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O. Hryniv

QUOTIENT TOPOLOGIES ON TOPOLOGICAL SEMILATTICES

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It is proven that for a closed σ-compact ideal I in a locally compact (Lawson) semilattice S the quotient S/I is a topological (Lawson) semilattice. Also we construct several counterexamples showing that the above result cannot be improved. One is an example of a countable subsemilattice $S \subset \mathbb{R}^4$ containing a closed ideal $I \subset S$ such that S/I fails to be a topological semilattice. The other is an example of a metrizable locally compact locally countable Lawson semilattice S of size $|S| = \aleph_1$ containing a closed discrete ideal I such that the quotient S/I fails to be a topological semilattice. Moreover, the quotient topology on S/I in the category of topological semilattices differs from the quotient topology in the category of Lawson semilattices. This answers in negative a question of J.Lawson and B.Madison.

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Доказано, что для замкнутого σ-компактного идеала I в локально компактной (лоусоновской) полурешетке S факторрешетка S/I являэтся топологической (лоусоновской) полурешеткой. Также мы строим ряд контрпримеров, показывающих, что этот результат не может быть улучшен. Один из них является примером счетной подполурешетки $S \subset \mathbb{R}^4$, содержащей замкнутый идеал $I \subset S$, такой, что S/I не является топологической полурешеткой. Другой является примером метризуемой локально компактной локально счетной полурешетки Лоусона S размера $|S| = \aleph_1$, содержащей замкнутый дискретный идеал I, такой, что факторрешетка S/I не является топологической полурешеткой. Более того, фактортопология на S/I в категории топологических полурешеток отличается от фактортопологии в категории полурешеток Лоусона. Это дает отрицательный ответ на вопрос Дж. Лоусона и Б. Медисона.

It is well known that for a closed normal subgroup H of a topological group G the quotient group G/H endowed with the quotient topology (that is the strongest topology making the quotient homomorphism $q\colon G \to G/H$ continuous) is a topological group as well. Surprisingly, but in the category of topological semigroups such a result is not valid even in the simplest case of the quotient semigroup S/I of a topological semigroup S by a closed ideal $I \subset S$, see [2], [7, 2.7]. In this paper we discuss this phenomenon in the category of *topological semilattices*, that is topological spaces S endowed with a continuous associative commutative idempotent operation $\vee\colon S\times S \to S$ (the operation $\vee$ is *idempotent* if $x \vee x = x$ for all $x \in S$). Each semilattice S carries a partial order $\leq$ induced by the semilattice operation: $x \leq y$ if $x \vee y = x$.

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Now let us recall the notion of the quotient space in the category of topological spaces. If I is a subset of a topological space S, then the *quotient* of S by I is the set $S/I = S\setminus I \cup \{I\}$ endowed with the strongest topology making the quotient map $q\colon S \to S/I$ continuous. This topology consists of sets $U \subset S/I$ whose preimages $q^{-1}(U)$ are open in S. It is well-known that the quotient space S/I is Hausdorff if S is regular and I is closed in S.

If S is endowed with a continuous semilattice operation and I is a closed ideal of S, then on the quotient space S/I one can introduce a unique semilattice operation $\vee_I\colon S/I\times S/I \to S/I$ such that the quotient map $q\colon S \to S/I$ is a semilattice homomorphism. The image of I under q is a one-point set consisting of the smallest element of S/I.

Then the question arizes whether the semilattice operation on S/I is continuous with respect to the quotient topology? The answer to this question is not definite. A positive result in this direction can be found in [2, p.50] (see also [5], [7]), where it is proved that the quotient semigroup S/I of a σ-compact locally compact semigroup S by a closed ideal I is a topological semigroup.

It turns out that in the case of topological semilattices the σ-compactness of S can be weakened to the σ-compactness of the ideal I.

Recall that *a Lawson semilattice* is a topological semilattice admitting a base of the topology consisting of subsemilattices.

Theorem 1. *Let S be a locally compact (Lawson) semilattice and let I be a σ-compact closed ideal of S, then S/I is a (Lawson) topological semilattice.*

Proof. Denote by $q\colon S \to S/I$ the quotient map. According to [7, 2.1] in order to prove that S/I is topological semilattice, it is sufficient to verify that the map $q\times q\colon S\times S \to S/I\times S/I$ is quotient. Consider a set $U \subset S/I \times S/I$ whose preimage $W = (q\times q)^{-1}(U)$ is an open set in $S\times S$. We need to show that U is open in $S/I \times S/I$. Given a point $(q(a), q(b)) \in U$, we shall find its neighborhood in U.

If $a, b \in S\setminus I$, then choose neighborhoods $O(a), O(b) \subset S\setminus I$ of the points a, b such that $O(a)\times O(b) \subset W\setminus I\times I$. Since the map $q\colon S \to S/I$ is quotient and $O(a) = q^{-1}\circ q(O(a))$, the image $q(O(a))$ is open in S/I. By analogy, $q(O(b))$ is open in S/I. Then their product $q(O(a))\times q(O(b))$ is an open neighborhood of (a, b) in $S/I\times S/I$ so $(q(a), q(b)) \in q(O(a))\times q(O(b)) \subset U$.

Consider another case $a, b \in I$. Then $I\times I \subset W$. Since the ideal I is locally compact and σ-compact there exists a sequence $\{K_n\}$ of compact subsets of I such that $I = \bigcup_{n\in\mathbb{N}} K_n$ and $K_n \subset \operatorname{int}(K_{n+1})$ for each $n \in \mathbb{N}$ (see [4, Ex. 3.8.C.]). Applying the Wallace Theorem (see [4, p. 220]) to the product $K_1 \times K_1 \subset W$, we could find an open neighborhood $V_1 \subset S$ such that the closure $\overline{V_1} \subset S$ is compact and $K_1\times K_1 \subset V_1\times V_1 \subset \overline{V_1}\times\overline{V_1} \subset W$. The set $K_2 \vee \overline{V_1}$ is compact and $K_2\vee\overline{V_1} \subset I$. Again by the Wallace Theorem, there exists a neighborhood $V_2 \subset S$ of $K_2\vee\overline{V_1}$ such that $(K_2\vee\overline{V_1})\times(K_2\vee\overline{V_1}) \subset V_2\times V_2 \subset \overline{V_2}\times\overline{V_2} \subset W$. Define recursively an increasing sequence of open subsets $\{V_n\}$ of S such that

- $\overline{V_n}$ is compact;
- $K_n \subset V_n \subset \overline{V_n}$ and $\overline{V_n}\times\overline{V_n} \subset W$;
- $K_{n+1}\vee\overline{V_n} \subset V_{n+1} \subset \overline{V_{n+1}}$.

The set $V = \bigcup_{n=1}^{\infty} V_n$ is an open neighborhood of I in S with $V\times V \subset W$ and $q(V)$ is an open neighborhood of $\{I\}$ in S/I, with $(q(a), q(b)) \in q(V)\times q(V) \subset U$.

If $a \in S\setminus I$ and $b \in I$ we use the same arguments as in the previous case, since $\{b\}$ is compact, we can apply the Wallace Theorem.

For the proof of the "Lawson part" we shall need two lemmas, the first of which belongs to D.Lawson [6]. The second lemma will substitute the Wallace Theorem in the preceding argument.

Lemma 1. *Every compact subset K of a locally compact Lawson semilattice S lies in a compact subsemilattice of S.*

Lemma 2. *For any compact subsemilattice K of a locally compact Lawson semilattice S and any neighborhood U of K there exists an open subsemilattice $O(K)$ of S with compact closure $\overline{O(K)}$ such that $K \subset \overline{O(K)} \subset U$.*

Proof. Without loss of generality we can assume that $\overline{U}$ is compact because for every open set V containing K there exists an open neighborhood U such that $K \subset U \subset \overline{U} \subset V$ and $\overline{U}$ is compact. By Lemma 1, there exists a compact subsemilattice L containing $\overline{U}$. It is known that each compact Lawson semilattice is isomorphic to a subsemilattice of the Tychonov cube $[0,1]^\tau$ with pointwise max as the semilattice operation [6]. So we can identify L with a subsemilattice of $[0,1]^\tau$. Observe that K and $L \setminus U$ are disjoint compact subsets of the Tychonov cube. Then by the definition of the product topology on $[0,1]^\tau$ there is a finite index set $A \subset \tau$ such that the projections $\mathrm{pr}_A(K)$, $\mathrm{pr}_A(L \setminus U)$ of K and $L \setminus U$ onto the A-face $[0,1]^A$ are disjoint (here $\mathrm{pr}_A\colon [0,1]^\tau \to [0,1]^A$ is the coordinate projection). Then the complement $W = [0,1]^A \setminus \mathrm{pr}_A(L \setminus U)$ is a neighborhood of the subsemilattice $\mathrm{pr}_A(K)$ in the finite-dimensional cube $[0,1]^A$.

Observe that $\mathrm{pr}_A^{-1}(W) \cap L \subset U$. It remains to find an open subsemilattice $V \subset \overline{V} \subset W$ containing $\mathrm{pr}_A(K)$ and to take the preimage $L \cap \mathrm{pr}_A^{-1}(V) \subset U$. For this endow the cube $[0,1]^A$ with the max metric $d((x_i)_{i\in A},(y_i)_{i\in A}) = \max_{i\in A}|x_i - y_i|$. It is easy to verify that for each $\varepsilon > 0$ the ε-neighborhood

$$B(\mathrm{pr}_A(K),\varepsilon) = \left\{x \in [0,1]^A : d(x,\mathrm{pr}_A(K)) < \varepsilon\right\}$$

is an open subsemilattice of $[0,1]^A$ (this follows from the inequality $d(\max(x,y),\max(x',y')) \le \max\{d(x,x'),d(y,y')\}$ holding for all $x,y \in [0,1]^A$). Then we can choose $\varepsilon > 0$ so small that the closure $\overline{V}$ of the the ε-neighborhood $V = B(\mathrm{pr}_A(K),\varepsilon)$ of $\mathrm{pr}_A(K)$ in $[0,1]^A$ lies in W. Its preimage $\mathrm{pr}_A^{-1}(V) \cap L = O(K)$ is the required neighborhood of K. Since $O(K)$ is an open subset of U and U is open in S, $O(K)$ is open in S. Moreover $\overline{O(K)} \subset \mathrm{pr}_A^{-1}(\overline{V}) \cap L \subset \mathrm{pr}_A^{-1}(W) \cap L \subset U$. □

Now we are able to continue the proof of Theorem 1. We need to show that for each point $x \in S/I$ and for each neighborhood U of x there exists an open subsemilattice $O(x)$ such that $x \in O(x) \subset U$. The semilattice S is Lawson, so there exists a base $\mathcal{B}(S)$ of the topology on S consisting of subsemilattices. Hence for $x \in S/I \setminus \{I\}$ and for any neighborhood U of x there exists an open subsemilattice $O(x) \in \mathcal{B}(S)$ such that $x \in O(x) \subset U \setminus I \subset U$.

For the point $x = \{I\}$ and a neighborhood U of $\{I\}$ let us consider the neighborhood $q^{-1}(U)$ of I in S. We shall construct an open subsemilattice $O \in \mathcal{B}(S)$ with $I \subset O \subset q^{-1}(U)$. Then the open subsemilattice $q(O)$ will satisfy the condition $\{I\} \in q(O) \subset U$.

As stated above $I = \bigcup_{n\in\mathbb{N}} K_n$ where K_n are compact subsets of I and $K_n \subset \mathrm{int}(K_{n+1})$ for each $n \in \mathbb{N}$. By Lemma 1, the compactum K_1 lies in a compact subsemilattice $L_1 \subset I$. Moreover by Lemma 2, there exists an open subsemilattice $O(L_1) \subset S$ with compact closure $\overline{O(L_1)}$ such that $L_1 \subset \overline{O(L_1)} \subset U$. Since I is a closed ideal in S, $\overline{O(L_1)} \cup I$ is a locally compact semilattice. By Lemma 1, the compact subset $\overline{O(L_1)} \cup K_2$ lies in a compact subsemilattice

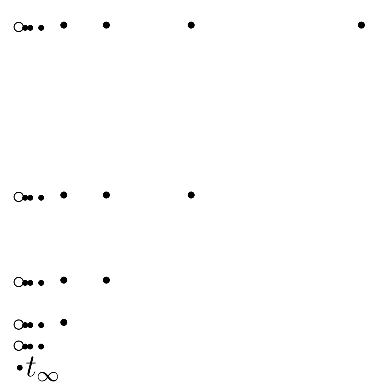
$L_2 \subset \overline{O(L_1)} \cup I \subset U$. By Lemma 2, L_2 lies in an open subsemilattice $\overline{O(L_2)} \subset S$ with compact closure $\overline{O(L_2)} \subset U$.

Continuing in this way we shall construct inductively sequences (L_n) of compact subsemilattices of S and $(O(L_n))$ of their neighborhoods such that $K_{n+1} \cup \overline{O(L_n)} \subset L_{n+1} \subset \overline{O(L_n)} \cup I$ and $L_{n+1} \subset O(L_{n+1}) \subset \overline{O(L_{n+1})} \subset U$. Then the open subsemilattice $O = \bigcup_{n=1}^{\infty} L_n \subset U$ is the desired neighborhood of I. □

Now we shall construct some examples showing that the conditions of the local and σ-compactness in Theorem 1 are essential. Our first example shows that the local compactness cannot be weakened to the Čech-completeness (we recall that a topological space X is *Čech-complete* if X is a G_δ-set in its Stone-Čech compactification, see [4, §3.9]).

Example 1. Consider the 4-dimensional Euclidean space $\mathbb{R}^4$ as a topological semilattice endowed with the coordinatewise maximum-operation. There is a subsemilattice $X \subset \mathbb{R}^4$ containing a closed ideal $I \subset X$ such that X/I endowed with the quotient topology fails to be a topological semilattice. Moreover, the space X, being a G_δ-subset of $\mathbb{R}^4$, is Čech-complete.

Proof. By T we denote the G_δ-subset $T = \{t_{n,m} : 0 \le n \le m\} \cup t_\infty$ of the plane, where $t_\infty = \{(0,0)\}$, $t_{n,m} = \left(\frac{1}{2^m}, \frac{1}{2^n}\right)$, for $n, m \in \mathbb{N}$, and endowed with a semilattice operation $t_{n,m} \vee t_{k,l} = \max(t_{n,m}, t_{k,l}) = (\max(\frac{1}{2^n}, \frac{1}{2^k}), \max(\frac{1}{2^m}, \frac{1}{2^l}))$. The set T looks as follows:



Observe that T is a subsemilattice of the plane $\mathbb{R}^2$ endowed with the coordinatewise max-operation. Let $S_0 = \{0, -2^{-n} : n \ge 0\}$ be an increasing sequence convergent to the point $c_\infty = 0 \in S_0$. Let us consider the product $X = T \times S_0 \times \mathbb{N}$ endowed with the semilattice operation

$$(t_{n,m}, c_i, j) \vee (t_{n',m'}, c_i', j') = (\max(t_{n,n'}, t_{m,m'}), \max(c_i, c_i'), \max(j, j')) .$$

The set X is a subsemilattice of the Euclidean space $\mathbb{R}^4$ with respect to coordinatewise max-operation and $I = T \times \{0\} \times \mathbb{N}$ is a closed ideal in X. Denote by $q \colon X \to X/I$ the quotient map and endow X/I with the quotient topology, i.e. the maximal topology making q continuous. To demonstrate that the semilattice operation on X/I is not continuous at $(\{I\}, \{I\})$ we will produce an open neighborhood U of I in X such that for any open neighborhood V of I in X, the product $V \vee V$ is not contained in U. Let $U = X \setminus \{(t_{n,m}, c_m, n) : m \ge n \ge 0\}$ and observe that U is an open neighborhood of I.

Suppose that V is an open set in X containing I, then V is a neighborhood of the point $(t_\infty, c_\infty, 1)$. Consequently there are $n_1, j_1 \in \mathbb{N}$ such that $(t_{n,m}, c_j, 1) \in V$ for all $m \ge n \ge n_1$

and $j \geq j_1$. Since V is a neighborhood of the point $(t_\infty, c_\infty, n_1)$, there exists $m_1 \geq \max\{j_1, n_1\}$ such that $(t_\infty, c_j, n_1) \in V$ for all $j \geq m_1$. Let us consider the product of two points $x = (t_{n_1,m_1}, c_{m_1}, 1)$ and $y = (t_\infty, c_{m_1}, n_1)$ from the set V:

$$x \vee y = (t_{n_1,m_1}, c_{m_1}, n_1) \notin U,$$

which implies $V \vee V \not\subset U$. □

Our next example answers in negative a question posed in [7, p.20]: if S is locally compact topological semigroup and I is a closed ideal of S, is S/I a topological semigroup? In the following example $S_0 = \{0, -2^{-n} : n \in \omega\}$ is the convergent sequence on the line.

Example 2. There is a discrete uncountable well-ordered set $(B, \leq)$ such that for the closed discrete ideal $I = B \times \mathbb{N} \times \{0\}$ in the product $S = B \times \mathbb{N} \times S_0$ the quotient S/I fails to be a topological semilattice. Also, S is metrizable, locally compact, and locally countable.

Proof. On the set $\mathbb{N}^\omega$ of all number sequences we consider the preorder: $(x_n) \leq^* (y_n)$ if $x_n \leq y_n$ for all sufficiently large n. Let $B \subset \mathbb{N}^\omega$ be a subset of increasing sequences which is unbounded in $\mathbb{N}^\omega$ with respect to the preoder $\leq^*$. The latter means that for each $(x_n) \in \mathbb{N}^\omega$ there is $(y_n) \in B$ such that $(y_n) \not\leq^* (x_n)$. For example we can take for B all the set $\mathbb{N}^\omega$.

Endow the set B with the discrete topology and a well-order $\prec$ and consider the set $S = B \times \mathbb{N} \times S_0$ endowed with the semilattice operation

$$(b, n, s) \vee (b', n', s') = (\max(b, b'), \max(n, n'), \max(s, s')).$$

Then $I = B \times \mathbb{N} \times \{0\}$ is a closed ideal in S and S is a locally compact locally countable metrizable space.

We claim that the quotient topology on S/I is not a semilattice topology. For this consider the closed subset $F = \left\{\left(b, n, \frac{1}{k}\right) : k \leq b_n\right\}$ of S/I (we recall that elements $b = (b_n)$ of B are sequences!) Assuming that the quotient topology is a semilattice topology we would find an open neighborhood $V \subset S/I$ of $\{I\}$ such that $V \vee V \subset (S/I) \setminus F$. For each $b \in B$ and $n \in \mathbb{N}$ find a number $k(b, n) \in \mathbb{N}$ such that $\left(b, n, \frac{1}{k}\right) \in V$ for all $k \geq k(b, n)$. Let a be the smallest element of B with respect to the well-order $\prec$. Since B is unbounded, there is a sequence $b = (b_n) \in B$ such that $(b_n) \not\leq^* (k(a, n))_{n=1}^\infty$. Consequently, there is $n \in \omega$ such that $b_n > \max(k(b, 1), k(a, n))$. Consider the points $\left(a, n, \frac{1}{k(a,n)}\right) \in V$ and $\left(b, 1, \frac{1}{k(b,1)}\right) \in V$. Their product must belong to $(S/I) \setminus F$. But $\left(a, n, \frac{1}{k(a,n)}\right) \vee \left(b, 1, \frac{1}{k(b,1)}\right) = \left(b, n, \frac{1}{\max(k(a,n), k(b,1))}\right) \in F$ by the definition of F and the inequality $b_n \geq \max(k(b, 1), k(a, n))$. □

The construction of Example 2 implies that I can be taken of cardinality $\mathfrak{b}$ equal to the smallest size $|B|$ of subset B of $\mathbb{N}^\omega$ which is not bounded with respect to the preoder $\leq^*$. It is clear that $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{c}$. The position of the cardinal $\mathfrak{b}$ on the interval $[\aleph_1, \mathfrak{c}]$ depends on some set-theoretic assumptions. In particular, $\mathfrak{b} = \mathfrak{c}$ under Martin Axiom but there are models of ZFC with $\aleph_1 = \mathfrak{b} < \mathfrak{c}$ [1],[3],[8].

In this context, one could ask if there is an example of a locally compact semilattice S and a closed discrete ideal $I \subset S$ of the smallest possible uncountable size $|I| = \aleph_1$ such that S/I fails to be a topological semilattice. The answer is positive if $\aleph_1 = \mathfrak{b}$. However, the same can be proved in ZFC alone.

Our last example will have some additional features. To describe them let us observe that in additon to the quotient topology a semilattice S/I carries at least two other natural topologies. One of them is the strongest topology on S/I making the quotient map $q\colon S \to S/I$ as well as the semilattice operation on S/I continuous. This topology will be called *the quotient topology in the category* TopSemilattice. Under *the quotient topology in the category* LawsonSemilattice we understand the strongest topology on S/I such that S/I is a Lawson semilattice and the quotient map $q\colon S \to S/I$ is continuous.

Example 3. Let $\exp_{<\omega}(\aleph_1)$ be the semilattice of all finite subsets of the uncountable cardinal $\aleph_1$ endowed with the discrete topology and the semilattice union operation $\cup$. Let $S = \exp_{<\omega}(\aleph_1) \times S_0$, where $S_0 = \left\{0, \frac{1}{n} : n \in \mathbb{N}\right\}$ is a convergent sequence. Endow the set S with the semilattice operation $(A, y) \vee (A', y') = (A \cup A', \min(y, y'))$ and consider the closed ideal $I = \exp_{<\omega}(\aleph_1) \times \{0\}$ in S. Then

(1) the quotient topology on S/I is not a semilattice topology;

(2) the quotient topology on S/I in the category of topological semilattices does not coincide with the quotient topology in the category of Lawson semilattices.

Proof. (1) Identifying $\aleph_1$ with a subset of $[0,1]$ we can find a metric d on $\aleph_1$, turning $\aleph_1$ into a metric separable space. In the semilattice S consider the closed subset

$$F = \left\{\left(\{a,b\}, \tfrac{1}{n}\right) \in S : a, b \in \aleph_1,\ a \neq b,\ \tfrac{1}{n} \geq d(a,b)\right\}$$

missing the ideal I. Then its complement $U = S \setminus F$ is an open neighborhood of I. Assuming that the quotient topology on S/I is a semilattice topology we would find an open neighborhood V of I in S such that $V \vee V \subset U$. For each element $(\{a\}, 0) \in I$ find $n(a) \in \mathbb{N}$ such that $\left(\{a\}, \frac{1}{n}\right) \in V$ for all $n \geq n(a)$. Then the set $\aleph_1$ can be written as a countable union $\aleph_1 = \bigcup_{n\in\mathbb{N}} A_n$, where $A_n = \{a \in \aleph_1 : n(a) = n\}$. Since $\aleph_1$ is uncountable, the set A_n is uncountable for some $n \in \mathbb{N}$. Being an uncountable subset of the separable metric space $(\aleph_1, d)$, the set A_n contains two distinct points $a, b \in A_n$ with $d(a,b) < \frac{1}{n}$. Then the point $\left(\{a,b\}, \frac{1}{n}\right) \in F$. On the other hand, $\left(\{a,b\}, \frac{1}{n}\right) = \left(\{a\}, \frac{1}{n}\right) \vee \left(\{b\}, \frac{1}{n}\right) \in V \vee V \subset U \setminus F$, which is a contradiction.

(2) In order to prove the second item we shall construct a semilattice topology τ on S/I such that the semilattice operation $\vee\colon S/I \times S/I \to S/I$ is continuous, and show that there is a τ-open neighborhood U of $\{I\}$ in S/I containing no subsemilattice $W \ni \{I\}$, open in the quotient topology.

Each point of the set $S\setminus\{I\}$ is isolated in τ, while the neighborhoods of the distinguished point $\{I\}$ of S/I are of the form:

$$U_{(a,\lambda)} = \left\{\{I\}, \left(A, \tfrac{1}{n}\right) : n \geq a\lambda^{|A|}\right\},$$

where a, λ run over natural numbers.

Let us show that τ is a semilattice topology. In order to prove the continuity of the operation $\vee$ on $(S/I, \tau)$ fix a pair of points $x_0, y_0 \in S/I$ and a neighborhood $U(x_0 \vee y_0)$ of their product and show that there exist neighborhoods $V(x_0)$, $V(y_0)$ such that $V(x_0) \vee V(y_0) \in U(x_0 \vee y_0)$.

If $x_0 = y_0 = x_0 \vee y_0 = \{I\}$, then we can find $a, \lambda \in \mathbb{N}$ with $U_{(a,\lambda)} \subset U(x_0 \vee y_0)$. Let $b = a$ and $\mu = \lambda^2$. We claim that $V_{(b,\mu)} \vee V_{(b,\mu)} \subset U_{(a,\lambda)}$, i.e. for any elements $\left(A, \frac{1}{n}\right), \left(B, \frac{1}{m}\right) \in V_{(b,\mu)}$ their product $\left(A \cup B, \frac{1}{\max(n,m)}\right)$ belongs to $U_{(a,\lambda)}$. Assume that $|A| \geq |B|$. Then $\max(n,m) \geq n \geq b\mu^{|A|} = b(\mu^{2|A|})^{\frac{1}{2}} \geq b(\mu^{|A\cup B|})^{\frac{1}{2}} = b(\mu^{\frac{1}{2}})^{|A\cup B|} = a\lambda^{|A\cup B|}$ and thus $\left(A \cup B, \frac{1}{\max(n,m)}\right) \in U_{(a,\lambda)} \subset U(x_0 \vee y_0)$.

If $x_0 = \left(A_0, \frac{1}{n_0}\right) \neq \{I\}$ and $y_0 = \{I\}$, then let $V(x_0) = \{x_0\}$ and $V(y_0) = U_{(b,\mu)}$, where $b = \max(n_0, a\lambda^{|A_0|})$ and $\mu = \lambda$. Take any $y = (B, \frac{1}{m}) \in U_{(b,\mu)}$ and consider the product $x_0 \vee y = \left(A_0 \cup B, \frac{1}{\max(n_0,m)}\right)$. Observe that $\max(n_0, m) \geq m \geq b\mu^{|B|} \geq a\lambda^{|A_0|}\lambda^{|B|} = a\lambda^{|A_0|+|B|} \geq a\lambda^{|A_0 \cup B|}$, so $\left(A_0 \cup B, \frac{1}{\max(n_0,m)}\right) \in U_{(a,\lambda)}$.

Therefore τ is a semilattice topology. It follows from the definition that the quotient topology τ' on S/I in the category TopSemilattice is stronger that τ. This implies that each set $U_{(a,\lambda)}$, $a, \lambda \in \mathbb{N}$, is open in τ'. Assuming that τ' is a Lawson topology we would find an open subsemilattice $L \subset U_{(1,2)}$ containing the point $\{I\}$. For each $a \in \aleph_1$ we can find $n(a) \in \mathbb{N}$ such that $\left(\{a\}, \frac{1}{n}\right) \in L$ for all $n \geq n(a)$. Because of uncountability of $\aleph_1$, for some $n \in \mathbb{N}$ the set $A_n = \{a \in \aleph_1 : n(a) = n\}$ is uncountable. Take any finite subset $F \subset A_n$ of size $|F| > n$. Then $\left(F, \frac{1}{n}\right)$ is the product of $\left(\{a\}, \frac{1}{n}\right)$, $a \in F$, in the semilattice L. Consequently $\left(F, \frac{1}{n}\right) \in L \subset U_{(1,2)}$ and $n > 2^{|F|} > 2^n$, which is a contradiction. □

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Ivan Franko Lviv National University,
Faculty of Mechanics and Mathematics,
olena_hryniv@ukr.net

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