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A GENERALIZATION OF COMPLETELY FACTORIZABLE GROUPS

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The locally graded p-groups G in which for some fixed cyclic subgroup A each subgroup $H \supseteq A$ has a complement are investigated.

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Исследуются локально ступенчатые p-группы G, в которых для некоторой фиксированной циклической подгруппы A каждая подгруппа $H \supseteq A$ имеет дополнение.

1. Introduction. Recall that a group G is said to be completely factorizable if for its every subgroup H there exists some subgroup K of G such that $G = HK$ and $H \cap K = 1$ [1]. Such an arbitrary K is called a complement to H in G. The complete constructive description of completely factorizable groups was obtained by N. V. Chernikova [1,2]. (For instance, by N. V. Chernikova's Theorem a p-group is completely factorizable iff it is elementary abelian). Remark that finite completely factorizable groups were first considered by Ph. Hall [3] who has given the known criterion for a finite group to belong to the class of completely factorizable groups.

It is reasonable to consider groups in which every subgroup containing a certain subgroup has a complement. It is known that in the case of arbitrary odd prime p every subgroup of a finite p-group G containing the fixed cyclic subgroup A has a complement in G iff A has an elementary abelian complement in G (V. A. Kreknin, A. V. Spivakovskiĭ, V.F.Malik [4]).

We recall that a group is said to be locally graded if its every nonidentity finitely generated subgroup has a proper subgroup of finite index [5]. The class of locally graded groups is very wide. It is known to include the classes of all locally finite, residually finite, locally soluble, linear groups, RN-groups (and, at the same time, all Kurosh-S. N. Chernikov's classes of groups belong to the class of locally graded groups).

The main results of the present paper are the following theorems.

Theorem 1. *Let G be a locally graded p-group such that each its subgroup containing fixed subgroup $A = \langle a \rangle$ has a complement in G. Then:*

(i) *G is locally finite.*

(ii) *G contains an elementary abelian normal subgroup K of finite index dividing $|\langle a \rangle|!$ such that $N \cap \langle a \rangle = 1$.*

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(iii) *G has a finite exponent.*

(iv) *G is nilpotent.*

The following Theorem 2 extends Theorem [4] mentioned above to the case of locally graded p-groups.

Theorem 2. *Let G be a group, $a \in G$, $p \in \mathbb{P}$ and $p \neq 2$. Then the following assertions are equivalent:*

(i) *Every subgroup of G containing $\langle a \rangle$ has a complement in G and G is a locally graded p-group.*

(ii) *$\langle a \rangle$ is a p-subgroup and has an elementary abelian p-complement in G.*

(iii) *$\langle a \rangle$ is a p-subgroup and there exists an elementary abelian p-subgroup B of G such that $G = \langle a \rangle B$.*

(iv) *Every subgroup of G containing $\langle a \rangle$ has a complement in G and G is a nilpotent metabelian locally finite p-group which has a normal elementary abelian subgroup of finite index.*

In connection with Theorems 1 and 2 we remark that the class of locally graded p-groups is wider than the class of locally finite p-groups as the classical E.S.Golod's examples [6] of infinite finitely generated residually finite p-groups show.

In virtue of the following theorem the condition $p \neq 2$ in Theorem 2 is essential.

Theorem 3. *For an arbitrary $l \in \mathbb{N}$ there exists a finite 2-group G with $G^{(3)} = 1$ such that every subgroup of G containing its some fixed cyclic subgroup has a complement in G and G' is nilpotent of class l.*

In what follows, $\mathbb{N}$ and $\mathbb{P}$ denote the sets of all positive integers and primes respectively. The symbols $\times$, $\leftthreetimes$, $\oplus$ are used to denote the direct and semidirect products and direct sum respectively. Let G be a group and $\varnothing \neq H \subseteq G$. Then $H_G = \bigcap_{g \in G} H^g$. The notations $H \leq G$ and $H < G$ mean that H is a subgroup of G and H is a subgroup of G different from G respectively. Then G' is the derived subgroup of G, $G^{(0)} = G$ and $G^{(n)} = (G^{(n-1)})'$ for $n \in \mathbb{N}$, $\Phi(G)$ means the Frattini subgroup. $J(G)$ is the intersection of all subgroups of finite index in G. (In consequence of Poincaré's Theorem $J(G)$ coincides with the intersection of all normal subgroups of finite index of G). The other notation is standard.

2. Preliminary results.

Lemma 1. *Let $G = AB$ and $A, B \leq G$, B is completely factorizable (in particular, elementary abelian) group and $A \leq H \leq G$. Then H has some complement $D \leq B$ in G.*

Proof. Indeed, by (S. N. Chernikov's) Lemma 1.8 [7] $H = A(H \cap B)$. Further, $H \cap B$ has some complement D in B. It is easy to see that D is a complement to H in G. □

Lemma 2. *Let $A \leq H \leq G$ and let ϕ be a homomorphism of H. If every subgroup of G containing A has a complement in G then every subgroup of H^ϕ containing A^ϕ has a complement in H^ϕ.*

Proof. Indeed in consequence of Lemma 1.8 [7] each subgroup L of H containing A has a complement in H. Further, let $A^\phi \leq L^\phi$ for some $K \leq H$ and let D be a complement to $L\mathrm{Ker}\,\phi$ in H. Then $H^\phi = K^\phi D^\phi$ and, obviously, $K^\phi \cap D^\phi = 1$. Thus D^ϕ is a complement to K^ϕ in H^ϕ. □

The following proposition is of independent interest.

Lemma 3. *Let $n \in \mathbb{N}$ and let the group G possess some local system of finitely generated subgroups H such that:*

(i) *$H/J(H)$ is periodic.*

(ii) *If $K \trianglelefteq H$ and $|\, H : K\,| < \infty$ then H/K is soluble and the derived length of H/K does not exceed n.*

(iii) *If $K \neq 1$ and $|\, H : K\,| < \infty$ then there exists $L < K$ such that $|\, K : L\,| < \infty$.*

Then G is soluble locally finite and $G^{(n)} = 1$.

Proof. In view of property (ii), obviously, $(H/J(H))^{(n)} = 1$. Thus $H/J(H)$ is a finitely generated periodic soluble group. Therefore by S. N. Chernikov's Theorem (see, for instance, [8], Proposition 1.1) $H/J(H)$ is finite.

Assume that $J(H) \neq 1$. By property (iii) there exists some subgroup $L \subset J(H)$ with $|\, J(H) : L\,| < \infty$. But then $|\, H : L\,| < \infty$ and $J(H) \subseteq L$. Contradiction.

Thus $J(H) = 1$. Consequently H is finite soluble with $H^{(n)} = 1$. Then in view of the arbitrariness of H, obviously, G is locally finite and $G^{(n)} = 1$. The lemma is proven. □

3. Proofs of theorems.

Proof of Theorem 1. Let G_ι, $\iota \in I$, be all finitely generated subgroups of G containing $\langle a \rangle$, $H = G_\iota^\phi$ be a finite homomorphic image of G_ι, $A = \langle a \rangle^\phi$, $N = \{g \in Z(H) | g^p = 1\}$, and $F = A \cap N$.

Evidently $AN = A \times D$ for some $D \leq N$. By Lemma 2, AN has some complement S in H. Since H is a finite p-group, $S_H \trianglelefteq H$ and moreover $N \cap S_H = 1$, obviously, $S_H = 1$. Further, it is clear that $L = D \times S$ is a complement to A in H and $D \subseteq L_H$. By Lemma 1.8 [7], $L_H = D \times (S \cap L_H)$. Since D is elementary abelian, it is easy to see that $\Phi(L_H) \subseteq S \cap L_H$. Further, obviously, $\Phi(L_H) \trianglelefteq H$. Therefore

$$\Phi(L_H) \subseteq (S \cap L_H)_H \subseteq S_H = 1.$$

Consequently, L_H is elementary abelian. Since $|H : L| = |A|$ it follows by [6, Theorem 12.2.2] that $|H : L_H|$ divides $|A|!$. So $|H : L_H|$ divides $|\langle a \rangle|!$. Therefore $H^{(m)} = 1$ for $m = |\langle a \rangle|! + 1$. Then in consequence of Lemma 3 G_ι is finite. Therefore (i) holds.

As it was proved G_ι has a normal elementary abelian subgroup of index dividing $|\langle a \rangle|!$. Let M_ι be a set of all such subgroups of G_ι. Let $M_\alpha \leq M_\iota$ iff $G_\alpha \subseteq G_\iota$. In the case $M_\alpha \leq M_\iota$ we define the projection $\pi_{\iota\alpha}$ from M_ι into M_α as follows: for an arbitrary $K \subseteq M_\iota$ $K^{\pi_{\iota\alpha}} = K \cap G_\alpha$. Obviously, the following holds:

1) for each M_α and M_β there exists M_γ such that $M_\alpha, M_\beta \leq M_\gamma$;

2) if $M_\alpha \leq M_\beta$, $M_\beta \leq M_\gamma$ then $\pi_{\gamma\alpha} = \pi_{\gamma\beta}\pi_{\beta\alpha}$;

3) $\pi_{\iota\iota}$ is the identity mapping of M_ι onto itself.

Consequently in view of [9] or [10, p.351-353] there exist $K_\iota \in M_\iota$, $\iota \in I$, such that $K_\alpha = K_\iota \cap G_\alpha$ whenever $G_\alpha \subseteq G_\iota$. Obviously, $K = \bigcup_{\iota \in I} K_\iota$ is an elementary abelian subgroup of G.

Let $g \in G$ and $a \in K$. Then $g \in G_\alpha$ for some $\alpha \in I$ and $a \in K_\alpha \trianglelefteq G_\alpha$. Therefore $a^g \in K_\alpha \subseteq K$. Thus $K \trianglelefteq G$. Let $n = \max_{\iota \in I} |G_\iota : K_\iota|$ and let $|G_\gamma : K_\gamma| = n$ for $\gamma \in I$ and $G_\gamma = \bigcup_{i=1}^n a_i K_\gamma$. Then $g \in G_\beta \supseteq G_\gamma$ for some $\beta \in I$. Since $K_\gamma \subseteq K_\beta$ and $|G_\beta : K_\beta| \leq |G_\gamma : K_\gamma|$, obviously, $G_\beta = \bigcup_{i=1}^n a_\iota K_\beta$. Consequently $g \in \bigcup_{i=1}^n a_i K$. Thus $G = \bigcup_{i=1}^n a_i K$. Since $K_\gamma = K_\alpha \cap G_\gamma$ whenever $G_\gamma \subseteq G_\alpha$ and, obviously, for $K = \bigcup_{\alpha \in \Lambda} K_\alpha$ $\Lambda = \{\alpha \in I | G_\gamma \subseteq G_\alpha\}$, we have $K_\gamma = K \cap G_\gamma$. Then, clearly, for distinct naturals $i, j \leq n$ we have $a_i K \neq a_j K$. Consequently $|G : K| = |G_\gamma : K_\gamma|$. Therefore $|G : K|$ divides $|\langle a \rangle|!$.

It is clear that (ii) implies (iii).

Finally, Baumslag's Theorem [10] asserts that a p-group is nilpotent iff it has a normal subgroup of finite index and finite exponent. In view of this theorem and (ii) G is nilpotent. □

Proof of Theorem 2. Let (i) hold. Then by Theorem 1 G is locally finite. Let G_ι, $\iota \in I$, be all finite subgroups of G containing a. In virtue of Theorem [4] mentioned above there exists some elementary abelian complement to $\langle a \rangle$ in G_ι. Let M_ι be a set of all such complements. Let $M_\alpha \leq M_\iota$ whenever $G_\alpha \subseteq G_\iota$. Presuppose that $M_\alpha \leq M_\iota$ and $B \in M_\iota$. In view of [7, Lemma 1.8], $B \cap G_\alpha \in M_\alpha$. Define the projection $\pi_{\iota\alpha}$ from M_ι into M_α by setting $B^{\pi_{\iota\alpha}} = B \cap G_\alpha$ for each $B \in M_\iota$. It is easy to see that conditions 1)-3) from the proof of Theorem 1 are fulfilled. Therefore, as in the case of mentioned proof, there exist corresponding B_ι, $\iota \in I$, such that $B_\alpha \subseteq B_\iota$ if $G_\alpha \subseteq G_\iota$. Evidently $\bigcup_{\iota \in I} B_\iota$ is an elementary abelian complement to $\langle a \rangle$ in G. Thus (i) implies (ii).

Obviously, (ii) implies (iii) and in view of Lemma 1 (iii) implies (ii). That is, assertions (iii) and (ii) are equivalent.

Let (ii) hold. Then in virtue of Lemma 1 the first part of assertion (iv) is valid. Further since $G = \langle a \rangle B$ and $|\langle a \rangle| < \infty$ we obtain $|G : B| < \infty$. Therefore in consequence of Poincaré's Theorem G/B_G is finite. Since $G/B_G = (\langle a \rangle B_G/B_G)(B/B_G)$ and $\langle a \rangle B_G/B_G$, B/B_G are p-groups, it follows G/B_G is a finite p-group. Therefore, obviously, G is a soluble p-group. Then by S.N.Chernikov's Theorem [8] mentioned above G is locally finite (and at the same time locally graded, of course). Therefore in view of Theorem 1 and N.Itô's Theorem [11] the second part of (iv) is valid too. (Recall that according to the last theorem for arbitrary group $G = AB$ $G^{(2)} = 1$ if $A' = B' = 1$). Thus (ii) implies (iv).

It is clear that (iv) implies (i).

The theorem is proven. □

Proof of Theorem 3. Let A be an n-dimensional vector space over a field consisting of two elements: $A = \langle a_1, a_2, ..., a_n \rangle = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus ... \oplus \langle a_n \rangle$, $n > 1$, and let t be a linear operator acting in A, which is represented by the equations: $t(a_i) = a_i + a_{i+1}$ if $i < n$; $t(a_i) = a_i$ if $i = n$ relative to the given basis $\{a_1, a_2, ..., a_n\}$. It is easy to see that the order of t equals 2^m where $m \in \mathbb{N}$. Let u be a linear operator relative to the basis $\{a_1, a_2, ..., a_n\}$ of A given by the equations: $u(a_1) = a_1$, $u(a_i) = t^{1-i}(a_i)$ for $i > 1$.

(i) We shall now prove some relations between t and u.

$$tut = u. \tag{1}$$

For if $i < n$ then $tut(a_i) = tu(a_i + a_{i+1}) = t(u(a_i) + u(a_{i+1})) = t(t^{1-i}(a_i) + t^{-i}(a_{i+1})) = t^{1-i}(t(a_i) + a_{i+1}) = t^{1-i}(a_i + a_{i+1} + a_{i+1}) = t^{1-i}(a_i) = u(a_i)$. If $i = n$ then $tut(a_n) = tu(a_n) = t(t^{1-n}(a_n)) = t^{1-n}(t(a_n)) = t^{1-n}(a_n) = u(a_n)$, as required.

It is easily seen that

$$t^{-1}ut^{-1} = u, \tag{2}$$

$$t^{j}ut^{j} = u, j \in \mathbb{Z}, \tag{3}$$

$$utu = t^{-1}. \tag{4}$$

For if $1 < i \leq n$, then $utu(a_i) = ut(t^{1-i}(a_i)) = ut^{2-i}(a_i) = t^{i-2}(t^{2-i}ut^{2-i})(a_i) = t^{i-2}u(a_i) = t^{i-2}(t^{1-i}(a_i)) = t^{-1}(a_i)$. If $i = 1$ then $utu(a_1) = ut(a_1) = t^{-1}(tut(a_1)) = t^{-1}(u(a_1)) = t^{-1}(a_1)$. Also

$$u^2 = 1 \tag{5}$$

For on one hand $utut = u(tut) = uu = u^2$. On the other hand $utut = (utu)t = t^{-1}t = 1$.

(ii) Let $T = \langle t \rangle$ and let $U = \langle u \rangle$, $F = T \leftthreetimes U$. The group F is isomorphic to the dihedral group and is a subgroup of a full linear group of non-singular operators acting in space A.

(iii) Let $k \in \mathbb{N}$, $1 \leq k \leq n$, and let A_k be a span of $\{a_1, a_2, ..., a_k\}$. We show next that *if $r, i \in \mathbb{N}$ and $r + i < n$, then $t^r(a_i) \in A_{r+i}$*. We proceed by induction on r. If $r = 1$, $i + 1 < n$ then $t(a_i) = a_i + a_{i+1} \in A_{i+1}$. Now suppose $t^r(a_i) \in A_{r+i}$, that is $t^r(a_i) = \sum_{j=1}^{r+i} i_j a_j$. Hence $t^{r+1}(a_i) = t(t^r(a_i)) = t(\sum_{j=1}^{r+i} i_j a_j) = \sum_{j=1}^{r+i} i_j t(a_j)$. Since $t(a_j) \in A_{j+1}$ and $1 \leq j \leq r + i < n$, for any j, $t(a_j) = a_j + a_{j+1} \in A_{r+j+1}$, $1 < j + 1 \leq r + j + 1$. This implies $t^{r+1}(a_i) = \sum_{j=1}^{r+i} i_j t(a_j) \in A_{r+i+1}$, as required.

(iv) Let $k \in \mathbb{N}$, $k < n$. *The subspace A_k is invariant relative to linear operator $t^{k-1}u$.* For if $k = 1$ the result is evident since $t^{k-1}u = u$, $A_1 = \langle a_1 \rangle$, $u(a_1) = a_1$. Let $k > 1$; $i \leq k$. Then $t^{k-1}u(a_i) = t^{k-1}(t^{1-i}(a_i)) = t^{k-i}(a_i)$. Because of (iii) $t^{k-i}(a_i) \in A_{k-i+i} = A_k$. Thus $t^{k-1}u(a_i) \in A_k$ and therefore the images of all basis vectors $\{a_1, a_2, ..., a_k\}$ of A_k belong to A_k and hence A_k is invariant relative to the operator $t^{k-1}u$.

(v) In what follows we shall construct an example of a group with the nonabelian derived subgroup such that every subgroup containing the given cyclic subgroup has a complement in the group.

Let A be an elementary abelian group of rank n: $A = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle$ and let $G = A \leftthreetimes F$ where F is a dihedral group defined in (ii). The group G is generated by $a_1, a_2, ..., a_n, t, u$ with the following relations: $a_i^2 = 1$, $i = 1, 2, ..., n$; $u^2 = 1$, $t^{s(m)} = 1$, $s(m) = 2^m$, $a_i a_j = a_j a_i$; $t^{-1}a_n t = a_n$, $t^{-1}a_i t = a_i a_{i+1}$, $i < n$; $ua_1u = a_1$, $ua_iu = t^{i-1}a_i t^{1-i}$, $i > 1$; $utu = t^{-1}$. The subgroup A corresponds to the vector space A considered above while the action of the linear operators t and u on the vector space corresponds to the conjugation of elements of A by t and u. Obviously, $|G| = |A||F| = 2^n|T||U| = 2^n 2^m 2 = 2^{m+n+1}$.

Let $D = A \leftthreetimes U$. Then it is clear that $G = TD$ and $T \cap D = 1$.

(vi) We show next that every $H \leq G$ such that $H \supseteq T$ has a complement in G. If $u \in H$ then since $t \in H$, we obtain $F \subseteq H$, hence by Lemma 1 H has a complement in G. If $u \notin H$, $H \cap D \nsubseteq A$ then since $|D : A| = 2$, obviously, $D = A(H \cap D)$. Since A is elementary abelian, evidently, for some $C \leq A$, $D = C(H \cap D)$ and $C \cap (H \cap D) = 1$. Consequently, because of $G = HD$ it is clear that C is a complement to H in G. The consideration of $u \notin H$, $H \cap D \subseteq A$ requires to establish the following previous result.

(vii) Let $V_k \leq A$ be generated by $\{a_{k+1}, a_{k+2}, ..., a_n\}$. In particular $V_0 = A$, $V_n = \{1\}$. *If B is invariant relative to t subgroup of A then $B = V_k$ for some k.*

Show first that $[B,t] = \{[b,t] | \ b \in B\}$ is an invariant relative to the element t subgroup of A. Since A is abelian and invariant relative to t for arbitrary b_1, $b_2 \in B \subset A$, it follows $[b_1,t][b_2,t] = [b_1b_2,t] \in [B,t]$. If $b \in B$ then $[b,t][b,t] = [b^2,t] = [1,t] = 1$. Consequently $[b,t]^{-1} \in [B,t]$. Hence $[B,t] \leq A$. If $b_3 \in [B,t]$ then $b_3 \in B$, $[b_3,t] \in [B,t]$ and $[B,t]$ is invariant relative to t.

Suppose there exists invariant relative to the element t $B \leq A$ such that $B \neq V_k$. Let B be such a group of the least order. Since $B \neq V_n, B \neq V_{n-1}$ there exists $x \in B$, $x = a_i \prod_{j>i} a_j^{\varepsilon(j)}$, $\varepsilon(j) \in \{0,1\}$, $i < n$, where i is the least index of all possible indices for elements in B. As we have already proved $[B,t]$ is invariant relative to t and since G is nilpotent, $[B,t] < B$. By choice of B, $[B,t] = V_k$ for some k. Consider $[x,t] = [a_i \prod_{j>i} a_j^{\varepsilon(j)}] = [a_i,t] \prod_{j>i} [a_j^{\varepsilon(j)},t] = a_{i+1} \prod_{j>i} a_{j+1}^{\varepsilon(j)} \in [B,t]$. Therefore $[B,t] \nsubseteq V_k$ for $k > i$. Hence $[B,t] = V_r$ for $r \leq i$. Since the index i is the least, $B \subset V_{i-1}$ and by the identity obtained $[B,t] \subset V_i$. Since $[B,t] = V_r$ when $r \leq i$, we have $[B,t] = V_i$. Hence it follows that $\prod_{j>i} a_j^{\varepsilon(j)} \in [B,t] \subset B$ and therefore $a_i = x(\prod_{j>i} a_j^{\varepsilon(j)})^{-1} \in B$. Thus $B \supset [B,t] = V_i$ and $a_i \in B$. Consequently $B \supset V_{i-1}$. Therefore $B = V_{i-1}$. Thus any invariant relative to t subgroup of A coincides with some V_k.

(viii) Consider the case $H \cap D \subseteq A$. Let $H \cap D = H \cap A = B$. Since $A \lhd G$ by [7, Lemma 1.8] $H = B \leftthreetimes T$. By (vii) $B = V_k$ for some k.

Consider $L = A_k \leftthreetimes \langle t^{k-1}u \rangle$ and show that L is complementary to H in G. Indeed this is so, for if $y \in H \cap L$ then $y = (t^{k-1}u)^\sigma b, b \in A_k$. If $\sigma = 0$ then $y \in H \cap A = B = V_k$. On the other hand $y \in L = A_k \leftthreetimes \langle t^{k-1}u \rangle$. Consequently $y \in V_k \cap L = 1, y = 1$. If $\sigma = 1$ then $y = t^{k-1}ub \in H$. Since $t \in H$, we have $ub \in H \cap D \subseteq A$. Therefore $u \in A$. Contradiction. This implies in any case $H \cap L = 1$. Furthermore $|HL| = |H||L| = |BT||A_k \langle t^{k-1}u \rangle| = |B||T||A_k||\langle t^{k-1}u \rangle| = 2^{n-k}2^m 2^k 2 = 2^{n+m+1} = |G|$. Thus $G = HL$ and $H \cap L = 1$. Thus in any case the subgroup H of G *does* have a complement in G.

(ix) Let $W = G'$. Since $[u,t] = ut^{-1}ut = (utu)^{-1}t = (t^{-1})^{-1}t = t^2$, it follows that $t^2 \in W$. Furthermore $[t, a_i] = a_{i+1}$ when $1 \leq i \leq n-1$. Therefore $V_1 \subset W$ which implies $W = V_1 \leftthreetimes \langle t^2 \rangle$. Consider the lower central series of W: $W = W_0 \supset W_1 \supset W_2 \supset \ldots \supset W_s \supset W_{s+1} = 1$. Since $t^2 \in W$ and $[t^2, a_i] = t^{-2}a_i t^2 a_i = t^{-1}(t^{-1}a_i t)t a_i = t^{-1}a_i a_{i+1} t a_i = a_i a_{i+1} a_{i+1} a_{i+2} a_i = a_{i+2}$; $2 \leq i \leq n-2$ we have $a_j \in W_1, j \geq 4$. Thus $V_3 \subset W_1$. Similarly $V_5 \subset W_2$, $V_7 \subset W_3, \ldots, V_{2k+1} \subset W_k$. Hence for $k < [\frac{n}{2}]$ $W_k \neq 1$. Therefore G' can be nilpotent of an arbitrary large class. □

Remark. In fact, in the proof of Theorem 3 for an arbitrary $n \in \mathbb{N}$ we have constructed the concrete group G such that for some its subgroup $T = \langle t \rangle$ every subgroup $H \supseteq T$ has a complement in G and G' is nilpotent of class $[\frac{n}{2}]$.

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