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THE DIRECT LIMIT OF METRIZABLE ANR'S IS AN ANR FOR STRATIFIABLE SPACES

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It is proven that the direct limit $\varinjlim X_n$ of a sequence $X_1 \subset X_2 \subset \dots$ metrizable A(N)R's is an A(N)R for stratifiable spaces.

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Доказано, что прямой предел $\varinjlim X_n$ последовательности $X_1 \subset X_2 \subset \dots$ метризуемых A(N)R-ов является A(N)R-ом для стратифицируемых пространств.

The theory of absolute extensors is one of the principal tools in infinite-dimensional topology. Initially the infinite-dimensional topology studied manifolds modeled on nice metrizable spaces like the separable Hilbert space ℓ_2 or the Hilbert cube $Q = [0,1]^\omega$. But later its methods were used for studying infinite-dimensional manifolds modeled on certain non-metrizable model spaces like $\mathbb{R}^\infty$ or Q^∞, see [4], [18].

Given a pointed topological space $(M, *)$, by M^∞ we denote the set

$$\{(x_i)_{i\in\omega} \in M^\omega : x_i = * \text{ for all but finitely many indices } i\}$$

endowed with the strongest topology inducing the original (product) topology on each n-power $M^n = \{(x_i)_{i\in\omega} : x_i = * \text{ for all } i \ge n\}$. It should be mentioned that for a homogeneous space M (like $\mathbb{R}$ or Q) the topological type of the space M^∞ is independent of the choice of the fixed point $* \in M$.

The topology of the space M^∞ is an example of a so-called inductive topology. We shall say that a topological space X carries the inductive topology with respect to a cover $\mathcal{C}$ of X if the topology of X is the strongest topology inducing the original topology on each space $C \in \mathcal{C}$. In other words, a subset $U \subset X$ is open (closed) in X if and only if for each $C \in \mathcal{C}$ the intersection $U \cap C$ is open (closed) in C. For example, a space X is a k-space if it carries the inductive topology with respect to the cover of X by compacta. A topological space X will be called an $\mathcal{M}_\omega$-space (resp. k_ω-space) if it carries the inductive topology with respect to some countable cover by closed metrizable (resp. compact) subspaces. It can be shown that each $\mathcal{M}_\omega$-space carries the inductive topology with respect to an increasing cover $\{X_n\}_{n\in\omega}$

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by closed metrizable subspaces of X. In this case we shall say that X is *the direct limit* of the sequence $X_1 \subset X_2 \subset \dots$ and write $X = \varinjlim X_n$.

In particular, the spaces $\mathbb{R}^\infty$ and Q^∞ are k_ω and $\mathcal{M}_\omega$. It is well known that $\mathbb{R}^\infty$ equipped with its natural linear structure is a locally convex linear topological space while Q^∞ is homeomorphic to a locally convex space (for example, to ℓ_2 carrying the bounded-weak topology). In fact, according to [1], [2] (see also [3]) each infinite-dimensional locally convex $\mathcal{M}_\omega$-space is homeomorphic either to $\mathbb{R}^\infty$ or to Q^∞. In this situation we can apply the Dugundji Theorem [11] and conclude that the spaces $\mathbb{R}^\infty$ and Q^∞ are absolute extensors for metrizable spaces.

We recall that a topological space X is an *absolute (neighborhood) extensor* for a class $\mathcal{C}$ of topological spaces (briefly X is an A(N)R[$\mathcal{C}$]) if each continuous map $f\colon B \to X$ defined on a closed subspace B of a space $C \in \mathcal{C}$ admits a continuous extension on the whole C (on a neighborhood of B in C). A topological space X is called an *absolute (neighborhood) retract* for a class $\mathcal{C}$ (briefly X is an A(N)R[$\mathcal{C}$]) if $X \in \mathcal{C}$ and X is a (neighborhood) retract in each space $Y \in \mathcal{C}$ containing X as a closed subspace. It is clear that a space $X \in \mathcal{C}$ is an A(N)R[$\mathcal{C}$] provided X is an A(N)E[$\mathcal{C}$]. The converse is true for some classes $\mathcal{C}$, in particular for the class $\mathcal{M}$ of metrizable spaces, see [15].

The Dugundji Theorem mentioned above asserts that each convex subset X of a locally convex linear topological space is an AE($\mathcal{M}$). This theorem was generalized by C. Borges [6] who proved that such an X is an absolute extensor for the class $\mathcal{S}$ of stratifiable spaces. The class $\mathcal{S}$ of stratifiable spaces contains the class $\mathcal{M}$ of all metrizable spaces and (unlike to $\mathcal{M}$) is closed with respect to many countable topological operations, see [6], [9], [13, §5]. In particular, a space is stratifiable if it carries the inductive topology with respect to a countable closed cover by stratifiable subspaces. This implies that each $\mathcal{M}_\omega$-space is stratifiable. Hence the spaces $\mathbb{R}^\infty$ and Q^∞ being homeomorphic to stratifiable locally convex spaces are absolute retracts in the class of stratifiable spaces.

In [17] E. Pentsak started studying infinite-dimensional manifolds modeled on the space ℓ_2^∞, the direct limit of Hilbert spaces. In spite of some similarity in definitions, there is an essential difference between the spaces $\mathbb{R}^\infty$ and ℓ_2^∞: the natural linear operations on ℓ_2^∞ fail to be continuous and thus ℓ_2^∞ is not a linear topological space. Moreover, according to [1], l_2^∞ is homeomorphic to no topological group and no closed convex subset of a linear topological space. In this situation we can apply neither the Dugundji nor Borges Theorems to conclude that l_2^∞ is an absolute extensor for stratifiable spaces.

In this paper we develop an alternative approach allowing us to prove that (inspite of absence of a linear structure) the space l_2^∞ is an absolute extensor for stratifiable spaces. The main result of this note is

Theorem. *An $\mathcal{M}_\omega$-space is an A(N)E[$\mathcal{S}$] if and only if it is an A(N)E[$\mathcal{M}$].*

We shall say that a topological space X is a *direct limit of metrizable A(N)Rs* if X carries the inductive topology with respect to an increasing closed cover $\{X_n\}_{n\in\mathbb{N}}$ by metrizable A(N)R[$\mathcal{M}$]s. According to [16], any such a space X is an A(N)E[$\mathcal{M}$]. This result together with Theorem yields

Corollary. *The direct limit of metrizable A(N)R's is an A(N)R for stratifiable spaces.*

Main Lemma. Recall that an *equiconnected map* on a topological space X is a continuous function $\lambda\colon X \times X \times [0,1] \to X$ such that $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$, and $\lambda(x, x, t) = x$ for every $x, y \in X$, $t \in [0,1]$. A subset U of a space X equipped with an equiconnected function λ is called *λ-convex*, provided $\lambda(U \times U \times [0,1]) \subset U$. An *equiconnected space* is a pair (X, λ) consisting of a topological space X and an equiconnected map λ on X. A topological space X is called *topologically convex* if it admits an equiconnected map λ such that the family of open λ-convex sets forms a base of the topology of X.

Extending the Borges Theorem [6] (asserting that each convex subset of a locally convex space is an AE[$\mathcal{S}$]) R. Cauty proved that each topologically convex space is an AE[$\mathcal{S}$]. This result of R. Cauty will be our main instrument in the proof of the AE-property of direct limits.

For a topological space X, by $\exp(X)$ the set of all closed subsets of X is denoted. A map $\mathcal{F}\colon Z \to \exp(X)$ of a topological space Z is called *upper semi-continuous* if for every open set $U \subset X$ the set $\mathcal{F}^{-1}(U) = \{z \in Z \mid \mathcal{F}(z) \subset U\}$ is open in Z. We will use the following well known fact: if X is a normal topological space and $\mathcal{F}\colon Z \to \exp(X)$ an upper semicontinuous map then for every closed subset $F \subset X$ the map $F \cap \mathcal{F}\colon Z \to \exp(F) \subset \exp(X)$ is upper semi-continuous, see [12].

For a map $\mathcal{F}\colon Z \to \exp(X)$ and a subset $A \subset Z$ let $\mathcal{F}(A) = \bigcup_{a \in A} \mathcal{F}(a)$. A map $\mathcal{F}\colon X \to \exp(X)$ is called *monotone* if $\mathcal{F}(\mathcal{F}(x)) \subset \mathcal{F}(x)$ for every $x \in X$.

Lemma. *Suppose that X is a normal topological space, $\lambda\colon X \times X \times [0,1] \to X$ an equiconnected function, and $\mathcal{F}\colon X \to \exp(X)$ an upper semi-continuous map such that $\lambda(x,y,t) \in \mathcal{F}(x) \cup \mathcal{F}(y)$ for every $(x,y,t) \in X \times X \times [0,1]$. Suppose that $X_1 \subset X_2 \subset \dots$ is a sequence of subsets of X such that for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with $\mathcal{F}(X_n) \subset X_m$. Then*

(1) the set $\bigcup_{n=1}^{\infty} X_n$ is λ-convex;

(2) the function $\lambda\colon \varinjlim X_n \times \varinjlim X_n \times [0,1] \to \varinjlim X_n$ is continuous;

(3) if $\mathcal{F}$ is monotone and X admits a base of open λ-convex subsets, then $\varinjlim X_n$ has a base of open λ-convex sets too.

Proof. The first statement follows trivially from $\lambda(x,y,t) \in \mathcal{F}(x) \cup \mathcal{F}(y)$ and

$$\mathcal{F}\left(\bigcup_{n=1}^{\infty} X_n\right) \subset \bigcup_{m=1}^{\infty} X_m.$$

The proofs of the remaining two statements are a bit more complicated. At first let us make a remark.

Since for every $x \in \varinjlim X_n$ we have $\mathcal{F}(x) \subset X_m$ for some m, it is legal to write $\mathcal{F}\colon \varinjlim X_n \to \exp(\varinjlim X_n)$. Let us show that this map is upper semicontinuous. Fix an open set $U \subset \varinjlim X_n$. Because $\varinjlim X_n$ has the direct limit topology, to show that $\mathcal{F}^{-1}(U)$ is open in $\varinjlim X_n$ it suffices to verify that the intersection $\mathcal{F}^{-1}(U) \cap X_n$ is open in X_n for every n. So, fix n, and find m with $\mathcal{F}(X_n) \subset X_m$. Then $\mathcal{F}^{-1}(U) \cap X_n = \{x \in X_n \mid \mathcal{F}(x) \subset U \cap X_m\}$ is open in X_n because the set $U \cap X_m$ is open in X_m and the restriction $\mathcal{F}|X_n\colon X_n \to \exp(X_m)$ is upper semi-continuous.

To show that the map $\lambda\colon \varinjlim X_n \times \varinjlim X_n \times [0,1] \to \varinjlim X_n$ is continuous, fix a triple $(x_0, y_0, t_0) \in \varinjlim X_n \times \varinjlim X_n \times [0,1]$ and a neighborhood $U \subset \varinjlim X_n$ of $\lambda(x_0, y_0, t_0)$. Find n such that $x_0, y_0 \in X_n$ and m such that $\mathcal{F}(X_n) \subset X_m$. Let $W \subset X$ be a neighborhood of $\lambda(x_0, y_0, t_0)$ such that $\overline{W} \cap X_m \subset U \cap X_m$ (here $\overline{W}$ is the closure of W in X). Observe that the set $\overline{W} \cap \left(\bigcup_{n=1}^{\infty} X_n\right)$ is closed in $\varinjlim X_n$. Then the map $\overline{W} \cap \mathcal{F}\colon \varinjlim X_n \to \exp(\varinjlim X_n)$ is upper semi-continuous. Consequently, the set $V = \{x \in \varinjlim X_n \mid \overline{W} \cap \mathcal{F}(x) \subset U\}$ is an open neighborhood of x_0 and y_0 in $\varinjlim X_n$. Using the continuity of the map $\lambda\colon X\times X\times[0,1]\to X$, we may find neighborhoods $W(x_0),W(y_0)\subset X$, $V(t_0)\subset[0,1]$ of the points x_0,y_0,t_0, respectively, such that $\lambda(W(x_0)\times W(Y_0)\times V(t_0))\subset W$. Let $V(x_0)=V\cap W(x_0)$, $V(y_0)=V\cap W(y_0)$. Evidently, $V(x_0),V(y_0)$ are open neighborhoods of x_0,y_0 in $\varinjlim X_n$. We claim that $\lambda(V(x_0)\times V(y_0)\times V(t_0))\subset U$. Indeed, fix any $(x,y,t)\in V(x_0)\times V(y_0)\times V(t_0)$ and remark that because of $x,y\in V$ we have $\overline{W}\cap\mathcal{F}(x)\subset U$ and $\overline{W}\cap\mathcal{F}(y)\subset U$. On the other hand, because $x\in W(x_0)$, $y\in W(y_0)$, $t\in V(t_0)$, we have $\lambda(x,y,t)\in\overline{W}$. Then, using the fact that $\lambda(x,y,t)\in\mathcal{F}(x)\cup\mathcal{F}(y)$ we get $\lambda(x,y,t)\in\overline{W}\cap(\mathcal{F}(x)\cup\mathcal{F}(y))=(\overline{W}\cap\mathcal{F}(x))\cup(\overline{W}\cap\mathcal{F}(y))\subset U$. Therefore, the map $\lambda\colon\varinjlim X_n\times\varinjlim X_n\times[0,1]\to\varinjlim X_n$ is continuous.

Now suppose that $\mathcal{F}$ is monotone and X has a basis of λ-convex open subsets. Fix any $x_0\in\bigcup_{n=1}^\infty X_n$ and a neighborhood $U\subset\varinjlim X_n$ of x_0. Find n such that $x_0\in X_n$ and m such that $\mathcal{F}(X_n)\subset X_m$. Let W be an open λ-convex neighborhood of x_0 in X such that $\overline{W}\cap X_m\subset U\cap X_m$. Because of the upper semi-continuity of the map $\overline{W}\cap\mathcal{F}\colon\varinjlim X_n\to\exp(\varinjlim X_n)$ the set $V=\{x\in\varinjlim X_n\mid\overline{W}\cap\mathcal{F}(x)\subset U\}$ is open in $\varinjlim X_n$. We claim that the set $O=W\cap V$ is a λ-convex open neighborhood of x_0 in $\varinjlim X_n$ such that $O\subset U$. Since $x=\lambda(x,x,t)\in\mathcal{F}(x)$, we get $x\in\overline{W}\cap\mathcal{F}(x)\subset U$ for every $x\in V\cap W$, and thus $O\subset U$. To prove that O is λ-convex, fix a triple $(x,y,t)\in O\times O\times[0,1]$. Since the set $W\ni x,y$ is λ-convex, $\lambda(x,y,t)\in W$. To see that $\lambda(x,y,t)\in V$, observe that $\mathcal{F}(\lambda(x,y,t))\subset\mathcal{F}(\mathcal{F}(x)\cup\mathcal{F}(y))=\mathcal{F}(\mathcal{F}(x))\cup\mathcal{F}(\mathcal{F}(y))\subset\mathcal{F}(x)\cup\mathcal{F}(y)$. Then $\overline{W}\cap\mathcal{F}(\lambda(x,y,t))\subset\overline{W}\cap(\mathcal{F}(x)\cup\mathcal{F}(y))=(\overline{W}\cap\mathcal{F}(x))\cup(\overline{W}\cap\mathcal{F}(y))\subset U$ and thus $\lambda(x,y,t)\in V$ (by the definition of V). Therefore $O=W\cap V$, being an itersection of two λ-convex sets, is λ-convex. □

Proof of Theorem. For an infinite set A by $l^2(A)=\{(x_a)_{a\in A}\in\mathbb{R}^A\mid\sum_{a\in A}x_a^2<\infty\}$ we denote the standard Hilbert space of density A, endowed with the norm $\|(x_a)_{a\in A}\|=\left(\sum_{a\in A}x_a^2\right)^{1/2}$. For two vectors $x=(x_a)_{a\in A}$ and $y=(y_a)_{a\in A}$ of $l^2(A)$ we write $x\le y$ if $x_a\le y_a$ for every $a\in A$, and set $\min\{x,y\}=(\min\{x_a,y_a\})_{a\in A}\in l^2(A)$.

Let $l^2_+(A)=\{x\in l^2(A)\mid x\ge0\}$ denote the positive cone of $l^2(A)$ and $S_+(A)=\{x\in l^2(A)\mid x\ge0,\ \|x\|=1\}$ its positive unit sphere.

For a subset $A'\subset A$ we identify $l^2_+(A')$ with the subspace $\{(x_a)_{a\in A}\in l^2_+(A)\mid x_a=0$ for $a\notin A'\}$ of $l^2_+(A)$.

Let $X\in\mathcal{M}_\omega$ be an A(N)E[$\mathcal{M}$]-space. Write $X=\varinjlim X_n$, where $X_0\subset X_1\subset X_2\subset\dots$ is a sequence of closed metrizable subsets of X. Applying Hausdorff's Theorem on extending metrics [14], we may find a continuous metric $\rho\le1$ on X which generates the topology of each X_n. According to Proposition 7.1 of [5, Ch.VI], there is an embedding $g\colon(X,\rho)\to S_+(A_0)$ of the metric space (X,ρ) into the positive unit sphere $S_+(A_0)$ of $l^2(A_0)$ for some set A_0. Let $A=A_0\cup\{0\}\cup\mathbb{N}$ and $A_n=A_0\cup\{0,\dots,n\}$ for $n\in\mathbb{N}$. Consider the embedding $f\colon(X,\rho)\to S_+(A)$ defined for $x\in X$ by $f(x)=(f(x)_a)_{a\in A}$, where

$$f(x)_a=\begin{cases}\frac{1}{\sqrt2}g(x)_a, & \text{if } a\in A_0;\\ 2^{-n}\rho(x,X_{n-1}), & \text{if } a=n\in\mathbb{N};\\ \left(\frac12-\sum_{n=1}^\infty(2^{-n}\rho(x,X_{n-1}))^2\right)^{1/2}, & \text{if } a=0.\end{cases}$$

Observe that

$$f(X)\cap l^2_+(A_n)=f(X_n)\ \text{ for every }\ n\in\mathbb{N}. \tag{1}$$

Let $Y = \{y \in l^2_+(A) \mid \exists x \in X \text{ with } 0 \le y \le f(x)\}$. Remark that $Y \cap S_+(A) = f(X)$, thus $f(X)$ is a closed subset in Y. This and (1) imply

$$f(X_n) \text{ is a closed subset in } Y \cap l^2_+(A_n) \text{ for every } n. \tag{2}$$

For $n \in \mathbb{N}$ let $Y_n = Y \cap l^2_+(A_n)$ and remark that $Y = \bigcup_{n=1}^\infty Y_n$ (because of (1)). Denote by τ the direct limit topology $\varinjlim Y_n$ on Y. We will apply Main Lemma and [7, 1.4] in order to show that the space (Y, τ) is an AE[$\mathcal{S}$]. For every $y \in Y$ let $\mathcal{F}(y) = \{x \in l^2_+(A) \mid 0 \le x \le y\}$. One can easily check that each set $\mathcal{F}(y)$ is a compactum lying in Y, and the map $\mathcal{F}\colon Y \to \exp(Y)$ is upper semi-continuous and monotone.

On Y let us consider the equiconnected function $\lambda\colon Y \times Y \times [0,1] \to Y$ defined for $(x, y, t) \in Y \times Y \times [0,1]$ by the formula

$$\lambda(x,y,t) = \begin{cases} (1-2t)\,x + 2t\,\min\{x,y\}, & \text{if } t \le \frac{1}{2}; \\ (2-2t)\,\min\{x,y\} + (2t-1)\,y, & \text{if } t \ge \frac{1}{2}. \end{cases}$$

Evidently, $\lambda(x,y,t) \in \mathcal{F}(x) \cup \mathcal{F}(y)$ for every (x,y,t). Observe also that $\mathcal{F}(Y_n) \subset Y_n$ for every n and that Y admits a base of open λ-convex sets. Thus it is legal to apply Main Lemma to conclude that the equiconnected function $\lambda\colon (Y,\tau) \times (Y,\tau) \times [0,1] \to (Y,\tau)$ is continuous and the space (Y,τ) admits a base of open λ-convex subsets. Finally, applying Theorem 1.4 of [7], we get (Y,τ) is an AE[$\mathcal{S}$].

Now let us return to our initial space X. It follows from (1) and (2) that the map $f\colon X = \varinjlim X_n \to (Y,\tau) = \varinjlim Y_n$ is a closed embedding. Using the fact that X is an A(N)E[$\mathcal{M}$], we may easily prove that X is an A(N)E[$\mathcal{M}_\omega$]. Since $(Y,\tau) \in \mathcal{M}_\omega$, this yields $f(X)$ is a (neighborhood) retract of $(Y,\tau) \in AE[\mathcal{S}]$, and consequently, X is an A(N)E[$\mathcal{S}$].

Concluding remarks and open questions.

1. For stratifiable k_ω-spaces our Theorem was known since R. Cauty [7, 4.2].

2. In light of Theorem, one could ask: Is every stratifiable ANE($\mathcal{M}$)-space an ANE($\mathcal{S}$)? The answer is “no”: any countable space with a unique non-isolated point and without non-trivial convergent sequences can serve as a suitable counterexample.

3. Suppose that $A_1 \subset A_2 \subset \dots$ is a sequence of countable sets with $A_n \ne A_{n+1}$ for every n. It follows from the proof of Lemma that the semilattice operation min is continuous on the direct limit $\varinjlim l^2_+(A_n)$. Applying [10] we conclude that each space $l^2_+(A_n)$ is homeomorphic to the separable Hilbert space l_2. Using this fact, one can easily show that the direct limit $\varinjlim l^2_+(A_n)$ is homeomorphic to the direct limit l^∞_2, considered by Pentsak [17]. Therefore we conclude that the space l^∞_2 supports the structure of a topological semilattice (even Lawson semilattice). This fact is of interest because the topology of the space l^∞_2 is not compatible with many other algebraical structures. For example, l^∞_2 is homeomorphic to no closed convex set in a linear topological space and no topological group (more generally, no closed multiplicative subset (or multiplicative subset with unity) in a topological group), see [1].

Question 1. Is the space l^∞_2 homeomorphic to a topological lattice? to a convex set in a linear topological space?

In connection with the second part of this question let us make the following observation. Denote by $Q = [-1,1]^\omega$ the Hilbert cube and by $s = (-1,1)^\omega$ its pseuso-interior. For $n \in \mathbb{N}$ let $Q_n = (1 - \frac{1}{n})Q = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]^\omega \subset Q$. Let λ denote the equiconnected function on $s \times Q$ generated by the convex structure of $s \times Q$, i.e. $\lambda(x,y,t) = (1-t)x + ty$. Evidently, the set $\bigcup_{n=1}^{\infty} s \times Q_n \subset s \times Q$ is convex. It can be easily shown that the direct limit $\varinjlim s \times Q_n$ is homeomorphic to l_2^∞. We claim that the equiconnected function λ is continuous on $\varinjlim s \times Q_n$. To see this, apply Main Lemma with the map $\mathcal{F}\colon s \times Q \to \exp(s \times Q)$ defined by $\mathcal{F}(x) = \frac{1}{2}x + \frac{1}{2}Q \times Q$. Therefore, l_2^∞ admits a kind of a convex structure, but that, of course, does not answer Question 1.

4. Corollary rises the following

Question 2. Suppose that a stratifiable space X is a direct limit $\varinjlim X_n$ of a sequence $X_1 \subset X_2 \subset \dots$ of closed subsets of X such that each X_n is an A(N)R[$\mathcal{S}$]. Is the space X equiconnected? an A(N)R[$\mathcal{S}$]?

5. Having in mind Question 2 and Main Lemma let us introduce a definition. For a topological space X let 2^X denote the set of all compact subsets of X. We call a space X *compactly equiconnected* if Y admits an equiconnected function $\lambda\colon X \times X \times [0,1] \to X$ and an upper semi-continuous map $\mathcal{F}\colon X \to 2^X$ such that $\lambda(x,y,t) \in \mathcal{F}(x) \cup \mathcal{F}(y)$ for every $(x,y,t) \in X \times X \times [0,1]$. Analysing the proof of Theorem, we see that every metrizable space embeds as a closed subset into a compactly equiconnected metrizable AR.

Question 3. Does every stratifiable space embeds as a closed subset into a compactly equiconnected stratifiable space? Equivalently, is every absolute retract for stratifiable spaces compactly equiconnected?

6. In connection with Theorem and Questions 2 and 3, let us consider the following surprising example. According to [8] there is a strongly countable-dimensional metrizable compactum K such that the free convex set $P_\infty(K)$ over K is not an AE for the class of all metrizable compacta and thus $P_\infty(K)$ is not an AE($\mathcal{S}$). Writing $K = \bigcup_{n=1}^{\infty} K_n$, where $K_1 \subset K_2 \subset \dots$ is a tower of finite-dimensional compacta, we see that the stratifiable convex (and thus equiconnected) space $P_\infty(K)$ which is not an AE($\mathcal{S}$) can be written as $P_\infty(K) = \bigcup_{n=1}^{\infty} P_\infty(K_n)$, where each $P_\infty(K_n) \subset P_\infty(K_{n+1})$ is a closed convex AE($\mathcal{S}$)-subspace of $P_\infty(K)$.

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