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DIAGONALIZATION OF MATRICES

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We consider the question of diagonalizability of matrices of given size over regular rings of finite stable range as well as "weak" diagonal reduction of arbitrary regular ring.

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Рассматривается вопрос диагонализации матриц заданного размера над регулярными кольцами конечного стабильного ранга, а также "слабая" диагональная редукция произвольного регулярного кольца.

1. Introduction. The theorem that each matrix over a ring R is equivalent to some diagonal matrix was proved when R is the ring of integers in 1861 by H. Smith [1]. It was gradually extended by Dickson [2], Wedderburn [3], van der Waerden [4], and Jacobson [5] to various commutative and noncommutative Euclidean domains and commutative principal ideal domains and then to noncommutative principal ideal domains by O. Teichmuller [6] in 1937 (then in somewhat sharper form in Asano [7] and Jacobson [8]). The theorem is also known for arbitrary principal ideal rings [9]. Such rings of course satisfy the ascending chain conditions. Helmer [15] showed that the chain condition can be replaced by the less restrictive hypothesis that R be adequate. Kaplansky [10] generalized this further by permitting zero-divisors provided that they are all in the Jacobson radical.

Following Kaplansky [10] a ring R is said to be an elementary divisor ring if for every matrix A over R there exist unimodular matrices P, Q such that $PAQ = D = (d_i)$, a diagonal matrix where $d_iR \cap Rd_i \supseteq Rd_{i+1}R$.

Particular attention will be devoted to the diagonal reduction of a matrix consisting of a single row or column. If every 1 by 2 matrix admits diagonal reduction we shall call R a right Hermite ring; if 2 by 1 matrices admit diagonal reduction, R is a left Hermite ring, and if both conditions hold, R is an Hermite ring. It is easy to see that an Hermite ring is a Bezout ring (a ring is a Bezout ring if every finitely generated 1-sided ideal is principal). Examples that either implication is reversible are provided by Gillman and Henriksen in [12]. In [13] Amitsur proved that a Bezout domain is Hermite.

Henriksen [16] proved that if R is a unit regular ring then every matrix over R is equivalent to a diagonal matrix. The diagonalizability question for rectangular matrices was answered by Menal and Moncasi [17] who showed that all rectangular matrices over a given regular ring R are equivalent to a diagonal matrix if and only if R is Hermite. If R

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is a separative regular ring then only a square matrix over R is equivalent to a diagonal matrix [18]. Menal and Moncasi [17] showed that the stable range of a right or left Hermite ring is at most 2. In [21] it is proved that a commutative Bezout ring R is Hermite if and only if R has stable range 2.

In [19-20] it is proved that any left distributive elementary divisor ring is a duoring. Kaplansky proved that if R is an elementary divisor ring then every finitely presented R-module is a direct sum of cyclic modules. In [11] the converse to Kaplansky's theorem for commutative ring is proved.

2. Definitions. Throughout, R will always denote a ring (associative, but not necessary commutative) with $1 \neq 0$. We shall write R_n for the ring of n by n matrices with elements in R. By a unit of a ring we mean an element with two-sided inverse. The units of R_n will be said to be unimodular. If $b = ca$ we say that a is a right divisor of b; equivalent conditions are $b \in Ra$ and $Rb \subseteq Ra$. We say that a is a total divisor of b if $RbR \subseteq Ra \cap aR$, or in words: everything in the two-sided ideal generated by b is right and left divisible by a. Note that an element is not necessary a total divisor of itself. If R is commutative then right, left and total divisibility all coincide.

An n by m matrix $A = (a_{ij})$ is said to be diagonal if $a_{ij} = 0$ for all $i \neq j$. We say that a matrix A admits diagonal reduction if there exist unimodular matrices P, Q such that PAQ is a diagonal matrix. We shall call two matrices A and B over a ring R equivalent (notation $A \sim B$) if there exist unimodular matrices P, Q such that $B = PAQ$. If every matrix over R is equivalent to a diagonal matrix, (d_{ij}), with the property that every d_{ii} is a total divisor of d_{i+1i+1} then R is an elementary divisor ring. We recall that R is said to be right (left) Hermite if every 1 by 2 (2 by 1) matrix admits a diagonal reduction, and if both, R is an Hermite ring. If every 1 by n (n by 1) matrix admits a diagonal reduction then R is a right (left) n-Hermite ring.

A row $(a_1, \dots, a_n)$ over a ring R is called right unimodular if $a_1R + \dots + a_nR = R$. If $(a_1, \dots a_n)$ is a right unimodular n-row over a ring R then we say that $(a_1, \dots, a_n)$ is reducible if there exists an $(n-1)$-row $(b_1, \dots, b_{n-1})$ such that the $(n-1)$-row $(a_1 + a_nb_1, \dots a_{n-1} + a_nb_{n-1})$ is right unimodular. A ring R is said to have stable range n if n is the least positive integer such that every right unimodular $(n+1)$-row is reducible. This number is denoted by $s.r.(R)$.

By a right (left) Bezout ring we mean a ring in which all finitely generated right (left) ideals are principal, and by a Bezout ring a ring which is both right and left Bezout.

R is said to be regular if for every $a \in R$ there exists an $x \in R$ such that $axa = a$. A regular ring is said to be unit regular if for any $a \in R$ there exists a unit $u \in R$ such that $aua = a$.

We denote by $GL_n(R)$ the group of units of R_n. We write $GE_n(R)$ for the subgroup of $GL_n(R)$ generated by the elementary matrices. The Jacobson radical of a ring R will be denoted by $J(R)$. Denote by $U(R)$ the group of units of R.

3. Diagonalization of matrices over a regular ring with finite stable range.

Theorem 1. *Let R be a regular ring with finite stable range n. Then for every $k \times m$ matrix A over R where $|k - m| = n$ there exist unimodular matrices $P \in GE_k(R), Q \in GE_m(R)$ such that PAQ is a diagonal matrix.*

Proof. In order to prove that A admits a diagonal reduction we proceed by induction on the minimum of k and m. If either $k = 1$ or $m = 1$ the result follows by Corollary 1 [24,

p. 140]. So, assume that $k, m \geq 2$, by symmetry we may suppose without loss of generality that $k < m$. By Corollary 1 [24, p.140] R is $n+1$-Hermite, then it is suffices to consider the matrix A of the form

$$\begin{pmatrix} & & & 0 \\ & B & & \vdots \\ & & & 0 \\ a_{k1} & \cdots & a_{km-1} & b \end{pmatrix},$$

where B is a $(k-1)$ by $(m-1)$ matrix. By induction there exist $P \in GE_{k-1}(R), Q \in GE_{m-1}(R)$ such that PBQ is a diagonal matrix.

Thus A is elementary equivalent to

$$D = \begin{pmatrix} & & & 0 \\ & P & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} A \begin{pmatrix} & & & 0 \\ & Q & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

where

$$D = \begin{pmatrix} a & 0 & \cdots & 0 & 0 \\ \vdots & & C & & \vdots \\ 0 & & \cdots & & 0 \\ b & c_1 & \cdots & c_{m-2} & c \end{pmatrix}$$

and C is a diagonal $(k-2)$ by $(m-2)$ matrix.

Since R is a regular ring, there exists an idempotent $e \in R$ such that $aR = eR$. Then $ad = e, a = eu$ for some $d, u \in R$.

Set

$$U = \begin{pmatrix} d & 0 & \cdots & 0 & 1-du \\ 0 & & \cdots & & 0 \\ \vdots & & I_{m-2} & & \vdots \\ 0 & & \cdots & & 0 \\ 1 & 0 & \cdots & 0 & -u \end{pmatrix},$$

where I_{m-2} is the identity $(m-2)$-matrix. Clearly, $U \in GE_m(R)$. Then

$$DU = \begin{pmatrix} e & 0 & \dots & 0 & 0 \\ 0 & & & & 0 \\ \vdots & & C & & \vdots \\ 0 & & & & 0 \\ t & s_1 & \dots & s_{m-2} & s \end{pmatrix}.$$

Let $s' \in R$ be such that $ss's = s$. By adding to the first column of DU its m-th column right multiplied by $-s't$, we may assume that $ss't = 0$. (Since $ss's = s,\ ss'$ is idempotent such that $ss'(1-ss') = (1-ss')ss' = 0$.)

Set

$$V = \begin{pmatrix} s & 0 & \dots & 0 & ss'-1 \\ 0 & & & & 0 \\ \vdots & & I_{k-2} & & \vdots \\ 0 & & & & 0 \\ 1+s's & 0 & \dots & 0 & s'ss' \end{pmatrix},$$

where I_{k-2} is the identity $(k-2)$-matrix. Clearly, $V \in GE_k(R)$. Then

$$VDU = \begin{pmatrix} se-t & * & \dots & * & 0 \\ 0 & & & & 0 \\ \vdots & & C & & \vdots \\ 0 & & & & 0 \\ (1+s's)e & s'ss's_1 & \dots & s'ss's_{m-2} & s's \end{pmatrix}.$$

By elementary column transformations we see that VDU and so A can be reduced to a matrix of the form $\begin{pmatrix} y & y_1 & \dots & y_{m-2} & 0 \\ 0 & & & & 0 \\ \vdots & & C & & \vdots \\ 0 & & & & 0 \\ e & 0 & \dots & 0 & s's \end{pmatrix}$.

Set $f = s's$, therefore f is an idempotent. Since R is a regular ring, $(1-f)e\phi(1-f)e = (1-f)e$, for some $\phi \in R$. Then $g = (1-f)e\phi(1-f)$ is an idempotent satisfying

$$(f+g)R = fR + eR, gf = fg = 0, ge = (1-f)e.$$

If we add to the first row of A the k-th row left multiplied by $-y\phi(1-f)$ we obtain the matrix

$$\begin{pmatrix} y_1(1-\phi)(1-f)e & * & \dots & * & 0 \\ 0 & & & & 0 \\ 0 & & C & & 0 \\ 0 & & & & 0 \\ e & 0 & \dots & 0 & f \end{pmatrix}.$$

Clearly, we have

$$\begin{pmatrix} y(1-\phi(1-f)e) & 0 \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e & 1 \end{pmatrix} \begin{pmatrix} 1 & \phi(1-f) \\ 0 & 1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} 1 & 0 \\ -(1-f)e & 1 \end{pmatrix} \begin{pmatrix} 1 & -\phi(1-f) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y(1-\phi(1-f)e) & 0 \\ 0 & f+g \end{pmatrix}$$

By elementary column transformation we see that A can be reduced to a matrix of the form

$$\begin{pmatrix} y(1-\phi(1-f)e) & * & \dots & * & 0 \\ 0 & & & & 0 \\ \vdots & & C & & \vdots \\ 0 & 0 & \dots & 0 & f+g \end{pmatrix}.$$

Since $\begin{pmatrix} y(1-\phi(1-f)e) & * \dots * \\ 0 & \\ \vdots & C \\ 0 & \end{pmatrix}$ is a $(k-1)\times(m-1)$-matrix, the result follows by induction. □

4. A "weaker" diagonal reduction of matrices over regular ring. Let R be a ring with $1 \neq 0$. The idea of this result is based on the following remark, valid in any ring: if a and b are right multiples of each other, then $(a, 0)$ and $(b, 0)$ are right associates. If $b = ax$ and $a = by$ then

$$(a, 0)\begin{pmatrix} x & 1 - xy \\ 1 & -y \end{pmatrix} = (b, 0)$$

where

$$\begin{pmatrix} x & 1 - xy \\ 1 & -y \end{pmatrix} \in GE_2(R).$$

Suppose $a_1R + \ldots + a_nR = dR$, thus $d = a_1x_1 + \ldots + a_nx_n$, $a_i = dy_i$, $i = 1, 2, \ldots, n$, for any $x_1, \ldots x_n, y_1, \ldots, y_n \in R$. Write X, Y for the row and column $X = (x_1, \ldots, x_n)$, $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ respectively. Then

$$(a_1, \ldots, a_n, 0)\begin{pmatrix} X & I - XY \\ I & -Y \end{pmatrix} = (d, 0, \ldots, 0)$$

for any identity matrix I.

By the induction it follows readily that the addition of m columns of zeros will permit the reduction of an m-rowed matrix. The precise result is stated in the following theorem [10].

Theorem 2. *Let R be a right Bezout ring. Let A be an m-owed matrix over R, and A_1 be the matrix obtained by adjoining m-columns of zeros to the matrix A. Then we can find a unimodular matrix U such that A_1U is triangular (that is, has zeros above the main diagonal).*

Now we are ready to prove the main result of this paper.

Theorem 3. *Let R be a regular ring and let A be an m-rowed matrix over R, and A_1 be the matrix obtained by adjoining m columns of zeros to A. Then we can find a unimodular matrices P, Q such that PA_1Q is a diagonal matrix.*

Proof. If $m = 1$ the result follows from Theorem 2. We proceed by induction on m. By the induction and Theorem 2 we have that A_1 is equivalent to the matrix

$$A_2 = \begin{pmatrix} x & 0 & \ldots & 0 & 0 \\ 0 & & X & & 0 \\ y_1 & y_2 & \ldots & y_{n+m-1} & z \end{pmatrix},$$

where X is a diagonal matrix. Let $e^2 = e \in R$ be an idempotent such that $xR = eR$, then $xt = e, x = eu$ for some $t, u \in R$. Set

$$Q_1 = \begin{pmatrix} t & 0 & \ldots & 0 & 1 - tu \\ 0 & & I_{n+m-2} & & 0 \\ 1 & & 0 & & -u \end{pmatrix} \in GE_{n+m}(R)$$

then A_2Q_1 is of the same form as A_2, but its (1,1)-entry is e, an idempotent. Hence we may assume that $x = x^2 = e$.

Let $z' \in R$ be such that $zz'z = z$. By adding to the first column of A_2 its $n+m$-th column right multiplied by $-z'y$ we may assume that $zz'y = 0$. Set

$$P_1 = \begin{pmatrix} z & 0 & zz'-1 \\ 0 & I_{m-2} & 0 \\ 1+z'z & 0 & z'zz' \end{pmatrix} \in GE_m(R),$$

then

$$P_1 A_2 Q_1 = \begin{pmatrix} ze-y & & * & & 0 \\ 0 & & X & & 0 \\ (1+z'z)e & z'zz'y_2 & \dots & z'zzy_{n+m-2} & z'z \end{pmatrix}.$$

By elementary column transformation we see that $P_1A_2Q_1$ and therefore A_1 can be reduced to

$$A_3 = \begin{pmatrix} c & * & 0 \\ 0 & X & 0 \\ e & 0 & f \end{pmatrix},$$

where $z'z = f$ and $f^2 = f$.

Let $\alpha \in R$ be such that $(1-f)e\alpha(1-f)e = (1-f)e$ then $g = (1-f)e\alpha(1-f)$ is $g^2 = g$ and

$$(f+g)R = fR + eR, \quad gf = fg = 0, \quad (1-f)e = ge.$$

If we add to the first row of the matrix A_3 the m-th row left multiplied by $-c\alpha(1-f)$ we obtain the matrix

$$A_4 = \begin{pmatrix} c(1-\alpha(1-f)) & * & 0 \\ 0 & X & 0 \\ e & 0 & f \end{pmatrix}.$$

On the other hand, we have

$$\begin{pmatrix} c(1-\alpha(1-f)) & 0 \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha(1-f) \\ 0 & 1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} 1 & 0 \\ -(1-f)e & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha(1-f) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c(1-\alpha(1-f)e) & 0 \\ 0 & f+g \end{pmatrix}.$$

Hence it is clear that A_4 is equivalent to

$$\begin{pmatrix} c(1-\alpha(1-f)e) & * & 0 \\ 0 & X & 0 \\ 0 & 0 & f+g \end{pmatrix}.$$

Since $\begin{pmatrix} c(1-\delta(1-f)e) & * \\ 0 & X \end{pmatrix}$ is a matrix with $m-1$ rows, the result follows by induction.

□

5. An elementary divisor ring with Lam's condition.

Theorem 4. *Let R be an elementary divisor ring in which for any element $a \in R$, we have $RaR = R$ implies that a is a unit of R. Then R is a distributive ring.*

Proof. Let $aR + bR = R$. Let $A = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$. Since R is an elementary divisor ring, there exist unimodular matrices $P, Q \in GL_2(R)$ such that

$$P \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} Q = \begin{pmatrix} z & 0 \\ 0 & c \end{pmatrix} \tag{1}$$

where $RcR \subseteq zR \cap Rz$.

Since $aR + bR = R$, $RaR + RbR = R$. By [23], then $RzR = R$. Under either of our hypotheses we have that z is a unit of R. Let $P = (p_{ij})$ and $Q = (q_{ij})$. By (1) we obtain that $(p_{11}a + p_{12}b)q_{11}$ is a unit of R, then $Ra + Rb = R$. By [20] R is a distributive ring. □

By [22] we have

Corollary 1. *An elementary divisor ring in which for any $a \in R$ we have $RaR = R$ implies that a is a unit of R, is a duoring.*

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