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GROUPS WITH THE MINIMAL CONDITION ON NON-"ABELIAN-BY-FINITE" SUBGROUPS

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We characterize groups in which no non-trivial section is perfect without infinite properly descending series of non-"abelian-by-finite" subgroups.

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Мы характеризуем группы, в которых никакое сечение не является совершенным без бесконечных собственно убывающих рядов не "почти абелевых" подгрупп.

0. We say that a group G satisfies the minimal condition on non-"abelian-by-finite" subgroups (for short Min-$\overline{AF}$) if for every properly descending chain $\{G_n | n \in \mathbb{N}\}$ of subgroups in G there exists a number $n_0 \in \mathbb{N}$ such that G_n is an abelian-by-finite subgroup for any integer $n \geq n_0$. Every minimal non-"abelian-by-finite" (i.e. non-"abelian-by-finite" group with abelian-by-finite proper subgroups) G satisfies Min-$\overline{AF}$. In a series of papers of V.V.Belyaev [1], B.Bruno [2-4], B.Bruno and R.E.Phillips [5] have proved that a minimal non-"abelian-by-finite" group is an indecomposable metabelian group or a Čarin group (see e.g. Čarin's example [6, p.152]). A group G is indecomposable if any two proper subgroups of G generate a proper subgroup of G. Note that earlier groups with the minimal condition on non-abelian subgroups have been studied by S. N. Černikov (see [7]) and V. P. Šunkov [8] and solvable groups with the minimal condition on non-"nilpotent-by-finite" subgroups by the author [9].

In this paper we characterize groups in which no non-trivial section is perfect and which satisfy Min-$\overline{AF}$. Namely, we prove

Theorem. *Let G be a group in which no non-trivial section is perfect. Then G satisfies Min-$\overline{AF}$ if and only if it is of one of the following types:*

(i) G is an abelian-by-finite group;

(ii) G contains a normal subgroup H of finite index such that

$$H = H_0 \cdot H_1 \cdot \ldots \cdot H_n \ (n \geq 1),$$

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where H_i is a metabelian HM^*-group, $H_i' = H' \leq H_0$ $(i = 1, \dots, n)$, H_0 is an abelian group with the divisible Černikov quotient group H_0/H' and if $k \neq s$ $(1 \leq k, s \leq n)$, then $\pi(H_k/H') \cap \pi(H_s/H') = \varnothing$.

Throughout this paper, p is a prime, $\mathbb{C}_{p^\infty}$ a quasicyclic p-group and G' the commutator subgroup of a group G, $\pi(G)$ a set of all primes which divide the orders of torsion elements in G.

Recall also one construction from [10]. Let $G = M \rtimes Q$ be a semidirect product of an abelian q'-subgroup M and a quasicyclic q-subgroup Q. Then M is a right $\mathbb{Z}Q$-module, where the action is induced by the conjugation of Q on M. As in [10], if $\{V_\lambda | \lambda \in \Lambda\}$ is a complete set of representatives for the isomorphism types of irreducible $\mathbb{Z}_pQ$-modules, we view V_λ as $\mathbb{Z}Q$-modules and denote by $E(V_\lambda)$ the $\mathbb{Z}Q$-injective hull of V_λ. Let

$$V_\lambda(n) = \{v \in E(V_\lambda) | p^n v = 0\} \text{ and } V_\lambda(\infty) = E(V_\lambda).$$

Then $V_\lambda(n)$ $(n = 0, 1, \dots, \infty)$ is determined up to isomorphism by λ and n (see [10]).

We will also use other standard terminology from [6].

1. For the next we need the following lemmas.

Lemma 1. *Let G be a group that satisfies Min-$\overline{AF}$ and H its subgroup. Then*

(i) H satisfies Min-$\overline{AF}$;

(ii) if H is normal in G, then the quotient group G/H satisfies Min-$\overline{AF}$;

(iii) if H is a normal non-"abelian-by-finite" subgroup, then G/H satisfies the minimal condition on subgroups.

Proof. Evident. □

Lemma 2. *Let G be a non-perfect (i.e. $G' \neq G$) group with abelian-by-finite proper normal subgroups. Then G satisfies Min-$\overline{AF}$ if and only if it is of one of the following types:*

(1) G is an abelian-by-finite group;

(2) G is a minimal non-"abelian-by-finite" group;

(3) $G = G' \rtimes S$, where $S \cong \mathbb{C}_{p^\infty}$, $G' = S_1 \times \cdots \times S_n$ $(n \geq 1)$ is a p'-subgroup and a group direct product of finitely many abelian Sylow p_i-subgroups S_i and a right $\mathbb{Z}S$-module S_i is a module direct sum of finitely many submodules each isomorphic to some $V_\lambda(m)$ $(i = 1, \dots, n; 1 \leq m \leq \infty)$;

(4) $G = A \rtimes S$ is a metabelian group, where S is a minimal non-"abelian-by-finite" p-group, A is a normal abelian p'-subgroup of G and G/S' is a group of type (3).

Proof. $(\Leftarrow)$ is immediate.

$(\Rightarrow)$ 1) First we assume that G/G' is not an indecomposable group. Then $G = AB$ is a product of two abelian-by-finite proper normal subgroups A and B. Since A (respectively B) contains an abelian G-invariant subgroup A_1 (respectively B_1) of finite index, we obtain that $G = A_1B_1$. As a consequence, G is a nilpotent group and therefore $G'K \neq G$ for any proper subgroup K of G. This means that K is an abelian-by-finite subgroup and in view of Theorem B of [1] G is the one.

2) Now let G/G' be an indecomposable group and so it is a cyclic p-group (in which case G is an abelian-by-finite group) or a quasicyclic p-group for some prime p. Assume that

$G/G' \cong \mathbb{C}_{p^\infty}$. If D is a proper abelian G-invariant subgroup of finite index in G', then G/DG'' is an abelian group, a contradiction. This means that G' is an abelian subgroup. Since G satisfies Min-$\overline{AF}$, it contains a subgroup S which is a minimal non-"abelian-by-finite" group. Hence $G = G'S$.

Suppose that $S \neq G$. Let $\overline{G} = G/(G' \cap S) = \overline{G}' \rtimes \overline{S}$. It is easy to see that $\overline{G}$ satisfies the minimal condition on normal subgroup Min-n and so Baer Theorem [6, Theorem 5.25] and Theorem 2.1 of [5] imply that G is a locally finite group. If $\overline{G}$ is a p-group, then it is Černikov (see [61, p.156, Corollary 2]). This yields that $\overline{G}$ is a nilpotent group and we obtain a contradiction. From this it follows that $\overline{G}'$ is a p'-subgroup. Our hypothesis and Theorem B of [1] give that G' is a π-subgroup for some finite set of primes π and $G = A \rtimes Q$, where either $Q = S$ is a minimal non-"abelian-by-finite" p-group or $Q \cong \mathbb{C}_{p^\infty}$, A is a p'-subgroup of G'.

Now assume that Q is a quasicyclic p-subgroup. Let $q \in \pi$ and B be a Sylow q-subgroup of G'. Since $B \rtimes Q$ is a non-"abelian-by-finite" group with Min-$\overline{AF}$, it also satisfies Min-n and therefore by Theorem A of [10] a right $\mathbb{Z}Q$-module B is a module direct sum of finitely many submodules each isomorphic to some $V_\lambda(n)$ $(1 \leq n \leq \infty)$. Thus G is a group of type (3).

If Q is a minimal non-"abelian-by-finite" group, then not difficult to see that G/S' is a group of type (3). The lemma is proved. □

Corollary 3. *If G is a group with Min-$\overline{AF}$ in which no non-trivial section is perfect, then it is countable and locally finite.*

If G' is a hypercentral subgroup and G/G' is a divisible Černikov p-group, then G is called an HM^*-group (see [11] and [9]). Any group of Heineken-Mohamed type (i.e. non-nilpotent group with all proper subgroups nilpotent and subnormal) is an HM^*-group.

Example 4. Let $p_1, \ldots, p_s, p$ be distinct primes, Y_i the splitting field of the polynomials $x^{p^n} - 1$ $(n \in \mathbb{N} \cup \{0\})$ over the field $\mathbb{Z}_{p_i}$, $A = Y_1 \oplus \cdots \oplus Y_s$ a ring direct sum. By Theorem 2.5 of [12] every Y_i has a nontrivial automorphism σ_i $(i = 1, \ldots, s)$. Then $R = A[x; \sigma_1, \ldots, \sigma_s]/(x^m)$ $(m \geq 2)$, where

$$(a_1, \ldots, a_s)x = x({a_1}^{\sigma_1}, \ldots, {a_s}^{\sigma_s})$$

for all elements $(a_1, \ldots, a_s) \in A$, is a semiperfect ring with the unit group

$$U(R) = (1 + J(R)) \rtimes ({Y_1}^* \times \cdots \times {Y_s}^*).$$

Moreover, $1 + J(R)$ is a nilpotent π-subgroup, where $\pi = \{p_1, \ldots, p_s\}$, and the multiplicative group Y_i^* of Y_i is a p_i'-subgroup which contains a quasicyclic p-subgroup H_i of finite index. Let $\overline{A}$ and X be a homomorphic image of A and x in R, respectively. Assume that $m = 2$. Then $(1 + Xf)^{-1} = 1 - Xf$ and the commutator

$$[1 + Xf, u] = (1 - Xf)u(1 + Xf)u^{-1} = 1 + X(u_1 - u)fu^{-1}$$

for all elements $f \in \overline{A}$ and $u \in {Y_1}^* \times \cdots \times {Y_s}^*$, where $uX = Xu_1$ for some $u_1 = (u_{11}, \ldots, u_{1s}) \in {Y_1}^* \times \cdots \times {Y_s}^*$. Since $\overline{A} = (u_1 - u)\overline{A}u^{-1}$ for some $u, u_1 \in H_1 \times \cdots \times H_s$ with $u_{1i} \neq 0$ for all i $(1 \leq i \leq s)$, we conclude that

$$[1 + J(R), H_1 \times \cdots \times H_s] = 1 + J(R).$$

From this it follows that

$$G = (1 + J(R)) \rtimes (H_1 \times \cdots \times H_s)$$

is an HM^*-group for any $m \geq 2$. If $s = 1$ and $m = 2$, then G is a Čarin group by Lemma 1 of [13].

Lemma 5. *Let G be an HM^*-group. Then G satisfies Max-$\overline{AF}$ if and only if it is metabelian.*

Proof. ($\Leftarrow$) Since the quotient group G/G' is Černikov and G' is an abelian subgroup, we conclude that G satisfies Min-$\overline{AF}$.

($\Rightarrow$) If G' is not abelian-by-finite, then in view of Lemma 2 it contains a subnormal non-"abelian-by-finite" subgroup S with all proper normal subgroups abelian-by-finite. But Lemma 2 yields that S is not a hypercentral group and we obtain a contradiction with the hypercentrality of G'. This means that G' is an abelian-by-finite subgroup and, as a consequence, it is abelian, as desired. □

2. Proof of Theorem. ($\Leftarrow$). Evident.

($\Rightarrow$). Suppose that G is a non-"abelian-by-finite" group. By Lemma 2 G, has a descending subnormal series

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = S,$$

where S is a group with all proper normal subgroups abelian-by-finite and by Lemma 1 G_j/G_{j+1} is a Černikov group ($j \in \{0, 1, \ldots, n-1\}$).

Let $x \in G_{n-1}$. Then $S^x \triangleleft G_{n-1}$ and consequently $S' \triangleleft G_{n-1}$. By D_{n-1} we denote a subgroup of finite index in G_{n-1} such that D_{n-1}/S' is the divisible part of G_{n-1}/S'. Then $D'_{n-1} = S'$. Since $D_{n-1}D^y_{n-1}/S'$ is a Černikov group for every $y \in G_{n-2}$ and S' not contains a proper S-invariant subgroup of finite index, we conclude that $D_{n-1} \triangleleft G_{n-2}$. By the same argument after a finite number of steps we obtain that G has a normal subgroup D of finite index such that $D' = S'$ and D/D' is a divisible Černikov group. Then

$$D = D_1 \cdot \ldots \cdot D_n (n \geq 1),$$

where D_s/D' is a divisible Černikov Sylow p_s-subgroup of D/D' ($s = 1, \ldots, n$) and $p_s \neq p_l$ if $s \neq l$ ($1 \leq s, l \leq n$). If D_k is an abelian-by-finite subgroup for some integer k ($1 \leq k \leq n$), then it is abelian. Assume that D_k is not abelian-by-finite. Then it contains a subnormal subgroup T with all proper normal subgroups abelian-by-finite. As before, we can prove that $T' = D'_k = S'$. Hence D_k is a metabelian HM^*-group. The theorem is proved.

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