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I. I. Reznykov

COMPOSITE AND NON-MONOTONIC GROWTH FUNCTIONS OF MEALY AUTOMATA

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We introduce the notion of composite growth function and provide examples that illustrate fundamental properties of these growth functions. We provide examples of Mealy automata that have composite non-monotonic growth functions of the polynomial growth order. We described examples of Mealy automata that have composite monotonic growth functions of intermediate and exponential growth. Questions concerning the relationship between the notions "composite" and "non-monotonic" of a Mealy automaton growth function are formulated.

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В статье введено понятие составной функции роста и представлены примеры, которые иллюстрируют основные свойства таких функций. Рассмотрены примеры автоматов Мили с составными немонотонными функциями роста полиномиального порядка. Описаны примеры автоматов Мили с составными монотонными функциями роста промежуточного и экспоненциального порядков. Поставлен вопрос о взаимосвязи понятий "составная" и "немонотонная" функция роста автомата Мили.

1. Introduction. The notion of growth was introduced in the middle of the last century [14], [10] and was applied to various geometric, topological and algebraic objects [2] [15]. Mainly, growth functions of studied objects are non-decreasing monotonic functions of a natural argument [2]. For example, the growth function of a semigroup (group) at a point n, $n \geq 0$, equals a number of different semigroup elements of length n. Obviously, the growth function of an arbitrary semigroup is a non-decreasing monotonic function.

Growth of Mealy automata have been studied since the 80th of the 20th century [3], [6], and it closely interrelated with growth of automatic transformation semigroups (groups) defined by them [6]. However, the growth functions of the Mealy automaton and the corresponding semigroup have different properties; for example, they may have different growth orders [12]. In the paper we consider a special type of growth functions of Mealy automata — composite growth functions.

A composite function is a function such that it can be described by different expressions on infinite non-overlapped intervals. There exist Mealy automata that have composite growth functions of various growth orders. Moreover, some of these automata have non-monotonic

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growth functions. There were not known Mealy automata that have non-monotonic growth functions (see the survey [7], and [13], [11], etc.).

Preliminaries of the theory of Mealy automata are listed in Section 2. The notion of composite function is introduced in Section 1. In Section 2 we provide several examples of Mealy automata that have non-monotonic growth functions of the polynomial growth order. In addition, the Mealy automata with composite growth functions such that one of its finite differences consists of doubled values, are provided in Section 3. The theorems concerning the main properties of these automata are formulated, and we list these theorems without proofs. They can be proved by using the technique similar to that of [11] (see also [12]). We are planning to publish the proofs of these theorems in subsequent papers. For convenience, the propositions, where the normal form of semigroup elements are formulated, are provided for the most complex of the considered automata. Moreover, questions concerning the composite growth functions are appeared, and some of them are mentioned in Section 4.

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2. Preliminaries. The basic notions of the theory of Mealy automata and the semigroup theory can be found in many books, for example [5], [4], [8], [3]. We use definitions from [12].

2.1. Mealy automata. Denote the set of all finite words over X_m, including the empty word ε, by the symbol X_m^*, and denote the set of all infinite (to right) words by the symbol X_m^ω. We write a function $\phi\colon X_m \to X_m$ as

$$\begin{pmatrix} \phi(x_0) & \phi(x_1) & \dots & \phi(x_{m-1}) \end{pmatrix}.$$

Moreover, we have in mind $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let $A = (X_m, Q_n, \pi, \lambda)$ be a *non-initial Mealy automaton* [9] with the finite set of states $Q_n = \{q_0, q_1, \ldots, q_{n-1}\}$, the input and output alphabets are the same and equal X_m, $\pi\colon X_m \times Q_n \to Q_n$ and $\lambda\colon X_m \times Q_n \to X_m$ are its transition and output functions, respectively. The function λ can be extended in a natural way either to the mapping $\lambda\colon X_m^* \times Q_n \to X_m^*$ or to the mapping $\lambda\colon X_m^\omega \times Q_n \to X_m^\omega$. The transformation $f_{\mathsf{q}}\colon X_m^* \to X_m^*$ ($f_{\mathsf{q}}\colon X_m^\omega \to X_m^\omega$), defined by the equality $f_{\mathsf{q}}(u) = \lambda(u, \mathsf{q})$, where $u \in X_m^*$ ($u \in X_m^\omega$), is called [5] *the automatic transformation* defined by A at the state q. The automaton A defines the set

$$F_A = \left\{f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}}\right\}$$

of automatic transformations over X_m^ω. Each automatic transformation defined by the automaton A can be written in the *unrolled form*

$$f_{q_i} = \left(f_{\pi(x_0,q_i)}, f_{\pi(x_1,q_i)}, \ldots, f_{\pi(x_{m-1},q_i)}\right) \sigma_{q_i},$$

where $i \in \{0, 1, \ldots, n-1\}$, and σ_{q_i} is the transformation over the alphabet X_m defined by the output function λ:

$$\sigma_{q_i} = \begin{pmatrix} \lambda(x_0, q_i) & \lambda(x_1, q_i) & \dots & \lambda(x_{m-1}, q_i) \end{pmatrix}.$$

Let us define the set of all n-state Mealy automata over the m-symbol alphabet by the symbol $A_{n\times m}$. The product of Mealy automata is introduced [3] over the set of automata with the same input and output alphabet X_m as their sequential applying. Therefore for the transformations f_{q_1,A_1} and f_{q_2,A_2}, $\mathsf{q}_1 \in Q_{n_1}$, $\mathsf{q}_2 \in Q_{n_2}$, the unrolled form of the product $f_{(\mathsf{q}_1,\mathsf{q}_2),A_1\times A_2}$ is defined by the equality:

$$f_{(\mathsf{q}_1,\mathsf{q}_2),A_1\times A_2} = f_{\mathsf{q}_1,A_1} f_{\mathsf{q}_2,A_2} = (g_0, g_1, \ldots, g_{m-1})\, \sigma_{\mathsf{q}_1,A_1}\sigma_{\mathsf{q}_2,A_2},$$

where $g_i = f_{\pi_1\left(\sigma_{\mathsf{q}_2,A_2}(x_i),\mathsf{q}_1\right),A_1} f_{\pi_2(x_i,\mathsf{q}_2),A_2}$, $i \in \{0, 1, \ldots, m-1\}$, and all transformations are applied from right to left.

The power A^n is defined for any automaton A and any positive integer n. Let us denote by $A^{(n)}$ the minimal Mealy automaton, equivalent to A^n. From the definition of the product it follows that $\left|Q_{A^{(n)}}\right| \leq |Q_A|^n$.

Definition 1 ([6]). The function γ_A of a natural argument, defined by

$$\gamma_A(n) = \left|Q_{A^{(n)}}\right|, \; n \in \mathbb{N},$$

is called *the growth function* of the Mealy automaton A.

2.2. Semigroups.

Definition 2. Let $A = (X_m, Q_n, \pi, \lambda)$ be a Mealy automaton. The semigroup

$$S_A = sg\left(f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}}\right)$$

is called the *semigroup of automatic transformations defined by* A.

Let S be a semigroup with the finite set of generators $G = \{s_0, s_1, \ldots, s_{k-1}\}$. The elements of the free semigroup G^+ are called *semigroup words* [8]. In the sequel, we identify them with the corresponding elements of S. Denote the length of a semigroup element s by the symbol $\ell(\mathsf{s})$.

Definition 3. The function γ_S of a natural argument such that

$$\gamma_S(n) = \left|\left\{\; s \in S \;\middle|\; \ell(s) \leq n \;\right\}\right|, \; n \in \mathbb{N},$$

is called the *growth function of S with respect to the system G of generators.*

Definition 4. The function $\widehat{\gamma}_S$ of a natural argument such that

$$\widehat{\gamma}_S(n) = \left|\left\{\; s \in S \;\middle|\; s = s_{i_1}s_{i_2}\ldots s_{i_n},\; s_{i_j} \in G,\; 1 \leq j \leq n \;\right\}\right|, \; n \in \mathbb{N},$$

is called the *spherical growth function of S with respect to the system G of generators.*

Definition 5. The function δ_S of a natural argument such that

$$\delta_S(n) = \left|\left\{\; s \in S \;\middle|\; \ell(s) = n \;\right\}\right|, \; n \in \mathbb{N},$$

is called the *word growth function of S with respect to the system G of generators.*

From Definitions 3, 4 and 5, the following inequalities hold for $n \in \mathbb{N}$:

$$\delta_S(n) \leq \widehat{\gamma}_S(n) \leq \gamma_S(n) = \sum_{i=0}^{n} \delta_S(i).$$

Similarly, from Definition 2 it follows [6] that

$$\gamma_A(n) = \widehat{\gamma}_{S_A}(n), \; n \in \mathbb{N}.$$

2.3. Growth functions. The growth of some object is defined by functions of a natural argument. One of the most used characteristics of these functions is the notion of growth order.

Definition 6. Let $\gamma_i \colon \mathbb{N} \to \mathbb{N}$, $i \in \{1, 2\}$, be arbitrary functions. The function γ_1 has *growth order not greater* than the function γ_2 (notation $\gamma_1 \preceq \gamma_2$), if there exist numbers $C_1, C_2, N_0 \in \mathbb{N}$ such that

$$\gamma_1(n) \leq C_1\gamma_2(C_2 n)$$

for any $n \geq N_0$.

Definition 7. Growth functions γ_1 and γ_2 are equivalent or have *the same growth order* (notation $\gamma_1 \sim \gamma_2$), if the inequalities $\gamma_1 \preceq \gamma_2$ and $\gamma_2 \preceq \gamma_1$ hold.

The equivalence class of the function γ is called *the growth order* and is denoted by the symbol $[\gamma]$. The growth order $[\gamma]$ is called

1. *polynomial*, if $[\gamma] = \left[n^d\right]$ for some $d > 0$;
2. *intermediate*, if $\left[n^d\right] < [\gamma] < [e^n]$ for all $d > 0$;
3. *exponential*, if $[\gamma] = [e^n]$.

It is often convenient to encode the growth function of a semigroup in a generating series.

Definition 8. Let S be a semigroup generated by a finite set G. The *growth series* of S is the formal power series

$$\Gamma_S(X) = \sum_{n \geq 0} \gamma_S(n) X^n.$$

The power series

$$\Delta_S(X) = \sum_{n \geq 0} \delta_S(n) X^n$$

can also be introduced; we then have $\Delta_S(X) = (1 - X)\Gamma_S(X)$. The series Δ_S is called the *word growth series* of the semigroup S.

The growth series of a Mealy automaton is introduced similarly:

Definition 9. Let A be an arbitrary Mealy automaton. The *growth series* of A is the formal power series

$$\Gamma_A(X) = \sum_{n \geq 0} \gamma_A(n) X^n.$$

3. Composite growth functions Let us introduce the concept of composite growth function in the following way. Let $\gamma \colon \mathbb{N} \to \mathbb{N}$ be an arbitrary function, and let $k \geq 1$ be a positive integer. Let us define the functions $\gamma_i \colon \mathbb{N} \to \mathbb{N}$, $i \in \{0, 1, \ldots, k-1\}$, by the equalities:

$$\gamma_i(n) = \gamma(k \cdot n + i), n \in \mathbb{N}.$$

We say that the function γ is *composite*, if there exists integer $k \geq 2$ such that at least two functions from the set

$$\{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}\}$$

can be defined by different expressions.

Let us fix the notation. Let A be an arbitrary Mealy automaton. Let us denote the semigroup of automatic transformations, defined by A, by the symbol S_A, and the growth functions of A and S_A by the symbols γ_A and γ_{S_A}, respectively. If γ_A is a composite function for some integer k, then let us denote its "parts" by the symbols $\gamma_{A,i}$, $i \in \{0, 1, \ldots, k-1\}$.

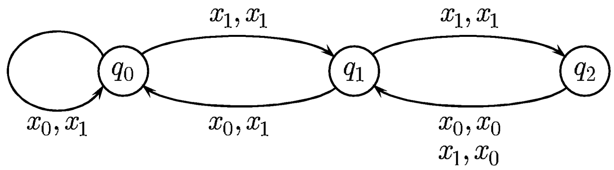

Fig. 1: The automaton A_1

Let γ be an arbitrary function, and let us denote the i-th finite difference of γ by the symbols $\gamma^{(i)}$, $i \geq 1$, i.e.

$$\begin{aligned}\gamma^{(1)}(n) &= \gamma(n) - \gamma(n-1),\\ \gamma^{(i)}(n) &= \gamma^{(i-1)}(n) - \gamma^{(i-1)}(n-1),\end{aligned}$$

where $i \geq 2$, $n \geq i+1$.

Let us consider an example of Mealy automaton with the composite growth function. Let A_1 be the 3-state Mealy automaton over the 2-symbol alphabet whose Moore diagram is shown on Figure 1. Its automatic transformations have the following unrolled forms:

$$f_0 = (f_0, f_1)\,(x_1, x_1)\,, \qquad f_1 = (f_0, f_2)\,(x_1, x_1)\,, \qquad f_1 = (f_1, f_1)\,(x_0, x_0)\,.$$

The following theorem holds:

Theorem 1. *1. The semigroup S_{A_1} has the following presentation:*

$$S_{A_1} = \left\langle\ f_0, f_1\ \middle|\ \begin{array}{l} f_1f_2 = f_0f_2,\ f_1^2f_i = f_1f_0f_2,\ i \in \{0,1\};\\ f_0^2f_1 = f_0^2,\ f_2f_0f_1 = f_2^2f_j = f_2f_0^2,\ j \in \{0,1,2\};\\ f_0^4 = f_0^3,\ f_0^3f_2 = f_0^3,\ f_0f_2f_0^2 = f_0f_1f_0f_2;\\ f_1f_0f_2f_0 = f_1f_0f_2,\ f_2f_0^3 = f_2f_0^2 \end{array} \right\rangle.$$

2. *The growth function γ_{A_1} is a composite function for $k = 2$, and is defined by the following equalities:*

$$\gamma_{A_1,0}(n) = 23 \cdot 2^{n-2} - 1, \qquad \gamma_{A_1,1}(n) = 32 \cdot 2^{n-2} - 1,$$

where $n \geq 2$, $\gamma_{A_1}(1) = 3$, $\gamma_{A_1}(2) = 8$, $\gamma_{A_1}(3) = 14$.

It follows from Theorem 1 that the growth function γ_{A_1} has the exponential growth order and can be written as

$$\gamma_{A_1}(n) = \begin{cases} 23 \cdot 2^{\frac{n-4}{2}} - 1, & \text{if } n \text{ is even;}\\ 32 \cdot 2^{\frac{n-5}{2}} - 1, & \text{if } n \text{ is odd;} \end{cases}$$

where $n \geq 4$. The normal form of elements of S_{A_1} is declared in the following proposition:

Proposition 1. *An arbitrary element* s *of* S_{A_1} *has the following normal form*

$$\mathsf{s} = \mathsf{s}' \cdot (f_0f_2)^{p_1}\,(f_1f_0)^{p_2}\,(f_0f_2)^{p_3}\,(f_1f_0)^{p_4} \ldots (f_0f_2)^{p_{2k-1}}\,(f_1f_0)^{p_{2k}} \cdot \mathsf{s}'',$$

where $\mathsf{s}' \in \{1, f_0, f_2\}$, $\mathsf{s}'' \in \{1, f_0, f_1, f_1^2, f_1f_0^3, f_0f_2^2, f_0f_2f_1f_0f_2\}$, *and* $k \geq 1$, $p_1, p_{2k} \geq 0$, $p_i > 0$, $i \in \{2, 3, \ldots, 2k-1\}$, $\ell(\mathsf{s}) \geq 1$.

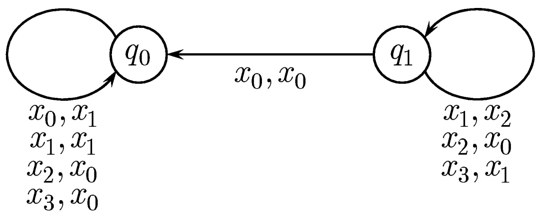

Fig. 2: The automaton A_2

4. Non-monotonic growth functions. The conception of a composite function allows us to construct easily non-monotonic functions. For example, let $k = 2$ and γ be a function such that $\gamma_1(n) = 1$ and $\gamma_2(n) = 2$. Obviously, γ is non-monotonic. Below we consider the 2-state Mealy automata over the 4-symbol alphabet, that have non-monotonic growth functions of constant, linear and square growth. There exist automata that have non-monotonic growth functions of other polynomial growth orders, but their consideration requires more technical details.

4.1. The automaton A_2 of constant growth. Let A_2 be the Mealy automaton defined by the Moore diagram on Figure 2. Its automatic transformations have the following unrolled forms:

$$f_0 = (f_0, f_0, f_0, f_0)\,(x_1, x_1, x_0, x_0)\,, \qquad f_1 = (f_0, f_1, f_1, f_1)\,(x_0, x_2, x_0, x_1)\,.$$

The automaton A_2 has a non-monotonic growth function of the constant growth order, and the graph of γ_{A_2} is shown on Figure 3. The following theorem holds:

Theorem 2. *1. The semigroup S_{A_2} has the following presentation:*

$$S_{A_2} = \left\langle\; f_0, f_1 \;\middle|\; \begin{array}{l} f_0^2 f_i = f_0 f_1^2 f_i = f_0^2,\ i \in \{0,1\};\ f_1^2 f_0^2 = f_0 f_1 f_0^2, \\ f_1 f_0 f_1 f_0^2 = f_1^4 = f_1^3 f_0,\ (f_1 f_0)^4 = (f_1 f_0)^2 \end{array} \;\right\rangle .$$

2. The growth function γ_{A_2} is a composite function for $k = 2$, and is defined by the following equalities:

$$\gamma_{A_2,0}(n) = 8, \quad \gamma_{A_2,1}(n) = 9,$$

where $n \geq 2$, $\gamma_{A_2}(1) = 2$, $\gamma_{A_2}(2) = 4$, $\gamma_{A_2}(3) = 7$.

4.2. The automaton A_3 of linear growth. Let us consider the automaton A_3, whose

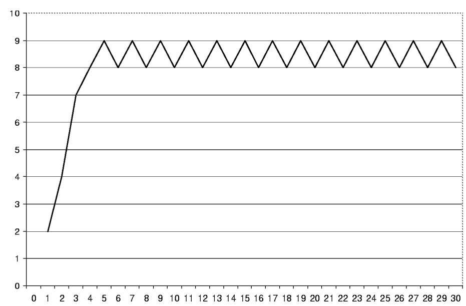

Fig. 3: The growth function of A_2

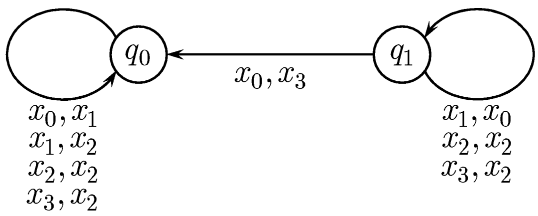

Fig. 4: The automaton A_3

Moore diagram is shown on Figure 4. Its automatic transformations have the following unrolled forms:

$$f_0 = (f_0, f_0, f_0, f_0)\,(x_1, x_2, x_2, x_2)\,, \qquad f_1 = (f_0, f_1, f_1, f_1)\,(x_3, x_0, x_2, x_2)\,.$$

The automaton A_3 have the non-monotonic linear growth function, and the graph of γ_{A_3} is shown on Figure 5. The following theorem holds:

Theorem 3. *1. The semigroup S_{A_3} has the following presentation:*

$$S_{A_3} = \left\langle\ f_0, f_1\ \middle|\ \begin{array}{l} f_0^2 f_i = f_1 f_0^2 = f_0^2,\ i \in \{0,1\};\ f_0 f_1 f_0 = f_0, \\ f_0 f_1^2 f_0 = f_0^2,\ f_1^3 f_0 f_1 = f_1 f_0 f_1^3 \end{array} \right\rangle.$$

2. The growth function γ_{A_3} is a composite function for $k = 2$, and is defined by the following equalities:

$$\gamma_{A_3,0}(n) = 4n, \quad \gamma_{A_3,1}(n) = 5n + 1,$$

where $n \geq 1$ and $\gamma_{A_3}(1) = 2$.

4.3. The automaton A_4 of square growth. Let A_4 be the automaton such that its Moore diagram is shown on Figure 6. Its automatic transformations have the following unrolled forms:

$$f_0 = (f_0, f_0, f_0, f_0)\,(x_0, x_0, x_1, x_1)\,, \qquad f_1 = (f_0, f_1, f_1, f_1)\,(x_2, x_3, x_0, x_1)\,.$$

The automaton A_4 has a non-monotonic growth function of square growth, and the graph of γ_{A_4} is shown on Figure 7. The properties of A_4 are formulated in the following theorem:

Theorem 4. *1. The semigroup S_{A_4} is infinitely presented and has the following presentation:*

$$S_{A_4} = \left\langle\ f_0, f_1\ \middle|\ \begin{array}{l} f_0^2 f_i = f_0^2,\ f_1 f_0 f_1^2 f_i = f_1 f_0 f_i,\ i \in \{0,1\}, \\ f_0 f_1^{2p+1} f_0 = f_0 f_1 f_0,\ p \geq 1 \end{array} \right\rangle. \tag{1}$$

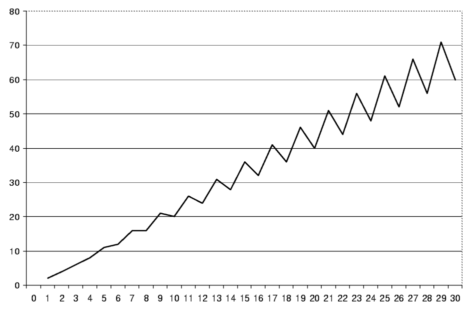

Fig. 5: The growth function of A_3

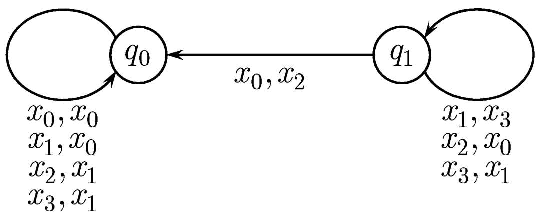

Fig. 6: The automaton A_4

2. *The growth function γ_{A_4} is a composite non-monotonic function defined by the following equalities:*

$$\gamma_{A_4,0}(n) = 4n^2 - 5n + 6, n \geq 2, \qquad \gamma_{A_4,1}(n) = \frac{7}{2}n^2 + \frac{3}{2}n + 2, n \geq 0$$

and $\gamma_{A_4}(2) = 4$. The function γ_{A_4} has the square growth order.

From the defining relations (1) the proposition follows:

Proposition 2. *An arbitrary element s of S_{A_4} admits a unique minimal-length representation as a word of one of the following forms*

$$f_0 f_1^{2p_1} (f_0 f_1)^{p_2} \cdot \mathsf{s}',$$

where $p_1 \geq 1$, $p_2 \geq 0$, $\mathsf{s}' \in \{1, f_1, f_0, f_0^2\}$, except the combination $p_1 = 1$, $p_2 = 0$, $\mathsf{s}' = 1$, or

$$f_1^{p_1} (f_0 f_1)^{p_2} \cdot \mathsf{s}',$$

where $p_1 \geq 0$, $p_2 \geq 1$, $\mathsf{s}' \in \{1, f_1, f_0, f_0^2\}$, or $p_2 = 0$, $p_1 \geq 0$, $\mathsf{s}' \in \{f_1, f_0, f_0^2\}$.

5. Growth functions with doubled finite differences. In this section we consider composite growth functions such that one of its finite differences consists of doubled values. We consider the sequence $\{B_m, m \geq 3\}$ of Mealy automata of polynomial growth, and two Mealy automata of the intermediate and the exponential growth orders.

5.1. The automata $\{B_m, m \geq 3\}$ of polynomial growth. Let B_m, $m \geq 3$, be the 2-state Mealy automaton over the m-symbol alphabet (Figure 8), and the unrolled forms of the automatic transformations $f_0 = f_{q_0,B_m}$ and $f_1 = f_{q_1,B_m}$ are defined in the following way:

$$f_0 = (f_0, f_0, f_1, f_0, \ldots, f_0, f_0)\,(x_1, x_0, x_2, x_3, \ldots, x_{m-2}, x_{m-1}),$$
$$f_1 = (f_0, f_1, f_1, f_1, \ldots, f_1, f_1)\,(x_1, x_2, x_3, x_4, \ldots, x_{m-1}, x_{m-1}).$$

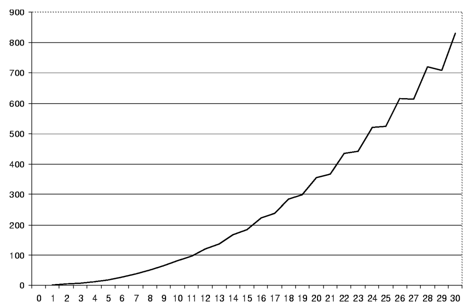

Fig. 7: The growth function of A_4

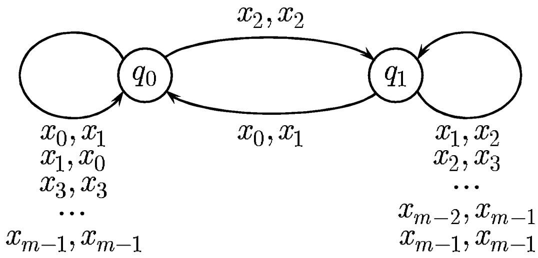

Fig. 8: The automaton B_m

Theorem 5. *1. For any $m \geq 3$ the semigroup S_m has the following presentation:*

$$S_{B_3} = \left\langle\ f_0, f_1 \ \middle|\ f_1^3 = f_0 f_1^2, f_1 f_0 f_1 = f_0^2 f_1 \ \right\rangle,$$

$$S_{B_4} = \left\langle\ f_0, f_1 \ \middle|\ f_1^4 = f_1 f_0 f_1^2,\ f_1 f_0^{p_1} f_1 f_0 f_1 = f_1 f_0^{p_1+2} f_1, p_1 \geq 0 \ \right\rangle,$$

$$S_{B_m} = \left\langle\ f_0, f_1 \ \middle|\ \begin{array}{l} \prod_{i=1}^{m-4} (f_1 f_0^{p_i}) f_1^4 = \prod_{i=1}^{m-4} (f_1 f_0^{p_i}) f_1 f_0 f_1^2, \\ \prod_{i=1}^{m-3} (f_1 f_0^{p_i}) f_1 f_0 f_1 = \prod_{i=1}^{m-3} (f_1 f_0^{p_i}) f_0^2 f_1, \\ p_i \geq 0,\ i \in \{1, 2, \ldots, m-3\} \end{array} \right\rangle.$$

All semigroups S_{B_m} for $m \geq 4$ are infinitely presented.

2. For $m \geq 3$ the growth function γ_{B_m} is defined for all $n \geq 1$ by the following equalities:

$$\gamma_{B_m}(n) = \sum_{i=0}^{m-2} \binom{n}{i} + \sum_{i=0}^{\left[\frac{n-m+1}{2}\right]} \binom{n-2i-1}{m-2} = \sum_{i=0}^{m-2} \binom{n}{i} + \sum_{i \geq 0} \binom{n-2i-1}{m-2}. \tag{2}$$

Here $[r]$ denotes the integer part of the real number r, and we assume that $\binom{n}{k} = 0$ if $k \geq n$ or $n < 0$.

The following proposition holds in the semigroup S_{B_m}.

Proposition 3. *The normal form of the element s of S_{B_m} is one of the following words*

$$f_0^{p_1} f_1 f_0^{p_2} f_1 \cdots f_0^{p_{k-1}} f_1 f_0^{p_k},$$

where $1 \leq k \leq m-1$, $p_i \geq 0$, $i \in \{1, 2, \ldots, k\}$, $\ell(\mathsf{s}) \geq 1$, and

$$f_0^{p_1} f_1 f_0^{p_2} f_1 \cdots f_0^{p_{m-2}} f_1 f_0^{2p_{m-1}} f_1 f_0^{p_m},$$

where $p_i \geq 0$, $i \in \{1, 2, \ldots, m\}$.

The corollary follows from Theorem 5.

Corollary 1. *1. For all $m \geq 3$ the function γ_{B_m} has the growth order $[n^{m-1}]$.*

2. The $(m-2)$-th finite differences of γ_{B_m} are defined by the equality

$$\gamma_{B_m}^{(m-2)}(n) = \left[\frac{n-m+1}{2}\right] + 2, \text{ where } n \geq m-1.$$

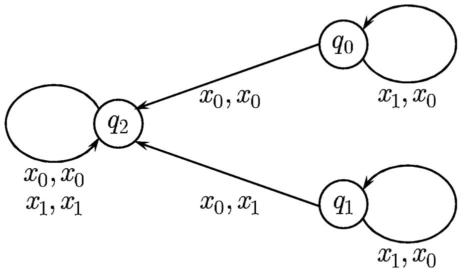

Fig. 9: The automaton A_5

It follows from (2) that for any $m \geq 4$ the equalities hold:

$$\gamma^{(1)}_{B_m}(n) = \gamma_{B_m}(n) - \gamma_{B_m}(n-1) = \sum_{i=0}^{m-3} \binom{n-1}{i} + \sum_{i \geq 0} \binom{n-2i-2}{m-3} = \gamma_{B_{m-1}}(n-1),$$

where $n \geq 2$. The growth function γ_{B_3} is defined by the equalities

$$\gamma_{B_3}(n) = \begin{cases} \frac{1}{4}n^2 + n + 1, & \text{if } n \text{ is even;} \\ \frac{1}{4}n^2 + n + \frac{3}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Hence, joining two last equalities, one has

$$\gamma^{(m-2)}_{B_m}(n) = \gamma^{(1)}_{B_3}(n-(m-3)) = \left\lceil \frac{n-(m-3)}{2} \right\rceil + 1 = \left\lceil \frac{n-m+1}{2} \right\rceil + 2,$$

for any $m \geq 3$, $n \geq m-1$. Therefore, the $(m-2)$-th finite difference of γ_{B_m} consists of doubled values, i.e. for any even integer n, $n \geq 0$, the equality holds:

$$\gamma^{(m-2)}_{B_m}(n+m) = \gamma^{(m-2)}_{B_m}(n+m-1).$$

5.2. The automaton A_5 of intermediate growth. Let A_5 be the 3-state Mealy automaton over the 2-symbol alphabet such that its Moore diagram is shown on Figure 9. The following theorem holds.

Theorem 6. *1. The semigroup S_{A_5} is an infinitely presented monoid, and has the following presentation:*

$$S_{A_5} = \left\langle\ 1, f_0, f_1 \ \middle|\ \begin{array}{l} f_0 f_1^{2^k-1} \cdot f_1^{p2^{k+1}} f_0 \prod_{i=k}^{1} \left(f_1^{2^i-1} f_0\right) = \\ = f_1^{p2^{k+1}} f_0 \prod_{i=k}^{1} \left(f_1^{2^i-1} f_0\right), k \geq 0, p \in \{0,1\}. \end{array} \right\rangle. \qquad (3)$$

2. The growth series $\Gamma_{A_5}(X)$ of A_5 and the growth series $\Gamma_{S_{A_5}}(X)$ of S_{A_5} coincide and are defined by the equality

$$\Gamma_{S_{A_5}}(X) = \Gamma_{A_5}(X) =$$
$$= \frac{1}{(1-X)^2}\left(1 + \frac{X}{1-X}\left(1 + \frac{X^2}{1-X^2} \cdot \left(1 + \frac{X^4}{1-X^4}\left(1 + \frac{X^8}{1-X^8}(1+\ldots)\right)\right)\right)\right).$$

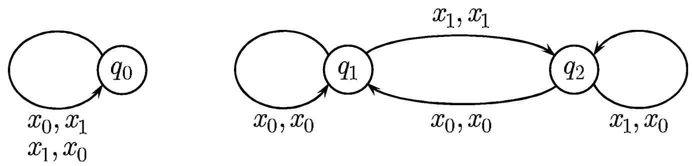

Fig. 10: The automaton A_6

The properties of the growth function γ_{A_5} are formulated in the following corollary.

Corollary 2. *1. The growth function γ_{A_5} has the intermediate growth order $\left[n^{\frac{\log n}{2\log 2}}\right]$.*

2. Let us define $\gamma^{(2)}_{A_5}(0) = \gamma^{(2)}_{A_5}(1) = \gamma^{(2)}_{A_5}(2) = 1$. The second finite difference of γ_{A_5} is defined by the following equality

$$\gamma^{(2)}_{A_5}(n) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \gamma^{(2)}_{A_5}(i),\ n \geq 3. \tag{4}$$

The system of defining relations (3) implies the following normal form.

Proposition 4. *Each semigroup element s of S_{A_5} can be written in the following normal form:*

$$f_1^{p_0} f_0 f_1^{2^{k-1}p_1 + \left(2^{k-1}-1\right)} f_0 \ldots f_1^{2^i p_{k-i} + \left(2^i - 1\right)} f_0 \ldots f_1^{4p_{k-2}+3} f_0 f_1^{2p_{k-1}+1} f_0 f_1^{p_k}$$

where $k \geq 0$, $p_i \geq 0$, $i \in \{0, 1, \ldots, k\}$.

The growth series for the second finite difference $\Delta^{(2)}\Gamma_{A_5}(X)$ can be easily constructed by using the expression for $\Gamma_{A_5}(X)$:

$$\Delta^{(2)}\Gamma_{A_5}(X) = \sum_{n\geq 3} \gamma^{(2)}_{A_5}(n)X^n + \gamma^{(2)}_{A_5}(0) + \gamma^{(2)}_{A_5}(1)X + \gamma^{(2)}_{A_5}(2)X^2 =$$
$$= (1-X)^2\Gamma_{A_5}(X) - (1-X)\gamma_{A_5}(0) - X(\gamma_{A_5}(1) - \gamma_{A_5}(0)) -$$
$$- X^2(\gamma_{A_5}(2) - 2\gamma_{A_5}(1) + \gamma_{A_5}(0)) + 1 + X + X^2 =$$
$$= 1 + \frac{X}{1-X}\left(1 + \frac{X^2}{1-X^2}\left(1 + \frac{X^4}{1-X^4}\left(1 + \frac{X^8}{1-X^8}(1+\ldots)\right)\right)\right).$$

The right-hand series of the last equality are the formal series for the numbers of partitions of n, $n \geq 1$, into "sequential" powers of 2, that is $\gamma^{(2)}_{A_5}(n)$ equals the cardinality of the set

$$\left\{\ p_0, p_1, \ldots, p_k\ \middle|\ k \geq 0, \sum_{i=0}^{k} p_i 2^i = n, p_i \geq 1, 0 \leq i \leq k\ \right\}.$$

Equality (4) is well-known for these partition numbers [1]. Therefore, the second finite difference of γ_{A_5} consists of doubled values, i.e. for any even integer n, $n \geq 2$, the equality holds:

$$\gamma^{(2)}_{A_5}(n) = \gamma^{(2)}_{A_5}(n-1).$$

5.3. The automaton A_6 of exponential growth. Let A_6 be the 3-state Mealy automaton over the 2-symbol alphabet such that its automatic transformations have the following unrolled forms:

$$f_0 = (f_0, f_0)(x_1, x_0), \qquad f_1 = (f_1, f_2)(x_0, x_1), \qquad f_2 = (f_1, f_2)(x_0, x_0).$$

The Moore diagram of A_6 is shown on Figure 10. The following theorem holds.

Theorem 7. *1. The semigroup S_{A_6} has the following presentation:*

$$S_{A_6} = \left\langle \; f_0, f_1, f_2 \;\middle|\; \begin{array}{l} f_0^2 = 1,\ f_2 f_1 = f_1 f_2 = f_2^2 = f_2 \\ f_1^2 = f_1,\ f_2 f_0 f_1 f_0 f_2 = f_1 f_0 f_1 f_0 f_2 \end{array} \; \right\rangle .$$

2. The growth series $\Gamma_{A_6}(X)$ of A_6 admits the description

$$\Gamma_{A_6}(X) = \frac{1}{(1-X)^2}\left(2X - 1 + \frac{1+X+X^3}{1-X^2-X^4}\right).$$

3. The growth series $\Gamma_{S_{A_6}}(X)$ of S_{A_6} is defined in the following way

$$\Gamma_{S_{A_6}}(X) = \frac{1}{(1-X)^2}\left(X + \frac{1+X+X^3}{1-X^2-X^4}\right).$$

Let us define the Fibonacci numbers by the symbols Φ_n, where $\Phi_n = \Phi_{n-1} + \Phi_{n-2}$, $n \geq 2$, and $\Phi_0 = \Phi_1 = 1$. It follows from Theorem 7 that the growth function γ_{A_6} can be written in close form and the following corollary holds.

Corollary 3. *The growth function γ_{A_6} is defined by the following equalities:*

$$\gamma_{A_6}(n) = \begin{cases} \Phi_{[\frac{n}{2}]+6} + \Phi_{[\frac{n}{2}]+4} - 2n - 18, & \text{if } n \text{ is even;} \\ \Phi_{[\frac{n}{2}]+6} + 2\Phi_{[\frac{n}{2}]+4} - 2n - 18, & \text{if } n \text{ is odd.} \end{cases} \tag{5}$$

The growth function γ_{A_6} has the exponential growth order.

Let n be any positive integer, and represent $n = 2k$, when n is even, and $n = 2k+1$, when n is odd. From (5) it follows that for any $k \geq 0$ the following equalities hold:

$$\gamma_{A_6}^{(1)}(2k+1) = \Phi_{k+4} - 2, \quad \gamma_{A_6}^{(1)}(2k+2) = 2\Phi_{k+3} - 2,$$

and, using the previous equalities, we have

$$\gamma_{A_6}^{(2)}(2k+1) = \Phi_{k+1}, \quad \gamma_{A_6}^{(2)}(2k+2) = \Phi_{k+1}.$$

Hence, the second finite difference γ_{A_6} consists of doubled values and for all even integer n the equality holds:

$$\gamma_{A_6}^{(2)}(n) = \gamma_{A_6}^{(2)}(n-1).$$

6. Final remarks. There are some questions that concern the composite non-monotonic growth functions of Mealy automata.

1. Does there exist a Mealy automaton such that its composite growth function includes "parts" of different growth orders?
2. Does there exist a Mealy automaton which has the non-monotonic growth function of the intermediate or the exponential growth order?
3. Does there exist a Mealy automaton such that its growth function is non-monotonic, but is not a composite function (in the sense of Section 3)?

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5, Krasnogvardeyskaya st., of. 2
02094, Kyiv, Ukraine
Illya.Reznykov@ikc5.com.ua

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