \in \{1,2\}$; $|B_{ijk}(a^{Au}(t),t)| \le B$, $i \in \{1,\dots,N\}, k \in \{1,2\}, j \in I_k$ (the existence of such constants follows from the continuity of the corresponding functions), and by local Lipschitzness we infer that $b_{ijk}(a^{Au}(t),t), H_i(a^{Au}(t),t)$ are Lipschitz with respect to ζ and t with some constants, which we denote by b_0 and H_0 respectively. From this and by (16) we immediately have the Lipschitzness of $B_{ijk}(a^{Au}(t),t)$ and $\big(\det B(a^{Au}(t),t)\big)^{-1}$ with respect to the arguments ζ and t, and let B_0 be their common Lipschitz constant. Finally, fix G such that $|g_i(x)| \le G$, $a_1^0 \le x \le a_2^0, i \in \{1,\dots,n\}$, and denote by r the Lipschitz constant of these functions.

By integral representation of $\varphi_i(\tau; x_j, t)$ and the Gronwall-Bellman lemma we have $|\Delta\varphi_i(\tau; x_j, t)| \le |\Delta x_j|\ e^{\lambda_0 \varepsilon_0}$, $i \in \{1,\dots,n\}$, $j \in \{1,2\}$. For simplicity of the exposition we denote the difference $\phi(x_2) - \phi(x_1)$ by $\Delta\phi(x_j)$ for any $\phi(x)$.

Let us determine the common Lipschitz constant of the functions $\mu_i^{\tilde{A}u}(t), i \in I_1$, $\nu_j^{\tilde{A}u}(t)$, $j \in I_2$, with fixed $u \in S$. For this, take ε_0 small enough for the inequality below to hold:

$$\Lambda\varepsilon_0 \le a_2^0 - a_1^0 - 2H\varepsilon_0. \tag{17}$$

Then we obtain $\chi_l\big(a_k^{Au}(t), t; \tilde{A}u\big) = 0$, $k \in \{1,2\}$, $l \notin I_k$, $0 \le t \le \varepsilon_0$, and therefore, equality (13) becomes of the following form:

$$W_l^{\tilde{A}u}\big(a_k^{Au}(t),t\big) = g_l\big(\varphi_l(0; a_k^{Au}(t),t)\big) + \int_0^t f_l\Big(\varphi_l(\tau; a_k^{Au}(t),t), \tau; (\tilde{A}u)\big(\varphi_l(\tau; a_k^{Au}(t),t),\tau\big)\Big)\, d\tau,$$

$k \in \{1,2\}$, $l \notin I_k$, $0 \le t \le \varepsilon_0$.

Next estimate $\Delta W_l^{\tilde{A}u}\big(a_k^{Au}(t_j),t_j\big)$. Letting $k=1$, $t_2>t_1$, we get

$$|\Delta W_l^{\tilde{A}u}\big(a_1^{Au}(t_j),t_j\big)| \le |\Delta g_l\big(\varphi_l(0;a_1^{Au}(t_j),t_j)\big)|+$$
$$+\int\limits_{t_1}^{t_2}\big|f_l\big(\varphi_l(\tau;a_1^{Au}(t_2),t_2),\tau,(\tilde{A}u)\big(\varphi_l(\tau;a_1^{Au}(t_2),t_2),\tau\big)\big)\big|\,d\tau+$$
$$+\int\limits_{0}^{t_1}\big|\Delta f_l\big(\varphi_l(\tau;a_1^{Au}(t_j),t_j),\tau,(\tilde{A}u)\big(\varphi_l(\tau;a_1^{Au}(t_j),t_j),\tau\big)\big)\big|\,d\tau \le r(H+\Lambda)|\Delta t_j|e^{\lambda_0\varepsilon_0}+$$
$$+F|\Delta t_j|+\varepsilon_0 f_0 p(H+\Lambda)|\Delta t_j|e^{\lambda_0\varepsilon_0}=\Big[F+e^{\lambda_0\varepsilon_0}(r+\varepsilon_0 f_0 p)(H+\Lambda)\Big]|\Delta t_j|,\quad p\ge 1.$$

Therefore, for all $t_1,t_2\in[0,\varepsilon_0]$ we obtain

$$\max_{i\in I_1,j\in I_2}\big\{|\mu_i^{\tilde{A}u}(t_2)-\mu_i^{\tilde{A}u}(t_1)|,|\nu_j^{\tilde{A}u}(t_2)-\nu_j^{\tilde{A}u}(t_1)|\big\}\le \mathrm{L}(r,p)|t_2-t_1|,$$

where

$$\mathrm{L}(r,p)=B_0HNB\big(\tilde{H}+2nb(G+FT)\big)+\frac{1}{\kappa}NB_0H\big(\tilde{H}+2nb(G+FT)\big)+$$
$$+\frac{1}{\kappa}NB\Big(H_0H+2nb_0H(G+FT)+2nb\Big[F+e^{\lambda_0\varepsilon_0}(r+\varepsilon_0 f_0 p)(H+\Lambda)\Big]\Big).$$

Define

$$L_1=\big(B_0NB+\frac{1}{\kappa}NB_0\big)\big(\tilde{H}+2nb(G+FT)\big)+\frac{1}{\kappa}NB\big(H_0+2nb_0(G+FT)\big),\ L_2=\frac{1}{\kappa}NB2nb.$$

Then rewrite $\mathrm{L}(r,p)=L_1H+L_2\Big[F+e^{\lambda_0\varepsilon_0}(r+\varepsilon_0 f_0 p)(H+\Lambda)\Big]$.

Let us now determine the conditions sufficient for the operator A to map S into itself, that is for Au to possess properties II,III provided $u\in S$.

First, consider condition II. For $(x,t)\in G_{\varepsilon_0}^{Au}$ we have

$$|(A_iu)(x,t)-\overline{g}_i(x)|\le\Big|\int\limits_{\chi_i(x,t;Au)}^{t} f_i\big(\varphi_i(\tau;x,t),\tau,(\tilde{A}u)(\varphi_i(\tau;x,t),\tau)\big)\,d\tau\Big|+$$
$$+|\vartheta_i(x,t;\tilde{A}u)-\overline{g}_i(x)|\le F\varepsilon_0+|\vartheta_i(x,t;\tilde{A}u)-\overline{g}_i(x)|.$$

If $\chi_i(x,t;Au)=0$, then $|\vartheta_i(x,t;\tilde{A}u)-\overline{g}_i(x)|=|g_i(\varphi_i(0;x,t))-\overline{g}_i(x)|\le r\Lambda\varepsilon_0$.
If $\chi_i(x,t;Au)>0$, then, letting for definiteness $(x,t)\in G_{\varepsilon_0 1}^{Au\,i}$, we obtain

$$|\vartheta_i(x,t;\tilde{A}u)-\overline{g}_i(x)|=|\mu_i^{\tilde{A}u}\big(\chi_i(x,t;Au)\big)-\mu_i^{\tilde{A}u}(0)|+|g_i(a_1^0)-\overline{g}_i(x)|\le$$
$$\le \mathrm{L}(r,p)\chi_i(x,t;Au)+r\big(\Lambda(t-\chi_i(x,t;Au))+H\chi_i(x,t;Au)\big)\le \mathrm{L}(r,p)\varepsilon_0+r\max\big\{\Lambda,H\big\}\varepsilon_0.$$

Thus, in order for property II to hold, it suffices to require that

$$\Big[F+\mathrm{L}(r,p)+r\max\big\{\Lambda,H\big\}\Big]\varepsilon_0\le\beta_0. \tag{18}$$

Next, consider condition III. Let $(x_j,t)\in G^{Au}_{\varepsilon_0}, j\in\{1,2\}$. To estimate $\Delta(A_iu)(x_j,t)$, we can obviously suppose that either $(x_j,t)\in G^{Au\,i}_{\varepsilon_0 g}, j\in\{1,2\}$, or $(x_j,t)\in G^{Au\,i}_{\varepsilon_0 k}, j\in\{1,2\}$, with k being the same for both j. In the first case we have

$$|\Delta(A_iu)(x_j,t)|\le\int\limits_0^t\bigl|\Delta f_i\bigl(\varphi_i(\tau;x_j,t),\tau,(\tilde{A}u)(\varphi_i(\tau;x_j,t),\tau)\bigr)\bigr|\,d\tau+|\Delta g_i(\varphi_i(0;x_j,t))|\le$$

$$\le\int\limits_0^t f_0\max\bigl\{|\Delta\varphi_i(\tau;x_j,t)|,p|\Delta\varphi_i(\tau;x_j,t)|\bigr\}\,d\tau+r|\Delta\varphi_i(0;x_j,t)|\le$$

$$\le\varepsilon_0 f_0 p|\Delta x_j|\,e^{\lambda_0\varepsilon_0}+r|\Delta x_j|\,e^{\lambda_0\varepsilon_0}.$$

Whereas, in the second case, letting for definiteness $k=1$, $x_1<x_2$, we obtain

$$|\Delta(A_iu)(x_j,t)|\le\int\limits_{\chi_i(x_1,t;Au)}^t\bigl|\Delta f_i\bigl(\varphi_i(\tau;x_j,t),\tau,(\tilde{A}u)(\varphi_i(\tau;x_j,t),\tau)\bigr)\bigr|\,d\tau+$$

$$+\int\limits_{\chi_i(x_2,t;Au)}^{\chi_i(x_1,t;Au)}\bigl|f_i\bigl(\varphi_i(\tau;x_2,t),\tau,(\tilde{A}u)(\varphi_i(\tau;x_2,t),\tau)\bigr)\bigr|\,d\tau+|\Delta\mu_i^{\tilde{A}u}\bigl(\chi_i(x_j,t;Au)\bigr)|\le$$

$$\le\varepsilon_0 f_0 p|\Delta x_j|\,e^{\lambda_0\varepsilon_0}+F|\Delta\chi_i(x_j,t;Au)|+\mathrm{L}(r,p)|\Delta\chi_i(x_j,t;Au)|.$$

Estimate the difference $\Delta\chi_i(x_j,t;Au)$. Suppose

$$(H+\Lambda)\varepsilon_0\le\frac{\gamma}{2\lambda_0},\tag{19}$$

then $|a_k^{Au}(\tau)-\varphi_i(\tau;x,t)|\le\dfrac{\gamma}{2\lambda_0}$ provided $\chi_i(x,t;Au)>0$ and $\varphi_i(\chi_i(x,t;Au);x,t)=$ $=a_k^{Au}(\chi_i(x,t;Au))$. Therefore,

$$|h_k\bigl(\tau,a^u(\tau),u(a^u(\tau),\tau)\bigr)-\lambda_i(\varphi_i(\tau;x,t),\tau)|\ge|h_k\bigl(\tau,a^u(\tau),u(a^u(\tau),\tau)\bigr)-\lambda_i(a_k^{Au}(\tau),\tau)|-$$

$$-|\lambda_i(a_k^{Au}(\tau),\tau)-\lambda_i(\varphi_i(\tau;x,t),\tau)|\ge\gamma-\lambda_0|a_k^{Au}(\tau)-\varphi_i(\tau;x,t)|\ge\gamma-\lambda_0\frac{\gamma}{2\lambda_0}=\frac{\gamma}{2}.$$

Since

$$\frac{d}{d\tau}\bigl(a_k^{Au}(\tau)-\varphi_i(\tau;x,t)\bigr)=h_k\bigl(\tau,a^u(\tau),u(a^u(\tau),\tau)\bigr)-\lambda_i(\varphi_i(\tau;x,t),\tau),$$

we conclude that

$$|\Delta\chi_i(x_j,t;Au)|\le\frac{2}{\gamma}\,|a_1^{Au}\bigl(\chi_i(x_1,t;Au)\bigr)-\varphi_i\bigl(\chi_i(x_1,t;Au);x_2,t\bigr)|=$$

$$=\frac{2}{\gamma}\,|\Delta\varphi_i\bigl(\chi_i(x_1,t;Au);x_j,t\bigr)|\le\frac{2}{\gamma}|\Delta x_j|\,e^{\lambda_0\varepsilon_0}.$$

Returning to the difference of the operator, we finally have

$$|\Delta(A_iu)(x_j,t)|\le\varepsilon_0 f_0 p|\Delta x_j|\,e^{\lambda_0\varepsilon_0}+F\frac{2}{\gamma}|\Delta x_j|\,e^{\lambda_0\varepsilon_0}+\mathrm{L}(r,p)\frac{2}{\gamma}|\Delta x_j|\,e^{\lambda_0\varepsilon_0}.$$

Thus, Au has property III if the following inequality is true:

$$e^{\lambda_0\varepsilon_0}\Big(\varepsilon_0 f_0 p + \max\big\{r, (F+\mathrm{L}(r,p))\frac{2}{\gamma}\big\}\Big) \le p. \qquad (20)$$

Suppose that all the conditions providing the inclusion $AS \subset S$ hold and clarify the restrictions on ε_0 sufficient for the operator to be contractive.

Let $u^j \in S, j \in \{1,2\},\ u^j = (u^j_1, \ldots, u^j_n)$, and $\rho(u^1,u^2) = \rho$. Then

$$|\Delta a_k^{Au^j}(t)| \le \int\limits_0^t \big|\Delta h_k\big(\tau, a^{u^j}(\tau), u^j(a^{u^j}(\tau),\tau)\big)\big|\, d\tau \le$$

$$\le \int\limits_0^t h_0 \max\big\{|\Delta a^{u^j}(\tau)|, |\Delta u^j(a_1^{u^j}(\tau),\tau)|, |\Delta u^j(a_2^{u^j}(\tau),\tau)|\big\}\, d\tau \le$$

$$\le \varepsilon_0 h_0 \max\big\{\max_{k,t}|\Delta a_k^{u^j}(t)|, \max_{i,x,t}|\Delta \overline{u}_i^j(x,t)|\big\} = \varepsilon_0 h_0 \rho, \quad 0 \le t \le \varepsilon_0.$$

First estimate $\Delta(\overline{\tilde{A}_i u^j})(x,t)$. Letting $x \le \min\{a_1^{Au^1}(t), a_1^{Au^2}(t)\}$, we obtain $|\Delta(\overline{\tilde{A}_i u^j})(x,t)| = |(\tilde{A}_i u^2)(a_1^{Au^2}(t),t) - (\tilde{A}_i u^1)(a_1^{Au^1}(t),t)| = |u_i^2(a_1^2(t),t) - u_i^1(a_1^1(t),t)|$, where $a_1^j(t) = \max\{a_1^{u^j}(t), a_1^{Au^j}(t)\}$. Thus, letting $a_1^1(t) < a_1^2(t)$, we derive that

$$|\Delta(\overline{\tilde{A}_i u^j})(x,t)| \le |u_i^2(a_1^2(t),t) - u_i^1(a_1^2(t),t)| + |u_i^1(a_1^2(t),t) - u_i^1(a_1^1(t),t)| \le \rho + p|a_1^2(t)-$$
$$-a_1^1(t)| \le \rho + p\big(|a_1^{u^2}(t) - a_1^{u^1}(t)| + |a_1^{Au^2}(t) - a_1^{Au^1}(t)|\big) \le$$
$$\le \rho + p(\rho + \varepsilon_0 h_0 \rho) = (1 + p + \varepsilon_0 h_0 p)\rho.$$

In the other cases, reasoning similarly, we arrive at the same result.

Next estimate $\Delta W_l^{\tilde{A}u^j}\big(a_k^{Au^j}(t_j), t_j\big)$. Letting $k=1$, $t_2 > t_1$, by (17) we have

$$|\Delta W_l^{\tilde{A}u^j}\big(a_1^{Au^j}(t_j), t_j\big)| \le |\Delta g_l\big(\varphi_l(0; a_1^{Au^j}(t_j), t_j)\big)| +$$

$$+ \int\limits_{t_1}^{t_2} \big|f_l\big(\varphi_l(\tau; a_1^{Au^2}(t_2), t_2), \tau, (\tilde{A}u^2)\big(\varphi_l(\tau; a_1^{Au^2}(t_2), t_2), \tau\big)\big)\big|\, d\tau +$$

$$+ \int\limits_0^{t_1} \big|\Delta f_l\big(\varphi_l(\tau; a_1^{Au^j}(t_j), t_j), \tau, (\tilde{A}u^j)\big(\varphi_l(\tau; a_1^{Au^j}(t_j), t_j), \tau\big)\big)\big|\, d\tau \le r|\Delta\varphi_l\big(0; a_1^{Au^j}(t_j), t_j\big)| +$$

$$+F|\Delta t_j| + \int\limits_0^{t_1} f_0\Big[\max\big\{\big|\Delta\varphi_l(\tau; a_1^{Au^j}(t_j), t_j)\big|, \big|\Delta\big(\tilde{A}u^1\big)\big(\varphi_l(\tau; a_1^{Au^j}(t_j), t_j), \tau\big)\big|\big\} +$$
$$+ \big|\Delta\big(\tilde{A}u^j\big)\big(\varphi_l(\tau; a_1^{Au^2}(t_2), t_2), \tau\big)\big|\Big]\, d\tau.$$

Since $|\Delta(\tilde{A}u^j)\big(\varphi_l(\tau; a_1^{Au^2}(t_2), t_2), \tau\big)| \le (1 + p + \varepsilon_0 h_0 p)\rho$ and

$$|\Delta\varphi_l(\tau; a_1^{Au^j}(t_j), t_j)| \le |\varphi_l(t_1; a_1^{Au^2}(t_2), t_2) - a_1^{Au^1}(t_1)| e^{\lambda_0\varepsilon_0} \le \big[|\varphi_l(t_1; a_1^{Au^2}(t_2), t_2)-$$
$$-a_1^{Au^2}(t_1)| + |a_1^{Au^2}(t_1) - a_1^{Au^1}(t_1)|\big] e^{\lambda_0\varepsilon_0} \le \big[(H+\Lambda)|\Delta t_j| + \varepsilon_0 h_0 \rho\big] e^{\lambda_0\varepsilon_0},$$

we find that

$$|\Delta W_l^{\tilde{A}u^j}\big(a_1^{Au^j}(t_j),t_j\big)| \le r(H+\Lambda)|\Delta t_j|e^{\lambda_0\varepsilon_0} + r\varepsilon_0 h_0\rho e^{\lambda_0\varepsilon_0} + F|\Delta t_j| +$$
$$+\varepsilon_0 f_0 p(H+\Lambda)|\Delta t_j|e^{\lambda_0\varepsilon_0} + \varepsilon_0^2 f_0 p h_0 \rho e^{\lambda_0\varepsilon_0} + \varepsilon_0 f_0(1+p+\varepsilon_0 h_0 p)\rho.$$

Hence, for all $t_1,\ t_2 \in [0,\varepsilon_0]$ we deduce that

$$\max_{i\in I_1, j\in I_2}\left\{|\mu_i^{\tilde{A}u^2}(t_2)-\mu_i^{\tilde{A}u^1}(t_1)|, |\nu_j^{\tilde{A}u^2}(t_2)-\nu_j^{\tilde{A}u^1}(t_1)|\right\} \le \mathrm{L}(r,p)|t_2-t_1| + \mathrm{L}^*(r,p)\rho,$$

where $\mathrm{L}^*(r,p) = L_1\varepsilon_0 h_0 + L_2\left[e^{\lambda_0\varepsilon_0}(r\varepsilon_0 h_0 + \varepsilon_0^2 f_0 p h_0) + \varepsilon_0 f_0(1+p+\varepsilon_0 h_0 p)\right]$.

Finally, estimate $\Delta(\overline{A_i u^j})(x,t)$. For this consider possible partial cases. If $(x,t)\in G_{\varepsilon_0 g}^{Au^1\ i}\cap G_{\varepsilon_0 g}^{Au^2\ i}$, then

$$|\Delta(\overline{A_i u^j})(x,t)| = |\Delta(A_i u^j)(x,t)| \le \int_0^t \left|\Delta f_i\big(\varphi_i(\tau;x,t),\tau,(\tilde{A}u^j)(\varphi_i(\tau;x,t),\tau)\big)\right| d\tau \le$$
$$\le \varepsilon_0 f_0(1+p+\varepsilon_0 h_0 p)\rho.$$

Now suppose that $(x,t)\in G_{\varepsilon_0 1}^{Au^1\ i}\cap G_{\varepsilon_0 1}^{Au^2\ i}$. Letting $\chi_i(x,t;Au^1) > \chi_i(x,t;Au^2)$, we obtain $|\Delta\chi_i(x,t;Au^j)| \le \dfrac{2}{\gamma}\ |a_1^{Au^2}(\chi_i(x,t;Au^1)) - a_1^{Au^1}(\chi_i(x,t;Au^1))| \le \dfrac{2}{\gamma}\ \varepsilon_0 h_0\rho$ and hence,

$$|\Delta(A_i u^j)(x,t)| \le \int_{\chi_i(x,t;Au^1)}^{t} \left|\Delta f_i\big(\varphi_i(\tau;x,t),\tau,(\tilde{A}u^j)(\varphi_i(\tau;x,t),\tau)\big)\right| d\tau +$$
$$+\int_{\chi_i(x,t;Au^2)}^{\chi_i(x,t;Au^1)} \left|f_i\big(\varphi_i(\tau;x,t),\tau,(\tilde{A}u^2)(\varphi_i(\tau;x,t),\tau)\big)\right| d\tau + |\mu_i^{\tilde{A}u^2}(\chi_i(x,t;Au^2)) -$$
$$-\mu_i^{\tilde{A}u^1}(\chi_i(x,t;Au^1))| \le \varepsilon_0 f_0(1+p+\varepsilon_0 h_0 p)\rho + F|\Delta\chi_i(x,t;Au^j)| + \mathrm{L}(r,p)|\Delta\chi_i(x,t;Au^j)| +$$
$$+\mathrm{L}^*(r,p)\rho = \varepsilon_0 f_0(1+p+\varepsilon_0 h_0 p)\rho +$$
$$+F\frac{2}{\gamma}\ \varepsilon_0 h_0\rho + \Big[L_1 H + L_2\Big[F + e^{\lambda_0\varepsilon_0}(r+\varepsilon_0 f_0 p)(H+\Lambda)\Big]\Big]\frac{2}{\gamma}\ \varepsilon_0 h_0\rho +$$
$$+\Big[L_1\varepsilon_0 h_0 + L_2\Big[e^{\lambda_0\varepsilon_0}(r\varepsilon_0 h_0 + \varepsilon_0^2 f_0 p h_0) + \varepsilon_0 f_0(1+p+\varepsilon_0 h_0 p)\Big]\Big]\rho =$$
$$= \Big[f_0(1+p+\varepsilon_0 h_0 p) + F\frac{2}{\gamma}h_0 + (L_1 H + L_2 F)\frac{2}{\gamma}h_0 + L_2 e^{\lambda_0\varepsilon_0}(r+\varepsilon_0 f_0 p)(H+\Lambda)\frac{2}{\gamma}h_0 +$$
$$+L_1 h_0 + L_2 e^{\lambda_0\varepsilon_0}(r h_0 + \varepsilon_0 f_0 p h_0) + L_2 f_0(1+p+\varepsilon_0 h_0 p)\Big]\varepsilon_0\rho.$$

Considering the case where $(x,t)\in G_{\varepsilon_0}^{Au^2}$ but $(x,t)\notin G_{\varepsilon_0}^{Au^1}$ (for definiteness $x < a_1^{Au^1}(t)$), we find that $\Delta(\overline{A_i u^j})(x,t) = (A_i u^2)(x,t) - (A_i u^1)(a_1^{Au^1}(t),t) = (A_i u^2)(x,t) - (A_i u^2)(a_1^{Au^1}(t),t) + \Delta(A_i u^j)(a_1^{Au^1}(t),t)$, where $|x - a_1^{Au^1}(t)| \le \varepsilon_0 h_0\rho$. Therefore, $|\Delta(\overline{A_i u^j})(x,t)| \le p\varepsilon_0 h_0\rho + |\Delta(A_i u^j)(a_1^{Au^1}(t),t)|$. However, $(a_1^{Au^1}(t),t)\in G_{\varepsilon_0}^{Au^1}\cap G_{\varepsilon_0}^{Au^2}$, the considered case.

Thus,

$$|\Delta(\overline{A_i u^j})(x,t)| \le \Big[p h_0 + f_0(1+p+\varepsilon_0 h_0 p) + F\frac{2}{\gamma}h_0 + (L_1 H + L_2 F)\frac{2}{\gamma}h_0 + L_2 e^{\lambda_0\varepsilon_0}\times$$
$$\times(r+\varepsilon_0 f_0 p)(H+\Lambda)\frac{2}{\gamma}h_0 + L_1 h_0 + L_2 e^{\lambda_0\varepsilon_0}(r h_0 + \varepsilon_0 f_0 p h_0) + L_2 f_0(1+p+\varepsilon_0 h_0 p)\Big]\varepsilon_0\rho,$$

and provided that ε_0 is small enough, the mapping $A\colon S \to S$ is contractive.

Fix p large and ε_0 small enough to satisfy conditions (17)–(20) as well as the inequality

$$\Big[ph_0 + f_0(1+p+\varepsilon_0 h_0 p) + F\frac{2}{\gamma}h_0 + \big(L_1H + L_2F\big)\frac{2}{\gamma}h_0 + L_2 e^{\lambda_0\varepsilon_0}(r + \varepsilon_0 f_0 p)\times$$
$$\times(H+\Lambda)\frac{2}{\gamma}h_0 + L_1h_0 + L_2e^{\lambda_0\varepsilon_0}(rh_0 + \varepsilon_0 f_0 p h_0) + L_2 f_0(1+p+\varepsilon_0 h_0 p)\Big]\varepsilon_0 < 1. \quad (21)$$

Then by the contractive mappings principle, we have the existence and uniqueness in S of a solution (u, a^u) to systems (10),(14), such that conditions (8),(9) hold.

As $u \in S$, u have the Lipschitz property with respect to x. Requiring that

$$(H+\Lambda)\varepsilon_0 \le \frac{a_2^0 - a_1^0}{2}, \quad (22)$$

for all $(x,t_1),(x,t_2) \in G^u_{\varepsilon_0}$, $t_1 < t_2$ and every fixed i we can find either a point $(x_1,t_1) \in G^u_{\varepsilon_0}$ such that $x_1 = \varphi_i(t_1;x,t_2)$, or a point $(x_1,t_2) \in G^u_{\varepsilon_0}$ such that $x = \varphi_i(t_1;x_1,t_2)$. In the first case we have $|u_i(x,t_1) - u_i(x,t_2)| \le |u_i(x,t_1) - u_i(x_1,t_1)| + |u_i(x_1,t_1) - u_i(x,t_2)| \le p|x-x_1| + F|t_1-t_2| \le (p\Lambda + F)|t_1 - t_2|$ and the same result in the second one. So u is also Lipschitz with respect to t with constant $p\Lambda + F$, i.e. $u \in S$ is a generalized solution of problem (1)-(4),(6).

On the other hand, each generalized solution of (1)-(4),(6) belongs to some set $S_{\overline{\varepsilon}\,\overline{\alpha}\,\overline{\beta}\,\overline{p}}$. Furthermore, we can reduce $\overline{\varepsilon}$ so that $\overline{\varepsilon} \le \varepsilon_0$, $\overline{\alpha} \le \alpha_0$, $\overline{\beta} \le \beta_0$, $\overline{p} \le p$, and then reducing ε_0 to $\overline{\varepsilon}$ yields $S_{\overline{\varepsilon}\,\overline{\alpha}\,\overline{\beta}\,\overline{p}} \subset S_{\varepsilon_0\alpha_0\beta_0 p}$. By this, the local uniqueness of a generalized solution is proved as well. □

4. Global solvability. Make some additional assumptions. Let $|h_k(t,\zeta,\omega)| \le H$, $k \in \{1,2\}$, in $[0,T] \times \mathbb{R}^{2+2n}$; $\lambda_i(x,t), i \in \{1,\dots,n\}$, be nondecreasing with respect to x; $h_1(t,\zeta,\omega) \le 0$, $h_2(t,\zeta,\omega) \ge 0$ in $[0,T]\times\mathbb{R}^{2+2n}$. Assume that there exist a summable function $M\colon [0,T] \to [0,\infty)$ and a continuous nondecreasing function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\operatorname{sgn}\psi(\alpha) = \operatorname{sgn}\alpha$, for all $\delta > 0$ $\int\limits_\delta^\infty \frac{d\alpha}{\psi(\alpha)} = \infty$, and in $\Omega^T \times \mathbb{R}^n$ the following inequality

$$|f_i(x,t,u)| \le M(t)\psi(\|u\|), \quad i \in \{1,\dots,n\},$$

holds, where $\Omega^T = \{(x,t) | 0 \le t \le T,\ a_1^0 - Ht \le x \le a_2^0 + Ht\}$, $\|u\| = \max\limits_{1\le i\le n} |u_i|$.

Let also

$$h_k(t,\zeta,\omega) - \lambda_i(\zeta'_k, t) \ne 0, \quad i \in \{1,\dots,n\},\ k \in \{1,2\}, \quad (23)$$
$$\det B(\zeta,t) \ne 0,$$

when $0 \le t \le T$, $a_1^0 - (H+\alpha_0)t \le \zeta_1, \zeta'_1 \le a_1^0$, $a_2^0 \le \zeta_2, \zeta'_2 \le a_2^0 + (H+\alpha_0)t$, $\|\omega\| \le P_0$, with $\alpha_0 > 0$ being any fixed constant, and where P_0 will be defined below.

For convenience set $\Omega^T_\zeta = \{(\zeta,t)|\, 0 \le t \le T,\ a_1^0 - Ht \le \zeta_1, \zeta_2 \le a_2^0 + Ht\}$,

$$\tilde q = \frac{2}{\gamma}L_2(H+\Lambda), \quad q = \max_{(\zeta,t)\in\Omega^T_\zeta}\left|\frac{1}{\det B(\zeta,t)}\right| N \max_{\substack{1\le i\le N, k\in\{1,2\}\\ j\in I_k, (\zeta,t)\in\Omega^T_\zeta}} |B_{ijk}(\zeta,t)|,$$

$$\hat b = 2n \max_{\substack{1\le i\le N,\, 1\le j\le n\\ k\in\{1,2\},\, (\zeta,t)\in\Omega^T_\zeta}} |b_{ijk}(\zeta,t)|, \quad \hat H = \max_{\substack{1\le i\le N\\ (\zeta,t)\in\Omega^T_\zeta}} |H_i(\zeta,t)|.$$

Theorem 2. *Under the conditions of Theorem 1, the assumptions of this section, and the requirements $\tilde{q} < 1$ and $q\hat{b} < 1$, problem (1)-(4),(6) has a unique generalized solution defined for $t \in [0, T]$ where T is arbitrarily large.*

Proof. If u is a generalized solution of problem (1)-(4), (6), then by (13) we obtain

$$|W_l^u(a_k^u(t), t)| \leq G + \int_0^t M(\tau)\psi(\max_{\substack{0\leq\theta\leq\tau \\ a_1^0 - H\theta \leq \xi \leq a_2^0 + H\theta}} \|\overline{u}(\xi,\theta)\|)\, d\tau, \ k \in \{1,2\},\ l \notin I_k$$

if $\chi_l\big(a_k^u(t), t; u\big) = 0$, and

$$|W_l^u(a_k^u(t), t)| \leq \max_{\substack{i\in I_1,\, j\in I_2 \\ 0\leq\tau\leq t}} \big\{|\mu_i^u(\tau)|, |\nu_j^u(\tau)|\big\} + \int_0^t M(\tau)\psi(\max_{\substack{0\leq\theta\leq\tau \\ a_1^0 - H\theta \leq \xi \leq a_2^0 + H\theta}} \|\overline{u}(\xi,\theta)\|)\, d\tau$$

if $\chi_l\big(a_k^u(t), t; u\big) > 0$. But from (11),(12) it may be concluded that

$$\max_{i\in I_1,\, j\in I_2} \big\{|\mu_i^u(t)|, |\nu_j^u(t)|\big\} \leq q\big(\hat{H} + \hat{b} \max_{k\in\{1,2\},\, l\notin I_k} |W_l^u(a_k^u(t), t)|\big).$$

Consequently, for $\chi_l\big(a_k^u(t), t; u\big) > 0$ we find that

$$\max_{i\in I_1,\, j\in I_2} \big\{|\mu_i^u(t)|, |\nu_j^u(t)|\big\} \leq q\hat{H} + q\hat{b} \max_{\substack{i\in I_1,\, j\in I_2 \\ 0\leq\tau\leq t}} \big\{|\mu_i^u(\tau)|, |\nu_j^u(\tau)|\big\} + q\hat{b}\int_0^t M(\tau)\psi(\max_{\substack{0\leq\theta\leq\tau \\ a_1^0 - H\theta \leq \xi \leq a_2^0 + H\theta}} \|\overline{u}(\xi,\theta)\|)\, d\tau,$$

from which it follows that

$$\max_{\substack{i\in I_1,\, j\in I_2 \\ 0\leq\tau\leq t}} \big\{|\mu_i^u(\tau)|, |\nu_j^u(\tau)|\big\} \leq \frac{q\hat{H}}{1 - q\hat{b}} + \frac{q\hat{b}}{1 - q\hat{b}} \int_0^t M(\tau)\psi(\max_{\substack{0\leq\theta\leq\tau \\ a_1^0 - H\theta \leq \xi \leq a_2^0 + H\theta}} \|\overline{u}(\xi,\theta)\|)\, d\tau.$$

Likewise, for $\chi_l\big(a_k^u(t), t; u\big) = 0$ we find that

$$\max_{i\in I_1,\, j\in I_2} \big\{|\mu_i^u(t)|, |\nu_j^u(t)|\big\} \leq q\hat{H} + q\hat{b}G + q\hat{b}\int_0^t M(\tau)\psi(\max_{\substack{0\leq\theta\leq\tau \\ a_1^0 - H\theta \leq \xi \leq a_2^0 + H\theta}} \|\overline{u}(\xi,\theta)\|)\, d\tau.$$

Thus,

$$\|u(x,t)\| \leq \max\big\{G,\ \max_{\substack{i\in I_1 \\ 0\leq\tau\leq t}} |\mu_i^u(\tau)|,\ \max_{\substack{j\in I_2 \\ 0\leq\tau\leq t}} |\nu_j^u(\tau)|\big\} + \int_0^t M(\tau)\psi(\max_{\substack{0\leq\theta\leq\tau \\ a_1^0 - H\theta \leq \xi \leq a_2^0 + H\theta}} \|\overline{u}(\xi,\theta)\|)\, d\tau \leq$$

$$\leq \max\Big\{G,\ \frac{q\hat{H}}{1 - q\hat{b}} + q\hat{b}G\Big\} + \Big(\frac{q\hat{b}}{1 - q\hat{b}} + 1\Big)\int_0^t M(\tau)\psi(\max_{\substack{0\leq\theta\leq\tau \\ a_1^0 - H\theta \leq \xi \leq a_2^0 + H\theta}} \|\overline{u}(\xi,\theta)\|)\, d\tau \leq$$

$$\leq \frac{q\hat{H}}{1 - q\hat{b}} + G + \frac{1}{1 - q\hat{b}} \int_0^t M(\tau)\psi(\max_{\substack{0\leq\theta\leq\tau \\ a_1^0 - H\theta \leq \xi \leq a_2^0 + H\theta}} \|\overline{u}(\xi,\theta)\|)\, d\tau.$$

Applying the estimate above we can assert that $\|u(x,t)\|$ does not exceed a constant $\varkappa$ defined by the equation

$$\int\limits_{\frac{q\hat{H}}{1-q\hat{b}}+G}^{\varkappa} \frac{d\alpha}{\psi(\alpha)} = \frac{1}{1-q\hat{b}} \int\limits_0^T M(\tau)\,d\tau.$$

Fix $P_0 > \varkappa$ and define $\Omega^T_{\alpha_0} = \{(x,t)|\, 0 \le t \le T,\, a_1^0 - (H+\alpha_0)t \le x \le a_2^0 + (H+\alpha_0)t\}$, $\Omega^T_{\zeta\alpha_0} = \{(\zeta,t)|\, 0 \le t \le T,\, a_1^0 - (H+\alpha_0)t \le \zeta_1, \zeta_2 \le a_2^0 + (H+\alpha_0)t\}$. Let $(x,t,u) \in \Omega^T_{\alpha_0} \times \{u : \|u\| \le P_0\}$, then from the continuity of the corresponding functions it follows that $|f_i(x,t,u)| \le F$, $|\lambda_i(x,t)| \le \Lambda$, $i \in \{1,\dots,n\}$, and by local Lipschitzness we conclude that $f_i(x,t,u), i \in \{1,\dots,n\}$ are Lipschitz with respect to x, u, and so are $\lambda_i(x,t), i \in \{1,\dots,n\}$, with respect to x with some constants which we denote by f_0 and λ_0 respectively. In a similar way, if $(t,\zeta,\omega) \in \Omega^T_{\zeta\alpha_0} \times \{\omega : \|\omega\| \le P_0\}$, then $h_k(t,\zeta,\omega), k \in \{1,2\}$, have the Lipschitz property with respect to all variables with some constant h_0. Note that F,Λ,f_0,λ_0 and h_0 are fixed values differing from the same constants defined in section 3.

Introduce a subspace $\tilde{S}$ of the space S by imposing on $a_k^u, k \in \{1,2\}$, some additional requirements of monotonicity. More precisely, we require that $a_1^u(t)$ be a nonincreasing function and $a_2^u(t)$ be a nondecreasing one. Taking into account the sign constancy of h_k, we see that the action of the operator A does not violate these properties. From now on, suppose that $u \in \tilde{S}$.

Determine conditions sufficient for $a_k^{Au}(t)$ to satisfy I.

$$\left|\frac{d}{dt}\, a_k^{Au}(t) - h_k\big(0,a^0,g(a^0)\big)\right| = |h_k\big(t,a^u(t),u(a^u(t),t)\big) - h_k\big(0,a^0,g(a^0)\big)| \le$$
$$\le h_0 \max_{k',i}\big\{t, |a_{k'}^u(t) - a_{k'}^0|, |u_i(a_{k'}^u(t),t) - g_i(a_{k'}^0)|\big\}.$$

By property I, we have

$$|a_{k'}^u(t) - a_{k'}^0| \le (|h_{k'}(0,a^0,g(a^0))| + \alpha_0)t \le (H+\alpha_0)\varepsilon_0,\ k' \in \{1,2\}.$$

Property II implies

$$|u_i(a_{k'}^u(t),t) - g_i(a_{k'}^0)| = |u_i(a_{k'}^u(t),t) - \overline{g}_i(a_{k'}^u(t))| \le \beta_0,\ i \in \{1,\dots,n\},\ k' \in \{1,2\}.$$

Thus, $\left|\frac{d}{dt}\, a_k^{Au}(t) - h_k\big(0,a^0,g(a^0)\big)\right| \le h_0 \max\big\{\varepsilon_0, (H+\alpha_0)\varepsilon_0, \beta_0\big\}$.

Fix $\beta_0 = \min\left\{\dfrac{\alpha_0}{h_0},\ P_0 - \varkappa\right\}$. Then from the condition

$$\varepsilon_0 \le \min\left\{\beta_0, \frac{\beta_0}{H+\alpha_0}\right\} \tag{24}$$

we get the inequality $\left|\frac{d}{dt}\, a_k^{Au}(t) - h_k\big(0,a^0,g(a^0)\big)\right| \le \alpha_0$, which, in its turn, ensures the Lipschitzness of the mapping $t \mapsto a_k^{Au}(t) - h_k\big(0,a^0,g(a^0)\big)t$.

Note that by (23) it follows that

$$h_k\big(t,a^u(t),u(a^u(t),t)\big) \ne \lambda_i(a_k^{Au}(t),t),\ k \in \{1,2\},\ i \in \{1,\dots,n\},\ 0 \le t \le \varepsilon_0,$$

and so, the operator A is well defined. Define γ and κ by the inequalities

$$|h_k(t,\zeta,\omega)-\lambda_i(\zeta'_k,t)|\geq\gamma>0,\ k\in\{1,2\},\ i\in I_k,\\ |\det B(\zeta,t)|\geq\kappa>0, \tag{25}$$

when $0\leq t\leq T,\ a_1^0-(H+\alpha_0)t\leq\zeta_1,\zeta'_1\leq a_1^0,\ a_2^0\leq\zeta_2,\zeta'_2\leq a_2^0+(H+\alpha_0)t,\ \|\omega\|\leq P_0$. The existence of such constants follows from the continuity of $h_k, k\in\{1,2\}$, $\lambda_i, i\in\{1,\dots,n\}$, and $b_{ijk}, i\in\{1,\dots,N\}, j\in\{1,\dots,n\}, k\in\{1,2\}$.

If $(\zeta,t)\in\Omega^T_{\zeta\alpha_0}$, reasoning in a similar manner as before, we conclude that $|b_{ijk}(\zeta,t)|\leq b$, $|H_i(\zeta,t)|\leq\tilde{H}$, $i\in\{1,\dots,N\}, j\in\{1,\dots,n\}, k\in\{1,2\}$, $|B_{ijk}(\zeta,t)|\leq B$, $i\in\{1,\dots,N\}$, $k\in\{1,2\}, j\in I_k$, and $b_{ijk}(\zeta,t)$, $H_i(\zeta,t)$ are Lipschitz with respect to all arguments with some constants, which we denote by b_0 and H_0 respectively. From this and by (25), we also have the Lipschitzness of $B_{ijk}(\zeta,t)$ and $\big(\det B(\zeta,t)\big)^{-1}$ with some common constant B_0.

Observe that the monotonicity of $\lambda_i(x,t)$ with respect to x gives the inequality

$$|\varphi_i(\tau;x_2,t)-\varphi_i(\tau;x_1,t)|\leq|x_2-x_1|,$$

and therefore, in the conditions for the local solvability of our problem we can let $e^{\lambda_0\varepsilon_0}=1$, i.e. we get conditions (17)-(19),(22),(24), with the other two being replaced by the inequalities

$$\varepsilon_0 f_0 p+\max\big\{r,(F+\mathrm{L}(r,p))\frac{2}{\gamma}\big\}\leq p,\quad p\geq 1,$$

where $\mathrm{L}(r,p)=L_1H+L_2\Big[F+(r+\varepsilon_0 f_0 p)(H+\Lambda)\Big]$, and

$$\Big[ph_0+f_0(1+p+\varepsilon_0h_0p)+(F+L_1H+L_2F)\frac{2}{\gamma}h_0+L_2(r+\varepsilon_0f_0p)(H+\Lambda)\frac{2}{\gamma}h_0+\\ +L_1h_0+L_2(rh_0+\varepsilon_0f_0ph_0)+L_2f_0(1+p+\varepsilon_0h_0p)\Big]\varepsilon_0<1.$$

Here and subsequently L_1 and L_2 denote the same constants as in the preceding section, but with $\varkappa$ in place of G.

After constructing the solution for $0\leq t\leq\varepsilon_1=\varepsilon_0$ we take as initial data the values of the functions $a_k^u(t)$ and $u_i(x,t)$ at $t=\varepsilon_1$. We thus arrive at the new problem

$$\frac{\partial v_i}{\partial t}+\lambda_i(x,t)\frac{\partial v_i}{\partial x}=f_i(x,t,v),\quad i\in\{1,\dots,n\}, \tag{26}$$

$$\frac{da_k^v}{dt}=h_k\big(t,a^v(t),v(a^v(t),t)\big),\quad k\in\{1,2\}, \tag{27}$$

$$a_k^v(0)=a_k^u(\varepsilon_1),\quad k\in\{1,2\}, \tag{28}$$

$$v_i(x,0)=u_i(x,\varepsilon_1),\quad a_1^u(\varepsilon_1)\leq x\leq a_2^u(\varepsilon_1),\ i\in\{1,\dots,n\}, \tag{29}$$

$$\sum_{k=1}^2\sum_{j=1}^n b_{ijk}(a^v(t),t)v_j(a_k^v(t),t)=H_i(a^v(t),t),\quad i\in\{1,\dots,N'\},\ N'=\sum_{k=1}^2\operatorname{card}I'_k, \tag{30}$$

where $I'_1=\big\{i:\ \lambda_i(a_1^u(\varepsilon_1),\varepsilon_1)>h_1\big(\varepsilon_1,a^u(\varepsilon_1),u(a^u(\varepsilon_1),\varepsilon_1)\big)\big\}$, $I'_2=\big\{i:\ \lambda_i(a_2^u(\varepsilon_1),\varepsilon_1)<h_2\big(\varepsilon_1,a^u(\varepsilon_1),u(a^u(\varepsilon_1),\varepsilon_1)\big)\big\}$. From (23) it follows that $I'_1=I_1$, $I'_2=I_2$.

Fix α_0 and β_0 at the same level as in the case $0 \le t \le \varepsilon_1$, and consider the metric space $S_{\varepsilon_2\,\alpha_0\,\beta_0\,p_2}$. Then we obtain the conditions on ε_2, p_2 for existence of a local solution to problem (26)-(30) when $\varepsilon_1 \le t \le \varepsilon_1 + \varepsilon_2$, which are determined as above but with ε_2,p_1 and p_2 instead of ε, r and p respectively:

$$\big[\Lambda + 2H\big]\varepsilon_2 \le a_2^u(\varepsilon_1) - a_1^u(\varepsilon_1), \tag{31}$$

$$\Big[F + \mathrm{L}(p_1,p_2) + p_1 \max\{\Lambda, H\}\Big]\varepsilon_2 \le \beta_0, \quad (H+\Lambda)\varepsilon_2 \le \frac{\gamma}{2\lambda_0},$$

$$\varepsilon_2 f_0 p_2 + \max\{p_1, (F + \mathrm{L}(p_1,p_2))\frac{2}{\gamma}\} \le p_2, \quad p_2 \ge 1, \tag{32}$$

$$\Big[p_2 h_0 + f_0(1 + p_2 + \varepsilon_2 h_0 p_2) + (F + L_1 H + L_2 F)\frac{2}{\gamma}h_0 + L_2(p_1 + \varepsilon_2 f_0 p_2)(H+\Lambda)\frac{2}{\gamma}h_0 +$$

$$+ L_1 h_0 + L_2(p_1 h_0 + \varepsilon_2 f_0 p_2 h_0) + L_2 f_0(1 + p_2 + \varepsilon_2 h_0 p_2)\Big]\varepsilon_2 < 1,$$

$$(H+\Lambda)\varepsilon_2 \le \frac{a_2^u(\varepsilon_1) - a_1^u(\varepsilon_1)}{2}, \quad \varepsilon_2 \le \min\left\{\beta_0, \frac{\beta_0}{H+\alpha_0}\right\}. \tag{33}$$

By the monotonicity of $a_k^u(t)$ we have $a_2^0 - a_1^0 \le a_2^u(\varepsilon_1) - a_1^u(\varepsilon_1)$, and hence, inequalities (31), (33) can be replaced by the following ones $\big[\Lambda + 2H\big]\varepsilon_2 \le a_2^0 - a_1^0$, $(H+\Lambda)\varepsilon_2 \le \frac{a_2^0 - a_1^0}{2}$. Now consider (32). After substituting the expression for $\mathrm{L}(p_1,p_2)$ we obtain the inequality

$$\varepsilon_2 f_0 p_2 + \max\left\{p_1, \Big[F + L_1 H + L_2 F + L_2(p_1 + \varepsilon_2 f_0 p_2)(H+\Lambda)\Big]\frac{2}{\gamma}\right\} \le p_2,$$

which, by the assumption $\tilde{q} = \dfrac{2}{\gamma}L_2(H+\Lambda) < 1$, can be strengthened as follows:

$$p_1 \ge \big[F + L_1 H + L_2 F + L_2(H+\Lambda)\big]\frac{2}{\gamma}(1-\tilde{q})^{-1}, \quad \varepsilon_2 f_0 p_2 \le 1, \quad 1 + p_1 = p_2.$$

Similarly, after finding the solution for $0 \le t \le \sum_{i=1}^{\sigma-1}\varepsilon_i$ we consider the metric space $S_{\varepsilon_\sigma\,\alpha_0\,\beta_0\,p_\sigma}$ in order to extend this solution to $\sum_{i=1}^{\sigma-1}\varepsilon_i \le t \le \sum_{i=1}^{\sigma}\varepsilon_i$, with values ε_σ, p_σ satisfying the conditions

$$\big[\Lambda + 2H\big]\varepsilon_\sigma \le a_2^0 - a_1^0, \qquad (H+\Lambda)\varepsilon_\sigma \le \frac{\gamma}{2\lambda_0}, \tag{34}$$

$$\Big[F + L_1 H + L_2 F + L_2(p_{\sigma-1}+1)(H+\Lambda) + p_{\sigma-1}\max\{\Lambda, H\}\Big]\varepsilon_\sigma \le \beta_0, \tag{35}$$

$$p_{\sigma-1} \ge \big[F + L_1 H + L_2 F + L_2(H+\Lambda)\big]\frac{2}{\gamma}(1-\tilde{q})^{-1}, \tag{36}$$

$$\varepsilon_\sigma f_0 p_\sigma \le 1, \qquad p_\sigma \ge 1, \qquad 1 + p_{\sigma-1} = p_\sigma, \tag{37}$$

$$\Big[p_\sigma h_0 + f_0(1 + p_\sigma) + h_0 + (F + L_1 H + L_2 F)\frac{2}{\gamma}h_0 + \tilde{q}h_0 p_{\sigma-1} +$$

$$+ \tilde{q}h_0 + L_1 h_0 + L_2 h_0 p_{\sigma-1} + 2L_2 h_0 + L_2 f_0(1 + p_\sigma)\Big]\varepsilon_\sigma < 1, \tag{38}$$

$$(H+\Lambda)\varepsilon_\sigma \le \frac{a_2^0 - a_1^0}{2}, \qquad \varepsilon_\sigma \le \min\left\{\beta_0, \frac{\beta_0}{H+\alpha_0}\right\}. \tag{39}$$

In order for (34)-(39) to be satisfied it is sufficient to require that

$$\big[\Lambda + 2H\big]\varepsilon_\sigma \le a_2^0 - a_1^0, \qquad (H+\Lambda)\varepsilon_\sigma \le \frac{\gamma}{2\lambda_0},$$
$$\Big[F + L_1 H + L_2 F + L_2(H+\Lambda)\Big]\varepsilon_\sigma \le \beta_0/2, \quad \Big[L_2(H+\Lambda) + \max\{\Lambda, H\}\Big]p_{\sigma-1}\varepsilon_\sigma \le \beta_0/2,$$
$$p_{\sigma-1} \ge \frac{\big[F + L_1 H + L_2 F + L_2(H+\Lambda)\big]\frac{2}{\gamma}}{1-\tilde{q}}, \quad f_0 p_\sigma \varepsilon_\sigma \le 1, \quad p_\sigma \ge 1, \quad 1 + p_{\sigma-1} = p_\sigma,$$
$$\Big[f_0 + h_0 + (F + L_1 H + L_2 F)\frac{2}{\gamma}h_0 + \tilde{q}h_0 + L_1 h_0 + L_2(2h_0 + f_0)\Big]\varepsilon_\sigma \le 1/4,$$
$$\Big[h_0 + f_0 + L_2 f_0\Big]p_\sigma\varepsilon_\sigma \le 1/3, \qquad \Big[\tilde{q}h_0 + L_2 h_0\Big]p_{\sigma-1}\varepsilon_\sigma \le 1/3,$$
$$(H+\Lambda)\varepsilon_\sigma \le \frac{a_2^0 - a_1^0}{2}, \qquad \varepsilon_\sigma \le \beta_0, \qquad \varepsilon_\sigma \le \frac{\beta_0}{H+\alpha_0}.$$

Rewrite the restrictions on ε_σ as follows:

$$\varepsilon_\sigma \le \frac{a_2^0 - a_1^0}{\Lambda + 2H}, \quad \varepsilon_\sigma \le \frac{\gamma}{2\lambda_0(H+\Lambda)}, \quad \varepsilon_\sigma \le \frac{\beta_0}{2}\Big[F + L_1 H + L_2 F + L_2(H+\Lambda)\Big]^{-1},$$
$$\varepsilon_\sigma \le \frac{\beta_0}{2p_{\sigma-1}}\Big[L_2(H+\Lambda) + \max\{\Lambda, H\}\Big]^{-1}, \quad \varepsilon_\sigma \le \frac{1}{f_0 p_\sigma},$$
$$\varepsilon_\sigma \le \frac{1}{4}\Big[f_0 + h_0 + (F + L_1 H + L_2 F)\frac{2}{\gamma}h_0 + \tilde{q}h_0 + L_1 h_0 + L_2(2h_0 + f_0)\Big]^{-1},$$
$$\varepsilon_\sigma \le \frac{1}{3p_\sigma}\Big[h_0 + f_0 + L_2 f_0\Big]^{-1}, \quad \varepsilon_\sigma \le \frac{1}{3p_{\sigma-1}}\Big[\tilde{q}h_0 + L_2 h_0\Big]^{-1},$$
$$\varepsilon_\sigma \le \frac{a_2^0 - a_1^0}{2(H+\Lambda)}, \quad \varepsilon_\sigma \le \beta_0, \quad \varepsilon_\sigma \le \frac{\beta_0}{H+\alpha_0},$$

which are imposed together with the requirements

$$p_\sigma = 1 + p_{\sigma-1}, \quad p_{\sigma-1} \ge \big[F + L_1 H + L_2 F + L_2(H+\Lambda)\big]\frac{2}{\gamma}(1-\tilde{q})^{-1}, \quad p_\sigma \ge 1. \tag{40}$$

From the recursion for p_σ we obtain the general equality $p_\sigma = \sigma + r$. Setting the constant r be so large that $r \ge \big[F + L_1 H + L_2 F + L_2(H+\Lambda)\big]\frac{2}{\gamma}(1-\tilde{q})^{-1}$, we guarantee satisfaction of the last two inequalities in (40) for all $\sigma \in \mathbb{N}$.

Now introduce the notation

$$\delta = \min\Bigg\{\frac{a_2^0 - a_1^0}{\Lambda + 2H}, \ \frac{\gamma}{2\lambda_0(H+\Lambda)}, \ \frac{1}{f_0}, \ \frac{a_2^0 - a_1^0}{2(H+\Lambda)}, \ \beta_0, \ \frac{\beta_0}{H+\alpha_0},$$
$$\frac{\beta_0}{2}\Big[F + L_1 H + L_2 F + L_2(H+\Lambda)\Big]^{-1}, \ \frac{\beta_0}{2}\Big[L_2(H+\Lambda) + \max\{\Lambda, H\}\Big]^{-1},$$
$$\frac{1}{4}\Bigg[f_0 + h_0 + (F + L_1 H + L_2 F)\frac{2}{\gamma}h_0 + \tilde{q}h_0 + L_1 h_0 + L_2(2h_0 + f_0)\Bigg]^{-1},$$
$$\frac{1}{3}\Big[h_0 + f_0 + L_2 f_0\Big]^{-1}, \ \frac{1}{3}\Big[\tilde{q}h_0 + L_2 h_0\Big]^{-1}\Bigg\}.$$

Then by the inequalities $p_\sigma > 1$, $p_\sigma > p_{\sigma-1}, \sigma \in \mathbb{N}$, where $p_0 = r$, we can assert that under $\varepsilon_\sigma = \dfrac{\delta}{p_\sigma}$ all the conditions on ε_σ are satisfied. However, $\sum_{\sigma=1}^{\infty} \varepsilon_\sigma = \sum_{\sigma=1}^{\infty} \frac{\delta}{p_\sigma} = \sum_{\sigma=1}^{\infty} \frac{\delta}{\sigma+r} = \infty$, but the value T is finite. Therefore, in a finite number of steps we reach the level $t = T$, which shows that the existence of a generalized solution of problem (1)-(4),(6) can be guaranteed for every time interval $[0, T]$.

Finally, suppose that the obtained solution is not unique. Transfer a zero-time reference into $\bar{t}$ defined as the greatest lower bound of the points t at which there is the violation of uniqueness. Under α_0, β_0 fixed above, by Theorem 1, we find $\bar{\varepsilon}$, $\bar{p}$ such that there exists a unique solution of our problem in the space $S_{\bar{\varepsilon}\,\alpha_0\,\beta_0\,\bar{p}}$ for $\bar{t} \le t \le \bar{t} + \bar{\varepsilon}$. By choosing $\bar{\varepsilon}$ and $\bar{p}$ we can ensure belonging of both our solutions to the space $S_{\bar{\varepsilon}\,\alpha_0\,\beta_0\,\bar{p}}$. This contradiction completes the proof. □

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Received 11.05.2004

Revised 14.12.2004