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POLYNOMIAL ASYMPTOTIC REPRESENTATIONS OF SUBHARMONIC FUNCTIONS WITH MASSES ON ONE RAY IN THE SPACE

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In the paper, a polynomial asymptotic representation of a subharmonic function in the space is obtained by a given polynomial asymptotics of Riesz measure of this function under the hypothesis of the concentration of the Riesz masses on one ray.

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В работе получено многочленное асимптотическое представление субгармонической в пространстве функции по заданной многочленной асимптотике меры Рисса при условии, что массы Рисса этой функции сосредоточены на одном луче.

The relation between polynomial asymptotic representations of subharmonic functions in the plane and distribution functions of their Riesz measures was rather intensively studied during the last decades. It was established [1] that there are many distinctions between this case and the case of functions of completely regular growth (one-term asymptotic representations). Note that subharmonic (entire) functions of several variables of completely regular growth were adequately investigated in the papers of many mathematicians (see the extensive bibliography in [5]). As in the plane, for the study of the functions of completely regular growth in the space various methods that used the subharmonicity of the first term of the asymptotics were applied. But it was shown in [1] that in the case of the plane the second and the next terms of asymptotics are not in general subharmonic functions. This circumstance causes the essential differences between the cases of one-term and n-term asymptotics. So all traditional investigation methods of the theory of functions of completely regular growth are not applicable. Some other tools must be used for functions in the plane, for example, the theory of singular integrals.

In this paper we consider subharmonic functions in $\mathbb{R}^m, m \geq 3$, the Riesz measures of which are concentrated only on one ray. It is the first attempt of the study of polynomial asymptotic representations for subharmonic functions in the space $\mathbb{R}^m,\ m \geq 3$. As for the plane, in this article the main tools of the investigation is the theory of singular integrals.

First of all we give the notions and notations that will be used later on .

Let us introduce the spherical coordinates in $\mathbb{R}^m,\ y \in \mathbb{R}^m, y = (t, \alpha_1, \dots, \alpha_{m-1})$:

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$$y_1 = t\sin\alpha_1\ldots\sin\alpha_{m-1};$$

$$y_2 = t\cos\alpha_1\ldots\sin\alpha_{m-1};$$

$$\ldots\ldots\ldots\ldots$$

$$y_m = t\cos\alpha_{m-1}.$$

S_{m-1} is the unit sphere with the center at the origin in $\mathbb{R}^m$.

Given a function $f(t), t>0$, we will say that f has *multi-term (polynomial, n-term) asymptotics* if it can be represented in the following way:

$$f(t) = \Delta_1 t^{\rho_1} + \Delta_2 t^{\rho_2} + \ldots + \Delta_n t^{\rho_n} + \kappa(t),\ t\to\infty,$$

where $\Delta_j, j\in\{1,2,\ldots,n\}$, are real constants; $0<[\rho_1]<\rho_n<\ldots<\rho_1$ (here $[a]$ denotes the integral part of a), and the function $\kappa(t)$ on the right is small in a certain sense compared to the previous term.

Similarly, we will understand the expression "polynomial asymptotics of a function $f(y)$, $y\to\infty$, $y\in\mathbb{R}^m$ ". In this case $t=|y|$ and the coefficients Δ_j are the functions of the angle coordinates only and do not depend on t.

$E\in\mathbb{R}^m$ is $C_{0,\alpha}$*-set* if it can be covered by the system of disks $K_j(x_j,r_j)=\{x: |x-x_j|<r_j\},\ j\in\{1,2,\ldots\}$, such that

$$\lim_{R\to\infty}\frac{1}{R^\alpha}\sum_{|x_j|<R} r_j^\alpha = 0.$$

Put

$$H_m(v,\gamma,p) = -(1-2v\cos\gamma+v^2)^{-\frac{m-2}{2}} + \sum_{j=0}^{p} B_j^m(\gamma)v^j,$$

where B_j^m are coefficients of the expansion in powers of v of the expression $(1-2v\cos\gamma+v^2)^{-\frac{m-2}{2}}$. As it was shown in [6], the coefficients B_j^m satisfy the inequality

$$|B_j^m| \le e^{m-3}j^{m-3} \tag{1}$$

and

$$t^{2-m}H_m\left(\frac{|x|}{t},\gamma,p\right) \le \begin{cases} C(m,p)\frac{|x|^p}{t^{m-2+p}}, & t\le 2|x|,\\ C(m,p)\frac{|x|^{p+1}}{t^{m-1+p}}, & t>2|x|.\end{cases}\quad t\ne 0. \tag{2}$$

Moreover, it is easy to see that for any fix $\eta>0$

$$\left|\frac{d}{dt}\left\{\frac{1}{t^{m-2}}H_m\left(\frac{|x|}{t},\gamma,p\right)\right\}\right| \le C(\eta,m)\frac{|x|^p}{t^{p+m-1}} \tag{3}$$

as long as $t\in(0,2|x|)$ and $\gamma\ge\eta$. If $t\in[2|x|,\infty)$ then

$$H_m\left(\frac{|x|}{t},\gamma,p\right) = \sum_{p+1}^{\infty} B_n^m(\gamma)\frac{|x|^n}{t^n} \tag{4}$$

and
$$\left|\frac{d}{dt}\left\{\frac{1}{t^{m-2}}H_m\left(\frac{|x|}{t},\gamma,p\right)\right\}\right| = \left|\sum_{p+1}^{\infty} B_n^m(\gamma)\frac{(n+m-2)|x|^n}{t^{n+m-1}}\right| \le C(m)\frac{|x|^{p+1}}{t^{p+m}}. \qquad (5)$$

Here the constants depend on the parameters which are listed in the brackets.

Let μ be the Riesz measure of a subharmonic function $u(x)$.

In this paper we will investigate the case of two-term asymptotic representations ($n=2$). This condition does not restrict the generality of our problem. It only makes the consideration easier.

Moreover without loss of generality we will consider the case of the space $\mathbb{R}^3$.

Theorem 1. *Let $u(x) \equiv u(r,\theta_1,\theta_2)$ be a subharmonic function of non-integer order in the space $\mathbb{R}^3 = \{y : y = (t,\alpha_1,\alpha_2)\}$, with Riesz's masses concentrated on the ray $\{\alpha_2 = 0\}$ outside some neighborhood of the origin $\{t < t_0\}, t_0 > 0$. Let for $t \ge t_0$*

$$\mu(t) := \mu(\{|x| < t\}) = \Delta_1 t^{\rho_1+1} + \Delta_2 t^{\rho_2+1} + \varphi(t), \qquad (6)$$

where $p = [\rho_1] < \rho_2 < \rho_1, \Delta_1 > 0$. Assume that for some $q \ge 1$ the function $\varphi(t)$ satisfies the following asymptotic estimate[1]

$$\int_T^{2T}\left|\frac{\varphi(t)}{t}\right|^q dt = o(T^{\rho_2 q+1}), \quad T \to \infty. \qquad (7)$$

Then the order of the function $u(x)$ is equal to ρ_1 and the following relation is true:

$$u(r,\theta_1,\theta_2) = \sum_{i=1}^{2}\Delta_i(\rho_i+1)r^{\rho_i}\int_{t_0}^{\infty}\zeta^{\rho_i-1}H(\frac{1}{\zeta},\theta_2,p)d\zeta + \psi(r,\theta_1,\theta_2).$$

Here the function $\psi(r,\theta_1,\theta_2) = o(r^{\rho_2})$ as $r \to \infty$, uniformly for $\overline{x} = \frac{x}{|x|}$ away from some $C_{0,2}$-set.

If in (7) $q > 1$ then

$$\int_{R\le r\le 2R,\theta_1\in[0,2\pi]} |\psi(r,\theta_1,\theta_2)|^q dr d\theta_1 = o(R^{\rho_2 q+1}), \quad R \to \infty,$$

uniformly for $\theta_2 \in [0,\pi]$.

Proof. In [4] it was shown that from (6) and (7) it is not difficult to obtain the following relation for the Riesz measure μ:

$$\mu(t) = \Delta_1 t^{\rho_1+1} + o(t^{\rho_1+1}), \quad t \to \infty.$$

Hence from the theorem of Riesz-Brelot [6] the order of the function $u(x)$ is ρ_1.

By the same theorem we see that the function $u(x)$ can be represented in the form

$$u(x) = J(x,p) + P(x),$$

[1] From [3], [8] and [4] it follows that it is natural to estimate the remainder term on the average in such kind of problems.

where the canonical potential is

$$J(x,p) = -\int\limits_0^\infty \left[\left(1 - 2\frac{|x|}{t}\cos\gamma + \frac{|x|^2}{t^2}\right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\gamma)\frac{|x|^n}{t^n} \right] \frac{d\mu(t)}{t},$$

γ is the angle between the vectors x and y, and $P(x)$ is a harmonic polynomial of the degree at most p.

Since $u(x)$ is the function of non-integer order, without loss of generality we can discard $P(x)$, and therefore

$$u(x) = J(x,p).$$

It is easy to see that $\cos\gamma = \cos\theta_2$, hence

$$u(x) = -\int\limits_0^\infty \left[\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2}\right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\theta_2)\frac{|x|^n}{t^n} \right] \frac{d\mu(t)}{t}.$$

Integrating by parts gives

$$u(x) = \left[-\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2}\right)^{-\frac{1}{2}} + \sum_0^p B_n^3(\theta_2)\frac{|x|^n}{t^n} \right] \frac{\mu(t)}{t}\Bigg|_0^\infty +$$

$$+\int\limits_0^\infty \mu(t)\frac{d}{dt}\left\{ \frac{1}{t}\left[\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2}\right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\theta_2)\frac{|x|^n}{t^n} \right] \right\} dt = I_1\Bigg|_0^\infty + I_2.$$

By the conditions of the theorem we have $\mu(t) \equiv 0$ in the neighborhood $\{t \le t_0\}$. Consequently $I_1\Big|_0 = 0$. By virtue of (2) the expression $I_1\Big|^\infty = 0$ and hence $I_1\Big|_0^\infty = 0$.

Next, taking into account expression (6) and using the assumption about the measure μ, we obtain

$$u(x) = \int\limits_0^\infty \mu(t)\frac{d}{dt}\left\{ \frac{1}{t}\left[\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2}\right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\theta_2)\frac{|x|^n}{t^n} \right] \right\} dt =$$

$$= \sum_{j=1}^2 \Delta_j \int\limits_{t_0}^\infty t^{\rho_1+1}\frac{d}{dt}\left\{ \frac{1}{t}\left[\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2}\right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\theta_2)\frac{|x|^n}{t^n} \right] \right\} dt +$$

$$+\int\limits_{t_0}^\infty \varphi(t)\frac{d}{dt}\left\{ \frac{1}{t}\left[\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2}\right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\theta_2)\frac{|x|^n}{t^n} \right] \right\} dt =$$

$$= \sum_{j=1}^2 \Gamma_j(x) + \psi(x),$$

where

$$\Gamma_j(x) = \Delta_j t^{\rho_j+1} \left\{ \frac{1}{t} \left[\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2} \right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\theta_2) \frac{|x|^n}{t^n} \right] \right\} \Bigg|_{t_0}^{\infty} -$$

$$-\Delta_j(\rho_j+1) \int_{t_0}^{\infty} t^{\rho_j-1} \left[\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2} \right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\theta_2) \frac{|x|^n}{t^n} \right] dt \tag{8}$$

and

$$\psi(x) = \int_{t_0}^{\infty} \varphi(t) \frac{d}{dt} \left\{ \frac{1}{t} \left[\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2} \right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\theta_2) \frac{|x|^n}{t^n} \right] \right\} dt. \tag{9}$$

From (2) and (8) we obtain the expressions for the main terms of the asymptotics of the function $u(x)$:

$$\Gamma_j = -\Delta_j(\rho_j+1)|x|^{\rho_j} \int_{t_0}^{\infty} s^{\rho_j-1} \left[\left(1 - 2\frac{1}{s}\cos\theta_2 + \frac{1}{s^2} \right)^{-\frac{1}{2}} - \sum_0^p B_n^3(\theta_2) \frac{1}{s^n} \right] ds =$$

$$= \Delta_j(\rho_j+1) r^{\rho_j} \int_{t_0}^{\infty} s^{\rho_j-1} H(\frac{1}{s}, \theta_2, p) ds.$$

Note that, as we know, V. Azarin [2] pioneered the establishment of the formula for Γ_1 (the formula for the indicator of subharmonic functions of completely regular growth).

Now we will estimate the remainder term $\psi(x)$. First of all we will consider this function far away from the ray where the measure of $u(x)$ is concentrated. Let the angle γ between the vector x and the ray $\{\alpha_2 = 0\}$ be greater than some $\eta > 0$. Then from (9) it follows that

$$|\psi(x)| \le \left(\int_0^{2|x|} + \int_{2|x|}^{\infty} \right) \left| \varphi(t) \frac{d}{dt} \left\{ \frac{1}{t} \left[\left(1 - 2\frac{|x|}{t}\cos\theta_2 + \frac{|x|^2}{t^2} \right)^{-\frac{1}{2}} - \right. \right. \right.$$

$$\left. \left. \left. - \sum_0^p B_n^3(\theta_2) \frac{|x|^n}{t^n} \right] \right\} \right| dt. \tag{10}$$

Using inequalities (3), (5) and estimate (7) we can obtain from (10) that

$$\psi(x) = o(|x|^{\rho_2}), \ |x| \to \infty \tag{11}$$

uniformly for $\frac{|x|}{x} \in S_2$.

Now let us estimate $\psi(x)$ near the ray where the measure of the function $u(x)$ is contained. To this end we decompose the ray $[1, \infty)$ into the intervals of the form $[2^k, 2^{k+1}), k \in \{0, 1, \dots\}$.

Assume that $x=(x_1,x_2,x_3), x_1=x_2=0,\ x_3\in[2^k,2^{k+1})$. Then from (9) we conclude that

$$\psi(x)=\int\limits_1^{2^{k-1}}+\int\limits_{2^{k-1}}^{2^{k+2}}+\int\limits_{2^{k+2}}^{\infty}\varphi(t)\frac{d}{dt}\left\{\frac{1}{t}\left[\left(1-2\frac{|x|}{t}\cos\theta_2+\frac{|x|^2}{t^2}\right)^{-\frac{1}{2}}-\sum_0^p B_n^3(\theta_2)\frac{|x|^n}{t^n}\right]\right\}dt=$$

$$=A_1^{(k)}+A_2^{(k)}+A_3^{(k)},$$

where the integral $A_2^{(k)}$ is understood in the sense of principal value for $x_3=t$.

Taking into account (5), we obtain

$$|x|^{-\rho_2}|A_3^{(k)}|\le|x|^{-\rho_2}\sum_{p+1}^{\infty}(n+1)|x|^n\int\limits_{2^{k+2}}^{\infty}\frac{|\varphi(t)|}{t}\frac{dt}{t^{n+1}}=$$

$$=|x|^{-\rho_2}\sum_{p+1}^{\infty}(n+1)|x|^n\sum_{j=k+2}^{\infty}\int\limits_{2^j}^{2^{j+1}}\frac{|\varphi(t)|}{t}\frac{dt}{t^{n+1}}\le$$

$$\le|x|^{-\rho_2}\sum_{p+1}^{\infty}(n+1)|x|^n\sum_{j=k+2}^{\infty}\frac{1}{2^{j(n+1)}}\left(\int\limits_{2^j}^{2^{j+1}}\left|\frac{\varphi(t)}{t}\right|^q dt\right)^{\frac{1}{q}}2^{j\frac{1}{q'}}.$$

Here $1/q+1/q'=1$. Consequently, in view of (7), it follows that

$$|x|^{-\rho_2}|A_3^{(k)}|\le|x|^{-\rho_2}\sum_{p+1}^{\infty}(n+1)|x|^n\sum_{j=k+2}^{\infty}o\left(2^{j(\rho_2-n)}\right)=$$

$$=\sum_{j=k+2}^{\infty}o\left(2^{(j-k)\rho_2}\right)\sum_{n=p+1}^{\infty}(n+1)2^{n(k-j)},$$

and hence

$$|x|^{-\rho_2}|A_3^{(k)}|\to 0, \tag{12}$$

when $|x|\to\infty$ uniformly for $\frac{x}{|x|}\in S_2$.

Consider the expression $A_1^{(k)}$:

$$A_1^{(k)}=\int\limits_1^{2^{k-1}}\varphi(t)\frac{d}{dt}\left\{\frac{1}{t}\left[\left(1-2\frac{|x|}{t}\cos\theta_2+\frac{|x|^2}{t^2}\right)^{-\frac{1}{2}}-\sum_0^p B_n^3(\theta_2)\frac{|x|^n}{t^n}\right]\right\}dt=$$

$$=\int\limits_{\frac{1}{|x|}}^{\frac{2^{k-1}}{|x|}}\frac{\varphi(s|x|)}{|x|}\left(-\frac{1}{s^2}\left(1-2\frac{\cos\theta_2}{s}+\frac{1}{s^2}\right)^{-\frac{3}{2}}\left(1-\frac{\cos\theta_2}{s}\right)+\sum_0^p B_n^3(\theta_2)\frac{n+1}{s^{n+2}}\right)ds=$$

$$=\int\limits_{\frac{1}{|x|}}^{\frac{2^{k-1}}{|x|}}\frac{\varphi(s|x|)}{|x|}\left[-\left(s^2-2s\cos\theta_2+1\right)^{-\frac{3}{2}}(s-\cos\theta_2)+\sum_0^p B_n^3(\theta_2)\frac{n+1}{s^{n+2}}\right]ds.$$

Note that $0 < s \leq \frac{1}{2}$, hence there exists a constant C such that

$$|A_1^{(k)}| \leq C \int\limits_{\frac{1}{|x|}}^{\frac{2^{k-1}}{|x|}} \left|\frac{\varphi(s|x|)}{x}\right| \frac{ds}{s^{p+2}} = \left(C \int\limits_1^{2^{k-1}} \frac{|\varphi(t)|}{t} \frac{dt}{t^{p+1}} \right) |x|^p \leq$$

$$\leq C|x|^p \sum_1^{k-1} \left(\int\limits_{2^j}^{2^{j+1}} \left(\frac{|\varphi(t)|}{t}\right)^q dt \right)^{\frac{1}{q}} 2^{-j(p+1)} 2^{j\frac{1}{q'}}.$$

(Again $1/q + 1/q' = 1$.) Using estimate (7) we obtain from here that

$$|A_1^{(k)}| = o\left(|x|^{\rho_2}\right), \quad |x| \to \infty, \tag{12}$$

uniformly for $\frac{x}{|x|} \in S_2$.

Let us go over to the expression $A_2^{(k)}$:

$$A_2^{(k)} = \int\limits_{2^{k-1}}^{2^{k+2}} \varphi(t) \left\{ -\frac{1}{t^2} \left(1 - 2\frac{|x|}{t} \cos\theta_2 + \frac{|x|^2}{t^2}\right)^{-\frac{3}{2}} \left(1 - \frac{|x|}{t} \cos\theta_2\right) + \right.$$

$$\left. + \sum_0^p B_n^3(\theta_2) \frac{|x|^n}{t^{n+2}} (n+1) \right\} dt =$$

$$= \int\limits_{2^{k-1}}^{2^{k+2}} \varphi(t) \left\{ -\frac{1}{t^2} \left(1 - 2\frac{|x|}{t} \cos\theta_2 + \frac{|x|^2}{t^2}\right)^{-\frac{3}{2}} \left(1 - \frac{|x|}{t} \cos\theta_2\right) \right\} dt +$$

$$+ \sum_0^p \int\limits_{2^{k-1}}^{2^{k+2}} \varphi(t) B_n^3(\theta_2) \frac{|x|^n}{t^{n+2}} (n+1) dt = A_{2,1}^{(k)} + A_{2,2}^{(k)}.$$

Estimates (5) and (7) give

$$|A_{2,2}^{(k)}| = o\left(|x|^{\rho_2}\right), \quad |x| \to \infty, \tag{14}$$

uniformly for $\frac{x}{|x|} \in S_2$.

Now we begin to estimate the expression $A_{2,1}^{(k)}$. It is easy to see that

$$A_{2,1}^{(k)} = \int\limits_{2^{k-1}}^{2^k} \varphi(t) \frac{t - x_3}{|t - x_3|^3} dt.$$

Moreover, $A_{2,1}^{(k)}$ and the integral

$$\tilde{A}(x) = \int\limits_0^{2\pi} \int\limits_{2^{k-1}}^{2^k} \frac{\varphi(t)}{t} \frac{t - x_3}{|t - x_3|^3} t dt d\alpha, \ x = (x_1, x_2, x_3),$$

have the same bounds. Evidently the latter integral is the Riesz transform in the plane of the function $g_k(t,\alpha) = \chi_k(t,\alpha)\frac{\varphi(t)}{t}$, where $\chi_k(t,\alpha)$ is the characteristic function of the ring $\{((t,\alpha) : 2^{k-1} \le t < 2^k, 0 < \alpha \le 2\pi\}$. We will use the following theorem for the estimate of $\tilde{A}$.

Theorem A ([7]). *Let Ω be a homogeneous function of the degree 0 such that*

$$\int_{S_{m-1}} \Omega(x)d\sigma = 0,$$

where $d\sigma$ is the area element of S_{m-1}, and

$$\sup_{|x-x'|\le\delta, |x|=|x'|=1} |\Omega(x) - \Omega(x')| = \omega(\delta), \quad \int_0^1 \frac{\omega(\delta)}{\delta} d\delta < \infty.$$

If $f \in L^p(\mathbb{R}^m), 1 \le p < \infty$ then the integral transformation

$$T_\varepsilon(f)(x) = \int_{|y|\ge\varepsilon} \frac{\Omega(y)}{|y|^m} f(x-y)dy, \quad \varepsilon > 0,$$

has the following properties:

a) $\lim_{\varepsilon\to 0} T_\varepsilon(f)(x)$ exists for almost every x.

b) Let $T^*(f)(x) = \sup_{\varepsilon>0} |T_\varepsilon(f)(x)|$. If $f \in L^1(\mathbb{R}^m)$ then $f \to T^* f$ has the weak type (1, 1).

c) If $1 < q < \infty$ then $\|T^* f\|_q \le C_q ||f||_q$ where the constant C_q depends only on the number q and the space dimension.

This theorem is valid for the Riesz transform. So applying Theorem A with $\Omega(y) = y/|y|, f = g_k$, we obtain that if $1 < q < \infty$ then in view of (7)

$$\int_{R\le r\le 2R, \theta_1\in[0,2\pi]} |\tilde{A}(r,\theta_1,\theta_2)|^q r dr d\theta_1 = o(R^{\rho_2 q+2}), \ R \to \infty,$$

uniformly for $\theta_2 \in [0,\pi]$. Hence

$$\int_{R\le r\le 2R, \theta_1\in[0,2\pi]} |A_{2,1}^{(k)}(r,\theta_1,\theta_2)|^q dr d\theta_1 = o(R^{\rho_2 q+1}), \ R \to \infty,$$

uniformly for $\theta_2 \in [0,\pi]$.

The latter relation and (11)–(14) prove the theorem for the case $1 < q < \infty$.

If $1 \le q < \infty$, then it follows from Theorem A that

$$\text{mes} E_k =: \text{mes}\{x : |A_{2,1}^{(k)}| > \varepsilon_k 2^{k\rho_2}\} \le \varepsilon_k^{-q} o(2^{2k}). \tag{15}$$

Consequently by (11) - (14) we conclude that outside the set $E = \bigcup E_k$ the remainder term

$$\psi(x) = o(|x|^{\rho_2}), \ |x| \to \infty.$$

Note, that if the sequence $\{\varepsilon_k\}_{k=0}^\infty$ decreases sufficiently slowly then from (15), as it is easy to see, it follows that E is $C_{0.2}$-set.

Theorem is completely proved. □

Remark 1. We have established that the exceptional set can contain the whole ray. It is natural since in the case $\mathbb{R}^m, m > 2$, the Riesz masses of a subharmonic function can in general completely fill out a ray. This fact distinguishes the space and plane cases.

For subharmonic functions of integer order the analogous fact takes place. Namely,

Theorem 2. *Let $u(x) \equiv u(r, \theta_1, \theta_2)$ be a subharmonic function of integer order in the space $\mathbb{R}^3 = \{y : y = (t, \alpha_1, \alpha_2)\}$ which Riesz masses concentrated on the ray $\{\alpha_2 = 0\}$ outside some neighborhood of the origin $\{t < t_0\}, 0 < t_0 < 1$. Let further for $t \geq t_0$*

$$\mu(t) = \Delta_1 t^{\rho_1+1} + \Delta_2 t^{\rho_2+1} + \varphi(t),$$

where $\rho_1 - 1 < \rho_2 < \rho_1, \Delta_1 > 0$, and the function $\varphi(t)$ satisfies estimate (7) for any $q \geq 1$. Then

$$u(r, \theta_1, \theta_2) = \Delta_1(\rho_1 + 1)r^{\rho_1} \left\{ \int_{t_0}^{1} \zeta^{\rho_1 - 1} H\left(\frac{1}{\zeta}, \theta_2, \rho_1 - 1\right) d\zeta + \int_{1}^{\infty} t^{\rho_1 - 1} H\left(\frac{1}{\zeta}, \theta_2, \rho_1\right) d\zeta \right\} +$$

$$+\Delta_2(\rho_2 + 1)r^{\rho_2} \int_{t_0}^{\infty} \zeta^{\rho_2 - 1} H(\frac{1}{\zeta}, \theta_2, \rho_2) d\zeta + \psi(r, \theta_1, \theta_2),$$

where the function $\psi(x) = o(r^{\rho_2}), r \to \infty$, uniformly for all $\overline{x} = \frac{x}{|x|}$ away from some $C_{0,2}$-set.

If $q > 1$ then

$$\int_{R \leq r \leq 2R, \theta_1 \in [0, 2\pi]} |\psi(r, \theta_1, \theta_2)|^q dr d\theta_1 = o(R^{\rho_2 q + 1}), \quad R \to \infty,$$

uniformly for $\theta_2 \in [0, \pi]$.

Remark 2. In the paper [2] the case of subharmonic functions of completely regular growth was studied. The exceptional set appeared in this work is more massive than $C_{0,2}$. So Theorems 1 and 2 give some generalization for the case of the one-term asymptotics as well.

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