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A HYPERCYCLIC COMPOSITION OPERATOR ON A HILBERT SPACE OF ENTIRE FUNCTIONS

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Let E be a separable Hilbert space. We consider a special Hilbert space of entire functions on E, $\mathcal{H}^2(E)$, and show that the operator of composition with translation $x \mapsto x+a$ is hypercyclic in $\mathcal{H}^2(E)$.

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Пусть $\mathcal{H}^2(E)$ — гильбертово пространство целых аналитических функций на сепарабельном гильбертовом пространстве E. Показано, что оператор композиции со сдвигом $x \mapsto x + a$ гиперциклический в пространстве $\mathcal{H}^2(E)$.

Let X be a Fréchet linear space. An operator $T\colon X \to X$ is called *hypercyclic* if there is a vector $x \in X$ whose *orbit* under T,

$$\operatorname{Orb}(T, x) = \{x, Tx, T^2x, \dots\},$$

is dense in X. Every such vector x is called *hypercyclic* for T. It is well known that a hypercyclic operator can exist only in separable infinite-dimensional spaces (see [6]). However, every separable infinite-dimensional Fréchet space admits a hypercyclic operator [3].

The investigation of hypercyclic operators has relation to invariant subspaces problem. It is easy to check that if every nonzero vector of X is hypercyclic for T, then T has no closed invariant subsets, and so no closed invariant subspaces as well. In his paper [10], Read shows that there exists a continuous linear operator on ℓ_1 for which every nonzero vector is hypercyclic. It is still an open problem whether there exists a linear continuous operator on a separable Hilbert space without closed invariant subspaces.

The study of hypercyclic operators started after Birkhoff result [2] that the operator of composition with translation $x \mapsto x + a$, $a \neq 0$, $T_a\colon f(x) \mapsto f(x + a)$ is hypercyclic in the space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$. Aron and Bès in [1] proved that the operator of composition with translation T_a is hypercyclic in the space of weakly continuous analytic functions on all bounded subsets of a separable Banach space X which are bounded on bounded subsets. In [4] Chan and Shapiro show that T_a is hypercyclic in various Hilbert spaces of entire functions on $\mathbb{C}$. More detaily, they considered Hilbert spaces

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of entire functions of one complex variable $f(z) = \sum_{n=1}^{\infty} f_n z^n$ with the norms $\|f\|_{2,\gamma}^2 = \sum_{n=1}^{\infty} \gamma_n^{-2}|f_n|^2$ for appropriated sequence of positive numbers and showed that if $n\gamma_n/\gamma_{n-1}$ is monotonically decreasing, then T_a is hypercyclic. The purpose of this paper is to show that the operator of composition with translation is hypercyclic in a special Hilbert space of analytic functions on a separable Hilbert space.

Further results on hypercyclic operators are described in [6]. For background on analytic functions on Banach spaces we refer the reader to [5].

Let E be a separable complex Hilbert space with inner product $(\cdot \mid \cdot)$ and an orthonormal basis (e_k). For every $x_1, \dots, x_n \in E$ let $\underbrace{x_1 \otimes \dots \otimes x_n}_{n}$ be a tensor product and

$$x_1 \otimes_s \dots \otimes_s x_n := \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)},$$

where S_n is the group of permutations on the set $\{1, \dots, n\}$.

Let $\otimes_s^n E$ be an n-symmetric Euclidian tensor product of E. It means that $\otimes_s^n E$ is an Euclidian space and the vectors $e_{[i]}^{(k)} := e_{i_1}^{k_1} \otimes_s \dots \otimes_s e_{i_n}^{k_n}$ form an orthogonal basis in $\otimes_s^n E$. Here we suppose that $[i]$ is an ordered multi-index, that is $i_1 < \dots < i_n$. The norm on $\otimes_s^n E$ can be defined by its value on the basis vectors $e_{[i]}^{(k)}$. Put

$$\left\| e_{[i]}^{(k)} \right\|_\eta^2 = \frac{k_1! \cdots k_n! n!}{(k_1 + \dots + k_n)!} = k_1! \cdots k_n!.$$

Let $\otimes_s^n E_\eta$ be the completion of $\otimes_s^n E$ with respect to $\|\cdot\|_\eta$. Denote by E^∞ the ℓ_2-sum of $\otimes_s^n E_\eta$, that is E^∞ is the direct orthogonal sum of Hilbert spaces $\otimes_s^n E_\eta$, $n = 0, \dots, \infty$, $\otimes_s^0 E := \mathbb{C}$. Let $\eta \colon E \to E^\infty$ be the map

$$\eta(x) = 1 + x + \frac{1}{2!} x^2 + \dots + \frac{1}{n!} x^n + \cdots ,$$

$x \in E$ and $\langle \cdot \mid \cdot \rangle$ be the inner product in E^∞ and x^n denotes the tensor power $\underbrace{x \otimes \dots \otimes x}_{n}$. Note, that E^∞ is a realization of the symmetric Fock space.

Let us consider functions on E, defined by

$$f_w(x) = \langle \eta(x) \mid w \rangle, \tag{1}$$

for any vector $w \in E^\infty$. It is possible to check (cf. [7]) that $f_w(x)$ is an analytic function on E^∞ for every $w \in E^\infty$ and, in particular,

$$\left\langle \eta(x) \mid e_{(i)}^{(k)} \right\rangle = x_{i_1}^{k_1} \dots x_{i_n}^{k_n}, \tag{2}$$

where $(x_j)_{j=1}^\infty$ are coordinates of x in E. Note, that for every $x, u \in E$,

$$\langle \eta(x) \mid \eta(u) \rangle = \sum_{n=0}^{\infty} \frac{\langle x^n \mid u^n \rangle}{(n!)^2} =$$

$$= \sum_{k_1 + \dots + k_n = n} \frac{1}{n!} \left(\frac{n!}{k_1! \dots k_n!} \right)^2 \|e_{i_1}^{k_1} \dots e_{i_n}^{k_n}\|^2 x_{i_1}^{k_1} \dots x_{i_n}^{k_n} u_{i_1}^{k_1} \dots u_{i_n}^{k_n} = \tag{3}$$

$$= \sum_{n=0}^{\infty} \frac{n!}{k_1! \dots k_n!} x_{[i]}^{(k)} u_{[i]}^{(k)} = \sum_{n=0}^{\infty} \frac{(x \mid u)^n}{n!} = e^{(x|u)}$$

and thus $\|\eta(x)\|^2 = e^{\|x\|^2}$.

Let $\mathcal{H}^2(E)$ denote the space of analytic functions which are defined by formula (1) with the norm $\|f_w\| := \|w\|_\eta$. From (2) it follows that the linear span of the range of $\eta(x)$ is dense in E^∞ and so every functional $\langle \cdot \mid w \rangle \in (E^\infty)^*$ generates an analytic function from $\mathcal{H}^2(E)$ by formula (2). Hence $\mathcal{H}^2(E) = (E^\infty)^*$. Notice that $K(x,u) := \langle \eta(x), \eta(u) \rangle$ is a reproducing kernel in means [8, p. 3] and according to [9], the space $\mathcal{H}^2(E)$ can be constructed as an infinite tensor product of

$$\mathcal{H}^2(\mathbb{C}) := \left\{ f \in H(\mathbb{C}) \ : \ \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dz < \infty \right\},$$

where dz is the Lebesgue measure on $\mathbb{C}$.

Let $w = \sum_{n=0}^{\infty} w_n$, $w_n \in \otimes_s^n E$, then $f_n(x) = \langle \eta(x) \mid w_n \rangle = \frac{1}{n!} \langle x^n \mid w_n \rangle$ is an n-homogeneous polynomial on E and the series

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

is the Taylor series expansion of f. So

$$\|f\|^2 = \sum_{n=0}^{\infty} \|f_n\|^2 := \sum_{n=0}^{\infty} \|w_n\|_\eta^2 < \infty.$$

Given $a \in E$, the operator $T_a \colon \mathcal{H}^2(E) \to \mathcal{H}^2(E)$ be defined as

$$T_a(f)(x) = f(x+a) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n f(x) a,$$

where $d^n f(x) a$ is the n-th order Fréchet directional derivative of f at a.

Theorem 1. *Let E be a separable Hilbert space and $0 \neq a \in E$. Then the operator*

$$T_a \colon \mathcal{H}^2(E) \to \mathcal{H}^2(E), \quad f \longrightarrow f(x+a),$$

is hypercyclic.

Our proof involves a technique of the so-called Hypercyclicity Criterion (see e.g. [6]). It may be stated as follows:

Theorem 2 [6]. *Let X be a complete, linear metric space, and $T \colon X \to X$ be linear, continuous. Suppose that there exist dense subsets X_0, Y_0 of X, a sequence (n_k) of positive integers, and a sequence of mappings (possibly nonlinear, possibly not continuous) $S_{n_k} \colon Y_0 \to X$ such that*

i) $T^{n_k} \to 0$, $k \to \infty$ pointwise on X_0.

ii) $S_{n_k} \to 0$, $k \to \infty$ pointwise on Y_0.

iii) $T^{n_k}S_{n_k} = \text{Identity}$ *on* Y_0.

Then T *is hypercyclic.*

Note that it is still an open problem whether every hypercyclic operator satisfies the Hypercyclicity Criterion. We are going to show that the operator T_a on $\mathcal{H}^2(E)$ satisfies the Hypercyclicity Criterion.

We will make use of the following two lemmas.

Lemma 1. *$\mathcal{B} = \{e^{\varphi} : \varphi \in E^*\}$ is a linearly independent subset of $\mathcal{H}^2(E)$.*

Proof. It is easy to see that every function $e^{\phi(x)}$, $\phi \in E^*$, is weakly continuous on bounded sets because it is a composition of two weakly continuous functions: a linear functional ϕ and e^t. For this case the lemma is proved in [1]. □

Lemma 2. *Let U be a non-empty open subset of E^*. Then*

$$S = \text{span}\,\{e^{\varphi} : \varphi \in U\}$$

is dense in $\mathcal{H}^2(E)$.

Proof. Since $e^{\phi(x)} = e^{(x|u)}$ for some $u \in E$, the proof immediately follows from formula (3) and the definition of $\mathcal{H}^2(E)$. □

Proof of Theorem 1. Let a be a fixed vector from E and $\varphi = (\cdot \mid u_{\varphi}) \in E^*$ for some $u_{\varphi} \in E$. Consider the function $g : E^* \to \mathbb{C}$ defined by

$$g(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n!} \varphi^n(a) = e^{(a|u_{\varphi})}.$$

It is clear that $g : E^* \to \mathbb{C}$ is continuous and non-constant. Therefore the sets $U := \{\varphi \in E^* : \|e^{(a|u_{\varphi})}\| = \|g(\varphi)\| < 1\}$, $V := \{\varphi \in E^* : \|e^{(a|u_{\varphi})}\| = \|g(\varphi)\| > 1\}$ are both open and non-empty. Hence, according to Lemma 2,

$$X_0 = \text{span}\,\{e^{\varphi} : \varphi \in U\} \tag{4}$$

$$Y_0 = \text{span}\,\{e^{\varphi} : \varphi \in V\} \tag{5}$$

are both dense subspaces of $\mathcal{H}^2(E)$.

Given $\varphi \in E^*$,

$$T(e^{\varphi}) = \sum_{n=o}^{\infty} \frac{1}{n!} d^n(e^{\varphi})a = \sum_{n=0}^{\infty} \frac{1}{n!} \varphi^n(a) e^{\varphi} = g(\varphi) e^{\varphi} = e^{(a|u_{\varphi})} e^{\varphi}.$$

By (4), $T^n \to 0,\ n \to \infty,$ pointwise on X_0.

By Lemma 1 there exists also a linear map $S : Y_0 \to Y_0$ determined by

$$S(e^{\varphi}) = [g(\varphi)]^{-1} e^{\varphi} \tag{6}$$

which by (5) and (6) satisfies $S^n \to 0,\ n \to \infty,$ pointwise on Y_0 and $TS = \text{id}_{Y_0}$ on Y_0.

By Theorem 2, T is hypercyclic. □

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