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ON THE NEVANLINNA THEORY FOR MEROMORPHIC FUNCTIONS ON ANNULI. I

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An extension of the Nevanlinna value distribution theory for meromorphic functions on annuli is proposed. The main characteristics are one-parameter and possess the same properties as in the classical case. Analogs of the Jensen and the First Fundamental Theorem for annuli are proved.

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Теория распределения значений Неванлинны распространяется на мероморфные в плоских круговых кольцах функции. Введённые характеристики зависят от одного параметра и обладают такими же свойствами, как и в классическом случае. Доказаны аналоги теорем Йенсена и первой основной теоремы теории распределения значений.

1. Introduction. The properties including value distribution of meromorphic functions in multiply connected domains of complex plane $\mathbb{C}$, were studied by many authors [1]-[10]. Here we propose a new approach to the Nevanlinna value distribution theory for meromorphic functions in doubly connected domains. In our extension of this theory the main characteristics of meromorphic functions are one-parameter and possess the same properties as in the classical case of a simply connected domain.

By the Doubly Connected Mapping Theorem [11] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \le r < R \le +\infty$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 < r < R < +\infty$. In the latter case the homothety $z \longmapsto z/\sqrt{rR}$ reduces the given domain to the annulus $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \longmapsto \frac{1}{z}$.

In this paper we prove an analog of the Jensen theorem [12](see also [13]-[16]) for an annulus, introduce the Nevanlinna and Ahlfors-Shimizu characteristics, study their properties and prove the First Fundamental Theorem of the value distribution theory for an annulus.

The Second Fundamental Theorem is the matter of the second part of our paper.

Some classical results of Nevanlinna theory were extended on δ-subharmonic functions [17]. A way to extend some results of this paper on the mentioned functions is unknown for the authors.

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2. Definitions and notations. Let f be a meromorphic function on $A=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$. We use the notations

$$m\left(R,\frac{1}{f-a}\right)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(Re^{i\theta})-a|}d\theta,\ \ m(R,f)=m(R,\infty)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(Re^{i\theta})|d\theta,$$

where $\log^+x=\max\{0,\log x\}$, $a\in\mathbb{C}$ and $\frac{1}{R_0}<R<R_0$. Let

$$m_0\left(R,\frac{1}{f-a}\right)=m\left(R,\frac{1}{f-a}\right)+m\left(\frac{1}{R},\frac{1}{f-a}\right),\quad 1<R<R_0$$

and

$$m_0(R,f)=m(R,f)+m\left(\frac{1}{R},f\right),\quad 1<R<R_0.$$

Put

$$N_1\left(R,\frac{1}{f-a}\right)=\int_{1/R}^{1}\frac{n_1\left(t,\frac{1}{f-a}\right)}{t}dt,\quad N_2\left(R,\frac{1}{f-a}\right)=\int_1^R\frac{n_2\left(t,\frac{1}{f-a}\right)}{t}dt,\quad 1<R<R_0,$$

where $n_1\left(t,\frac{1}{f-a}\right)$ is the counting function of poles of the function $\frac{1}{f(z)-a}$ in $\{z:t<|z|\le 1\}$ and $n_2\left(t,\frac{1}{f-a}\right)$ is the counting function of poles of this function in $\{z:1<|z|\le t\}$. Denote also

$$N_1(R,f)=N_1(R,\infty)=\int_{1/R}^{1}\frac{n_1(t,\infty)}{t}dt,\quad N_2(R,f)=N_2(R,\infty)=\int_1^R\frac{n_2(t,\infty)}{t}dt,$$

where $n_1(t,\infty)$ is the counting function of poles of the function f in $\{z:t<|z|\le 1\}$ and $n_2(t,\infty)$ is its counting function of poles in $\{z:1<|z|\le t\}$. Let

$$N_0\left(R,\frac{1}{f-a}\right)=N_1\left(R,\frac{1}{f-a}\right)+N_2\left(R,\frac{1}{f-a}\right),\quad N_0(R,f)=N_1(R,\infty)+N_2(R,\infty).$$

Definition 1. Let f be a meromorphic function on $A=\{z:\frac{1}{R_0}<|z|<R_0\}$, where $1<R_0\le+\infty$. The function

$$T_0(R,f)=m_0(R,f)-2m(1,f)+N_0(R,f),\quad 1<R<R_0, \tag{1}$$

is called the *Nevanlinna characteristic* of f.

The function $T_0(R,f)$ is nonnegative, continuous, nondecreasing and convex with respect to $\log R$. These facts will be proved below.

Let f be a nonconstant meromorphic function on A. By $F_{1/t}$ and F_t, where $t>1$, denote the pieces of the Riemann surfaces over the Riemann sphere such that $F_{1/t}$ is the range of the restriction of f on the annulus $\{z:\frac{1}{t}<|z|\le 1\}$ and F_t is the range of the restriction of f on the annulus $\{z:1<|z|\le t\}$.

Denote by $A_1(t)$ and $A_2(t)$ respectively the areas of the $F_{1/t}$ and F_t divided by π, the area of the Riemann sphere.

Definition 2. Let f be a nonconstant meromorphic function on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. The function

$$\overset{\circ}{T}_0(R, f) = \int\limits_1^R \frac{A_1(t)}{t} dt + \int\limits_1^R \frac{A_2(t)}{t} dt, \quad 1 < R < R_0,$$

is called the *spherical characteristic* of f.

We will show that the function $\overset{\circ}{T}_0(R, f)$ has the properties like its classical Ahlfors-Shimizu counterpart $\overset{\circ}{T}(R, f)$ ([13],[14]). Namely, $\overset{\circ}{T}_0(R, f)$ is continuous, nonnegative, nondecreasing and convex with respect to $\log R$ for $R > 1$.

Let

$$\overset{\circ}{m}(r, a) = \frac{1}{2\pi} \int\limits_0^{2\pi} \log \frac{1}{k(f(re^{i\theta}), a)} d\theta - \frac{1}{2\pi} \int\limits_0^{2\pi} \log \frac{1}{k(f(e^{i\theta}), a)} d\theta, \quad \frac{1}{R_0} < r < R_0,$$

and

$$\overset{\circ}{m}_0(R, a) = \overset{\circ}{m}(R, a) + \overset{\circ}{m}\left(\frac{1}{R}, a\right), \quad 1 < R < R_0,$$

where $k(w, a)$ is the chordal distance beetween two points w and a on the Riemann sphere.

3. Main results. We prove the following theorems.

Theorem 1. *Let f be a nonidentical zero meromorphic function on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Then*

$$N_0\left(R, \frac{1}{f}\right) - N_0(R, f) =$$
$$= \frac{1}{2\pi} \int\limits_0^{2\pi} \log \left| f\left(\frac{1}{Re^{i\theta}}\right) \right| d\theta + \frac{1}{2\pi} \int\limits_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \frac{1}{\pi} \int\limits_0^{2\pi} \log |f(e^{i\theta})| d\theta \tag{2}$$

for every R such that $1 < R < R_0$.

This is a generalization of Jensen's theorem for annuli.

Theorem 2. *Let f be a nonconstant meromorphic function on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, $1 < R_0 \le +\infty$. Let $T_0(R, f)$ be its Nevanlinna characteristic. Then*

$$T_0\left(R, \frac{1}{f - a}\right) = T_0(R, f) + O(1), \quad 1 < R < R_0,$$

for every fixed $a \in \mathbb{C}$.

This is an analog of the First Fundamental Theorem for annuli.

Theorem 3. *Let f be a nonconstant meromorphic function on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \le +\infty$ and let $\overset{\circ}{T}_0(R, f)$ be the spherical characteristic of the function f. Then*

$$\overset{\circ}{T}_0(R, f) = \overset{\circ}{m}_0(R, a) + N_0\left(R, \frac{1}{f - a}\right) \tag{3}$$

for all R such that $1 < R < R_0$, and for all $a \in \overline{\mathbb{C}}$.

Theorem 4. *Let f be a meromorphic function on $A = \{z : 0 < |z| < +\infty\}$ and let $T_0(R, f)$ be its Nevanlinna characteristic. Then*

$$\liminf_{R\to+\infty} \frac{T_0(R, f)}{\log R} < +\infty \tag{4}$$

if and only if the function f has a continuation to a rational function in $\mathbb{C}$.

4. Connections with the classical characteristics of meromorphic in $\mathbb{C}$ functions. If the function f is meromorphic in $\mathbb{C}$ and $T(R, f)$ is its classical Nevanlinna characteristic, then we have

(a) $$N_0(R, f) = N(R, f) + N\left(\frac{1}{R}, f\right) - 2N(1, f) \quad \text{for } R > 1,$$

where

$$N(R, f) = \int_0^R \frac{n(t, f) - n(0, f)}{t} dt + n(0, f)\log R$$

and $n(t, f)$ is the counting function of poles of the function f in $\{z : |z| \le t\}$,

(b) $$T_0(R, f) = T(R, f) + T\left(\frac{1}{R}, f\right) - 2T(1, f) \quad \text{for } R > 1,$$

(c) $$T(R, f) - 2T(1, f) \le T_0(R, f) \le T(R, f).$$

Indeed, suppose that $f(0) \ne \infty$. We have $n_1(t, f) = n(1, f) - n(t, f)$, $0 < t < 1$ and $n_2(t, f) = n(t, f) - n(1, f)$, $t > 1$. Then

$$N_0(R, f) = \int_{1/R}^1 \frac{n(1, f) - n(t, f)}{t} dt + \int_1^R \frac{n(t, f) - n(1, f)}{t} dt = \int_{1/R}^1 \frac{n(1, f)}{t} dt - \int_{1/R}^1 \frac{n(t, f)}{t} dt +$$

$$+ \int_1^R \frac{n(t, f)}{t} dt - \int_1^R \frac{n(1, f)}{t} dt = n(1, f)\log R - \int_0^1 \frac{n(t, f)}{t} dt + \int_0^{1/R} \frac{n(t, f)}{t} dt + \int_0^R \frac{n(t, f)}{t} dt -$$

$$- \int_0^1 \frac{n(t, f)}{t} dt - n(1, f)\log R = N(R, f) + N\left(\frac{1}{R}, f\right) - 2N(1, f).$$

The case when the function f has the pole at the origin can be proved similarly. Since $T(R, f) = m(R, f) + N(R, f)$, relation (b) follows immediately from (a).

5. Proof of Theorem 1. Let f be a nonidentical zero meromorphic function on A. We denote

$$A^t = \{z : 1 < |z| < t\}, \quad 1 < t < R_0,$$

$$A_\tau = \{z : \tau < |z| < 1\}, \quad \frac{1}{R_0} < \tau < 1.$$

Suppose that the function f has neither zeroes nor poles on $\partial A^t \cup \partial A_\tau$. By the argument principle,

$$\frac{1}{2\pi i}\int\limits_{\partial A^t}\frac{f'(z)}{f(z)}dz = n_2\left(t,\frac{1}{f}\right) - n_2(t,f), \quad \frac{1}{2\pi i}\int\limits_{\partial A_\tau}\frac{f'(z)}{f(z)}dz = n_1\left(\tau,\frac{1}{f}\right) - n_1(\tau,f).$$

In the case when the function f has either zeroes or poles on $\{z : |z| = 1\}$ consider the domains

$$A^t_{(\varepsilon)} = A^t \cap \left(\bigcup_{|z_j|=1}\{z : |z - z_j| > \varepsilon\}\right), \quad A^{(\varepsilon)}_\tau = A_\tau \cap \left(\bigcup_{|z_j|=1}\{z : |z - z_j| > \varepsilon\}\right),$$

where z_j are either zeroes or poles of f on $\{z : |z| = 1\}$, ε is positive, does not exceed $\min\{t-1, 1-\tau\}$ and such that the discs of radius ε centered at z_j contain neither zeroes nor poles of f except z_j.

Applying the argument principle to $A^t_{(\varepsilon)}$ and $A^{(\varepsilon)}_\tau$, we obtain

$$\frac{1}{2\pi i}\int\limits_{\partial A^t_{(\varepsilon)}}\frac{f'(z)}{f(z)}dz = n_2\left(t,\frac{1}{f}\right) - n_2(t,f), \quad \frac{1}{2\pi i}\int\limits_{\partial A^{(\varepsilon)}_\tau}\frac{f'(z)}{f(z)}dz = n^{(\varepsilon)}_1\left(\tau,\frac{1}{f}\right) - n^{(\varepsilon)}_1(\tau,f), \tag{5}$$

where $n^{(\varepsilon)}_1(\tau, \frac{1}{f-a})$ denotes the counting function of a-points of f in the domain $A^{(\varepsilon)}_\tau$.

Since $\partial A^t_{(\varepsilon)} = \{z : |z| = t\} \cup \{z : |z| = 1,\ |z-z_j| > \varepsilon\} \cup \left(\bigcup_{|z_j|=1}\{z : 1 \le |z|,\ |z - z_j| = \varepsilon\}\right)$, taking into consideration the directions in which the integrals are taken, we have

$$\int\limits_{\partial A^t_{(\varepsilon)}}\frac{f'(z)}{f(z)}dz = \int\limits_{\{|z|=t\}}\frac{f'(z)}{f(z)}dz - \int\limits_{\{|z|=1,\ |z-z_j|>\varepsilon\}}\frac{f'(z)}{f(z)}dz - \int\limits_{\bigcup\limits_{|z_j|=1}\{z:1\le|z|,\ |z-z_j|=\varepsilon\}}\frac{f'(z)}{f(z)}dz. \tag{6}$$

The latter integral of (6) is equal to $\pi i(n^1\left(\frac{1}{f}\right) - n^1(f)) + o(1)$, $\varepsilon \to 0$, where $n^1(0)$ and $n^1(\infty)$ are the numbers of zeroes and poles of the function f on $\{z : |z| = 1\}$ respectively. Thus,

$$\lim_{\varepsilon\to 0}\int\limits_{\bigcup\limits_{|z_j|=1}\{1\le|z|,\ |z-z_j|=\varepsilon\}}\frac{f'(z)}{f(z)}dz = \pi i\left(n^1\left(\frac{1}{f}\right) - n^1(f)\right). \tag{7}$$

We deduce from (5), (6) and (7) that there exists

$$\lim_{\varepsilon\to 0}\int\limits_{\{|z|=1,\ |z-z_j|>\varepsilon\}}\frac{f'(z)}{f(z)}dz = \text{v. p.}\int\limits_{\{|z|=1\}}\frac{f'(z)}{f(z)}dz.$$

Therefore, if $\varepsilon \to 0$ the first part of (5) gives

$$n_2\left(t,\frac{1}{f}\right) - n_2(t,f) + \frac{1}{2}\left(n^1\left(\frac{1}{f}\right) - n^1(f)\right) = \frac{1}{2\pi i}\int\limits_{\{|z|=t\}}\frac{f'(z)}{f(z)}dz - \text{v. p.}\frac{1}{2\pi i}\int\limits_{\{|z|=1\}}\frac{f'(z)}{f(z)}dz. \tag{8}$$

Similarly, its second part gives

$$n_1\left(\tau,\frac{1}{f}\right)-n_1(\tau,f)-\frac{1}{2}\left(n^1\left(\frac{1}{f}\right)-n^1(f)\right)=\text{v. p.}\frac{1}{2\pi i}\int\limits_{\{|z|=1\}}\frac{f'(z)}{f(z)}dz-\\ -\frac{1}{2\pi i}\int\limits_{\{|z|=\tau\}}\frac{f'(z)}{f(z)}dz. \tag{9}$$

Let $f(z_0)=1$, $|z_0|=1$. We define in $\overline{A^R}$ with radial slits, the beginnings of which are z_0, zeroes and poles of f in A^R, the function $\log f(z)$ as follows. Let $\log f(z_0)$ be determined. Put

$$\log f(z)=\log f(z_0)+\int\limits_{z_0}^{z}\frac{f'(\zeta)}{f(\zeta)}d\zeta,$$

where the integral is taken along a path joining z_0 and z in the domain mentioned above. In this way the function $\log f(z)$ is determined in $\overline{A^R}$ except zeroes and poles of f in $\overline{A^R}$ as well as the made slits.

Dividing (8) on t and integrating over t from 1 to R, we obtain

$$\int\limits_1^R\frac{n_2\left(t,\frac{1}{f}\right)}{t}dt-\int\limits_1^R\frac{n_2(t,f)}{t}dt+\frac{1}{2}\left(n^1\left(\frac{1}{f}\right)-n^1(f)\right)\log R=\\ =\frac{1}{2\pi i}\int\limits_1^R\left(\int\limits_{\{|z|=t\}}\frac{f'(z)}{f(z)}dz\right)\frac{dt}{t}-\frac{1}{2\pi i}\log R\left(\text{v. p.}\int\limits_{\{|z|=1\}}\frac{f'(z)}{f(z)}dz\right). \tag{10}$$

Applying the Fubini theorem to the first integral of the right side of (10) we have

$$\frac{1}{2\pi i}\int\limits_1^R\left(\int\limits_{\{|z|=t\}}\frac{f'(z)}{f(z)}dz\right)\frac{dt}{t}=\frac{1}{2\pi i}\int\limits_1^R\left(\int\limits_0^{2\pi}\frac{f'(te^{i\theta})}{f(te^{i\theta})}tie^{i\theta}d\theta\right)\frac{dt}{t}=$$

$$=\frac{1}{2\pi}\int\limits_0^{2\pi}\left(\int\limits_1^R\frac{f'(te^{i\theta})}{f(te^{i\theta})}e^{i\theta}dt\right)d\theta=\frac{1}{2\pi}\int\limits_0^{2\pi}(\log f(Re^{i\theta})-\log f(e^{i\theta}))d\theta.$$

Thus, (10) can be written as follows:

$$\int\limits_1^R\frac{n_2\left(t,\frac{1}{f}\right)}{t}dt-\int\limits_1^R\frac{n_2(t,f)}{t}dt+\frac{1}{2}\left(n^1\left(\frac{1}{f}\right)-n^1(f)\right)\log R=\\ =\frac{1}{2\pi}\int\limits_0^{2\pi}(\log f(Re^{i\theta})-\log f(e^{i\theta}))d\theta-\frac{1}{2\pi i}\log R\left(\text{v. p.}\int\limits_{\{|z|=1\}}\frac{f'(z)}{f(z)}dz\right). \tag{11}$$

The case of the domain A_r is similar. Only the direction of slits is the origin. Integrating (9) over τ from r to 1 we obtain

$$\int_r^1 \frac{n_1\left(\tau, \frac{1}{f}\right)}{\tau} d\tau - \int_r^1 \frac{n_1(\tau, f)}{\tau} d\tau + \frac{1}{2}\left(n^1\left(\frac{1}{f}\right) - n^1(f)\right)\log r =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} (\log f(re^{i\theta}) - \log f(e^{i\theta}))d\theta + \frac{1}{2\pi i}\log\frac{1}{r}\left(\text{v. p.} \int_{\{|z|=1\}} \frac{f'(z)}{f(z)}dz\right). \tag{12}$$

Hence, taking the real parts of both sides of (11) and (12), adding them and putting $r = \frac{1}{R}$, we find

$$\int_{\frac{1}{R}}^1 \frac{n_1\left(\tau, \frac{1}{f}\right)}{\tau} d\tau + \int_1^R \frac{n_2\left(t, \frac{1}{f}\right)}{t} dt - \int_{\frac{1}{R}}^1 \frac{n_1(\tau, f)}{\tau} d\tau - \int_1^R \frac{n_2(t, f)}{t} dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\frac{1}{Re^{i\theta}}\right)\right| d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \frac{1}{\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta.$$

This is (2) according to the definition of $N_0(R, f)$.

6. An analog of the Cartan theorem and the properties of $\boldsymbol{N_0}$, $\boldsymbol{T_0}$, $\boldsymbol{\overset{\circ}{T}_0}$. It follows from the definition of the function $N_0(R, f)$ that it is nonnegative, continuous, nondecreasing and convex with respect to $\log R$, $1 < R < R_0 \leq +\infty$.

Lemma 1. *Let f be a meromorphic function on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, $1 < R_0 \leq +\infty$, and $T_0(R, f)$ be determined by relation (1). Then*

$$T_0(R, f) = \frac{1}{2\pi}\int_0^{2\pi} N_0\left(R, \frac{1}{f - e^{i\theta}}\right) d\theta$$

for all R such that $1 < R < R_0$.

Lemma 1 is an analog of the Cartan theorem. Its proof can be done in the same way (see, for example, [15, p.p.33–36] or [16]).

As a corollary of this theorem we obtain that the Nevanlinna characteristic $T_0(R, f)$ is nonnegative, continuous, nondecreasing and convex with respect to $\log R$.

Lemma 2. *Let f, f_1, f_2 be nonconstant meromorphic functions on $A = \{z : \frac{1}{R_0} < |z| < R_0\}$, $1 < R_0 \leq +\infty$. Let T_0 be the Nevanlinna characteristic and let $\overset{\circ}{T}_0$ be the spherical characteristic. Then*

(i) $$T_0(R, f) = T_0\left(R, \frac{1}{f}\right);$$

(ii) $$T_0(R, f(z)) = T_0\left(R, f\left(\frac{1}{z}\right)\right);$$

(iii) $$T_0(R, f_1 \cdot f_2) \le T_0(R, f_1) + T_0(R, f_2) + O(1),$$
$$T_0\left(R, \frac{f_1}{f_2}\right) \le T_0(R, f_1) + T_0(R, f_2) + O(1),$$
$$T_0(R, f_1 + f_2) \le T_0(R, f_1) + T_0(R, f_2) + O(1), \quad 1 < R < R_0;$$

(iv) $$T_0(R, f) = \overset{\circ}{T}_0(R, f) + O(1),$$

where $1 < R < R_0$.

Proof. Statement (i) follows immediately from (2) in view of the equality

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+\frac{1}{|f(re^{i\theta})|}d\theta$$

for all r such that $1/R_0 < r < R_0$.

The statement of (ii) follows from the definition of the characteristic $T_0(R, f)$. Indeed, it is easy to verify that

$$N_0(R, f(z)) = N_0\left(R, f\left(\frac{1}{z}\right)\right) \text{ and } m(R, f(z)) = m\left(\frac{1}{R}, f\left(\frac{1}{z}\right)\right)$$

for all R such that $1 < R < R_0$.

Using the inequalities $\log^+ ab \le \log^+ a + \log^+ b$, $\log^+(a+b) \le \log^+ a + \log^+ b + \log 2$ for all positive a, b ([16, p. 9]),

$$N_0(R, f_1 \cdot f_2) \le N_0(R, f_1) + N_0(R, f_2), \quad N_0(R, f_1 + f_2) \le N_0(R, f_1) + N_0(R, f_2),$$

and property (i) we obtain (iii).

According to [13], $\log^+|f| \le -\log k(f, \infty) \le \log^+|f| + \frac{1}{2}\log 2$. Then in view of the definitions of the functions $T_0(R, f)$ and $\overset{\circ}{T}_0(R, f)$, we have

$$|T_0(R, f) - \overset{\circ}{T}_0(R, f)| \le 2\log 2.$$

This completes the proof of Lemma 2. □

7. Proof of Theorem 2. Taking into account (i) of Lemma 2, it is sufficient to prove that $T_0(R, f-a) = T_0(R, f) + O(1)$ for any f meromorphic on A and $a \in \mathbb{C}$. The proof of this fact is the same as in the case of the classical Nevanlinna theory for the functions meromorphic in $\mathbb{C}$. Namely, the inequality

$$|\log^+|f-a| - \log^+|f|| \le \log^+|a| + \log 2$$

(see for example [15, p. 25] or [16, p. 9]) and the identity $N_0(R, f-a) = N_0(R, f)$ are used. The principal role plays Theorem 1 as an analog of Jensen's theorem.

8. Proof of Theorem 3. At first we prove an analog of the Ahlfors-Shimizu formula. This can be done in the same way as in [13]. We present the general scheme of the proof. Let f be a meromorphic function on A and let

$$d\mu(a) = \frac{|a|d|a|d\alpha}{\pi(1+|a|^2)^2}, \qquad a = |a|e^{i\alpha},$$

be the area element of the Riemann sphere divided by π. Note that

$$\int\limits_{|a|<+\infty} d\mu(a) = \frac{1}{\pi}\int\limits_0^{+\infty}\int\limits_0^{2\pi} \frac{|a|d|a|d\alpha}{(1+|a|^2)^2} = 2\int\limits_0^{+\infty} \frac{|a|d|a|}{(1+|a|^2)^2} = \int\limits_0^{+\infty} \frac{d(|a|^2)}{(1+|a|^2)^2} = 1. \tag{13}$$

The Jensen formula (2) for the function $f(z) - a$ has the form

$$\begin{gathered} N_0\left(R, \frac{1}{f-a}\right) - N_0(R, f) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log\left|f\left(\frac{1}{Re^{i\theta}}\right) - a\right| d\theta + \\ + \frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(Re^{i\theta}) - a| d\theta - \frac{1}{\pi}\int\limits_0^{2\pi} \log|f(e^{i\theta}) - a| d\theta. \end{gathered} \tag{14}$$

Multiplying (14) by $d\mu(a)$, integrating it with respect to a and using (13) we have

$$\int\limits_{|a|<+\infty} N_0\left(R, \frac{1}{f-a}\right) d\mu(a) - N_0(R, f) = \frac{1}{2\pi}\int\limits_{|a|<+\infty} \left(\int\limits_0^{2\pi} \log\left|f\left(\frac{1}{Re^{i\theta}}\right) - a\right| d\theta\right) d\mu(a) +$$

$$+\frac{1}{2\pi}\int\limits_{|a|<+\infty} \left(\int\limits_0^{2\pi} \log|f(Re^{i\theta}) - a| d\theta\right) d\mu(a) - \frac{1}{\pi}\int\limits_{|a|<+\infty} \left(\int\limits_0^{2\pi} \log|f(e^{i\theta}) - a| d\theta\right) d\mu(a).$$

The following equalities hold for all R, $1 < R < R_0$ and for all r, $\frac{1}{R_0} < r < R_0$,

$$\int\limits_{|a|<+\infty} N_0\left(R, \frac{1}{f-a}\right) d\mu(a) = \int\limits_1^R \frac{A_1(t)}{t} dt + \int\limits_1^R \frac{A_2(t)}{t} dt,$$

$$\frac{1}{2\pi}\int\limits_{|a|<+\infty} \left(\int\limits_0^{2\pi} \log|f(re^{i\theta}) - a| d\theta\right) d\mu(a) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log\sqrt{1 + |f(re^{i\theta})|^2} d\theta.$$

These equalities can be proved by the method described in [13]. So, the analog of the Ahlfors-Shimizu formula is the following

$$\begin{gathered} \int\limits_1^R \frac{A_1(t)}{t} dt + \int\limits_1^R \frac{A_2(t)}{t} dt - N_0(R, f) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log\sqrt{1 + \left|f\left(\frac{1}{Re^{i\theta}}\right)\right|^2} d\theta + \\ + \frac{1}{2\pi}\int\limits_0^{2\pi} \log\sqrt{1 + |f(Re^{i\theta})|^2} d\theta - \frac{1}{\pi}\int\limits_0^{2\pi} \log\sqrt{1 + |f(e^{i\theta})|^2} d\theta. \end{gathered} \tag{15}$$

Relation (15) is (3) for $a = \infty$. Rewriting (15) for the function $F(z) = \frac{1+\overline{a}f(z)}{f(z)-a}$ we obtain (3) for $a \in \mathbb{C}$.

9. Proof of Theorem 4. Let f be a rational function. Since for each rational function f

$$\liminf_{R\to+\infty} \frac{T(R,f)}{\log R} < +\infty$$

([15, p. 50]), in view of (c) from §4 relation (4) holds.

Let (4) hold. Then $n_1(\frac{1}{t}, f) + n_2(t, f)$ is a bounded function for $t > 1$. Indeed, suppose that it is false. Thus for any $m \in \mathbf{N}$ there exists some real number R_m such that $n_1(\frac{1}{t}, f) + n_2(t, f) \geq m$ for $t \geq R_m$. Without loss of generality we can assume that $n_1(t, f) \geq m$ for $t \geq R_m$. Then

$$T_0(R,f) + 2m(1,f) \geq N_1(R,f) = \int_1^R \frac{n_1(\frac{1}{t},f)}{t}dt \geq \int_{R_m}^R \frac{n_1(\frac{1}{t},f)}{t}dt \geq m\int_{R_m}^R \frac{dt}{t} = m\log\frac{R}{R_m}$$

for $R \geq R_m$. And we obtain a contradiction with (4). Thus the function f has a finite number of poles. As $T_0(R, f) = T_0(R, \frac{1}{f})$ the function f has also a finite number of zeroes.

Let $r(z)$ be a rational function, zeroes and poles of which coincide with zeroes and poles of the function f with regard to their multiplicities. The function $g(z) = f(z)/r(z)$ has neither zeroes nor poles in $\mathbb{C}\backslash\{0\}$.

Using (iii) of Lemma 2 and (4) we obtain that

$$\liminf_{R\to+\infty} \frac{T_0(R,g)}{\log R} < +\infty. \tag{16}$$

Consider the Laurent expansion

$$g(z) = \sum_{k=-\infty}^{+\infty} c_k z^k.$$

We have

$$\left|\sum_{k=-\infty}^{-1} c_k z^k\right| \leq \sum_{k=1}^{+\infty} |c_k| \quad \text{for } |z| > 1.$$

The Laurent series converges absolutely and uniformly in any closed annulus in $\mathbb{C}\backslash\{0\}$, and, in particular, on the circle $\{z : |z| = 1\}$. Therefore, $\sum\limits_{k=1}^{+\infty} |c_k| < +\infty$ and $\sum\limits_{k=-\infty}^{-1} c_k z^k = O(1)$ for $|z| > 1$.

Then

$$|P(z)| \leq |g(z)| + O(1) \quad \text{for } |z| > 1,$$

where $P(z) = \sum\limits_{k=0}^{+\infty} c_k z^k$ and, consequently,

$$\int_0^{2\pi} \log^+|P(Re^{i\theta})|d\theta \leq \int_0^{2\pi} \log^+|g(Re^{i\theta})|d\theta + O(1) \quad \text{for } R > 1. \tag{17}$$

According to (16) and (17) we have $\liminf_{R\to+\infty} \frac{T(R,P(z))}{\log R} < +\infty$. Then ([15, p. 51]) there exists a positive integer m_0 such that $c_k = 0$ for $k \geq m_0$.

Consider the function $g(1/z)$. Using (ii) of Lemma 2 we obtain as above that $c_{-k} = 0$ for $k \geq n_0$, where n_0 is some positive integer.

Hence, g is a rational function and therefore the function f is rational as well. This completes the proof of Theorem 4.

Corollary. *Let f be a meromorphic function in $\{z : 0 < |z| < +\infty\}$. Then $f = \text{const}$ if and only if $T_0(R, f) = 0$.*

Proof. If $f = \text{const}$ then the equality $T_0(R, f) = 0$ follows immeadiately from the definition of the characteristic $T_0(R, f)$.

Let $T_0(R, f) = 0$. It follows from Theorem 4 that the function f is rational. According to (c) from §4, the classical Nevanlinna characteristic of f is bounded. Therefore [13], $f =$ const. □

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