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ON THE FOURIER SERIES OF THE ZETA-FUNCTION LOGARITHM ON THE VERTICAL LINES

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The Jensen-Littlewood theorem for a rectangle is generalized. The generalization is applied to the study of Fourier's series of the Riemann zeta-function logarithm on the vertical lines.

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Обобщена теорема Литтлвуда-Иенсена для прямоугольника. Это обобщение применено к изучению ряда Фурье логарифма дзета-функции на вертикальных прямых.

Introduction and main results. The Riemann zeta-function is defined as

$$\zeta(s)=\sum_{n\in\mathbb{N}}\frac{1}{n^s}\quad\text{or}\quad \zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},\quad \mathrm{Re}s>1,$$

where the product is over all prime p.

This function was first considered by Leonhard Euler for real s in 1737. He also represented it as the product over the primes. G. F. B. Riemann showed (1859) that $\zeta(s)$ had a meromorphic continuation to $\mathbb{C}$ with a single pole at $s=1$.

In his fundamental paper [1] (see also [2]) J. Littlewood established for a rectangle an analogue of the well-known Jensen theorem and deduced from it, in particular, that

$$\int\limits_\sigma^1 N(\eta,T)\,d\eta=O\left(T\log\frac{1}{\sigma-\frac{1}{2}}\right),\quad \sigma>1/2,\quad T\to\infty,$$

where $N(\sigma,T)$ is the number of zeroes of the Riemann zeta-function $\zeta(s)$ whose imaginary part γ satisfies $0<\gamma<T$ and the real part is greater than σ. Further, this result was improved [2].

Our purpose is to generalize the Jensen-Littlewood theorem for a rectangle and apply the generalized theorem for the study of the zeta-function and its zeroes.

If $f(s)$ is a holomorphic function in the rectangle $R=\{s=\sigma+it: 0<t<T,\,\alpha<\sigma<\beta\}$ then the function $\log|f(s)|$ is subharmonic in R, $\Delta\log|f(s)|\geq 0$ in the sense of distributions

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on R where Δ is the Laplace operator and $\frac{1}{2\pi}\Delta \log|f(s)| = \sum_\rho \delta(s-\rho)$. Here $\{\rho\}$ is the sequence of zeroes of f taking into account their multiplicities and $\delta(s-\rho)$ is the Dirac measure concentrated at the point ρ.

Denote the measure $\frac{1}{2\pi}\Delta \log|f(s)|$ by μ, consider the orthogonal system $\left\{e^{i\frac{2\pi}{T}kt}\right\}$, $k \in \mathbb{Z}$, on $[0,T]$ and the Fourier-Stielties coefficients

$$N_k(\alpha,T) = \iint\limits_R e^{-i\frac{2\pi}{T}kt}\, d\mu(s) = \sum_{\rho_j \in R} e^{-i\frac{2\pi}{T}k\gamma_j}, \quad \gamma_j = \mathrm{Im}\rho_j.$$

If $f(s) = \zeta(s)$, $\alpha = \sigma$ and $\beta \geq 1$ we have $N_0(\sigma,T) = N(\sigma,T)$. The coefficients $N_k(\alpha,T)$, $k \in \mathbb{Z}$, determine completely the distribution of zeroes of the function f in R.

Define $\log\zeta(s)$ as usually [1], [2] (see also $\mathbf{1^0}$ below) and denote its Fourier coefficients by

$$l_k(\sigma,T) = \frac{1}{T}\int\limits_0^T e^{-i\frac{2\pi}{T}kt} \log\zeta(\sigma+it)\, dt, \quad k \in \mathbb{Z}.$$

Generalizing the Jensen-Littlewood theorem, we establish connections between $N_k(\sigma,T)$ and $l_k(\sigma,T)$ and some their properties.

By the Parseval equality and the Hausdorff-Young inequality the coefficients $l_k(\sigma,T)$ are connected with the integral means of $\log\zeta(s)$. It is easy to prove the following

Proposition. i) *The Riemann Hypothesis* (RH) *for the zeta-function is equivalent to the following assertion.*

For any fixed σ, $1/2 < \sigma < 1$, and any fixed $T > 0$ there exists $C(\sigma,T)$ such that

$$\left(\frac{1}{T}\int\limits_0^T \big|\log|\zeta(\sigma+it)|\big|^q\, dt\right)^{1/q} \leq C(\sigma,T) \tag{1}$$

for all $q \geq 1$.

ii) *For the validity of* RH *the following condition is sufficient.*

For any fixed σ, $0 < \sigma < 1$, and any fixed $T > 0$, there exists $c(\sigma,T)$ such that

$$\|l_k(\sigma,T)\|_p \leq c(\sigma,T) \tag{2}$$

for all p, $1 < p \leq 2$, where

$$\|l_k(\sigma,T)\|_p = \left(\sum_{k\in\mathbb{Z}} |l_k(\sigma,T)|^p\right)^{1/p}.$$

Indeed, if RH holds then $\log|\zeta(\sigma+it)|$ is continuous on $[0,T]$, $1/2 < \sigma < 1$ and we have (1) with $C(\sigma,T) = \max\left\{\big|\log|\zeta(\sigma+it)|\big| : 0 \leq t \leq T\right\}$.

Conversely if we have (1) for all $q \geq 1$ then $\sup_{0<t<T}\big|\log|\zeta(\sigma+it)|\big| < +\infty$, $1/2 < \sigma < 1$, and we obtain $\zeta(\sigma+it) \neq 0$.

Using the Hausdorf-Young inequality we obtain (1) from (2).

Note also that with the use of the Parseval equality, the results of A. Selberg [3], M. Balazard and A. Ivič [4] give

$$\sum_{k\in\mathbb{Z}} \left|l_k(\frac{1}{2},T)\right|^2 = \log\log T + O\left(\sqrt{\log\log T}\right), \quad T\to\infty,$$

and

$$\sum_{k\in\mathbb{Z}} |l_k(\sigma,T)|^2 = \sum_{k=1}^{\infty}\frac{1}{k^2}\sum_p \frac{1}{p^{2k\sigma}} + r(\sigma,T),$$

for fixed σ, $1/2<\sigma<1$, where p denotes primes, $r(\sigma,T)=O(T^{c(\sigma)-1})$, $0<c(\sigma)<1$, $T\to+\infty$.

Put $K=2\pi k/T$, $k\in\mathbb{Z}$, $T>0$.

Theorem 1. *The following relations hold:*

$$l_k(\sigma,T) = \frac{2\pi}{T}e^{K\sigma}\int_\sigma^\beta e^{-K\eta}N_k(\eta,T)\,d\eta +$$

$$+\frac{i}{T}e^{K\sigma}\int_\sigma^\beta e^{-K\eta}\left(\log\zeta(\eta+iT)-\log\zeta(\eta)\right)\,d\eta + e^{K(\sigma-\beta)}l_k(\beta,T), \quad k\in\mathbb{Z}, \tag{3}$$

$$\frac{2\pi}{T}\int_\sigma^\beta N_k(\eta,T)\,d\eta = l_k(\sigma,T) + K\int_\sigma^\beta l_k(\eta,T)\,d\eta +$$

$$+\frac{i}{T}\int_\sigma^\beta \left(\log\zeta(\eta)-\log\zeta(\eta+iT)\right)\,d\eta - l_k(\beta,T), \quad k\in\mathbb{Z}, \tag{4}$$

$0<\sigma<\beta<1$.

For $\beta\ge 1$ the relations are slightly modified (see the remark below), because $\zeta(s)$ has the pole at $s=1$.

If $k=0$ both of the relations give the Jensen-Littlewood theorem.

We prove also the following properties of the Fourier coefficients $l_k(\sigma,T)$.

Theorem 2. *For fixed $T>0$ the Fourier coefficients $l_k(\sigma,T)$ are continuous functions of σ. They are bounded for $\sigma\ge\sigma_0>1/2$, $T\ge 1$, by a constant depending of σ_0. The Fourier coefficient $l_0(\sigma,T)$ is bounded if $\sigma\ge 1/2$, $T\ge 1$.*

1^0. Preliminary Lemmas.

Lemma 1. *([5]) If $u(t)\ge 0$ on $[0,T]$ and $\mathrm{I}=\frac{1}{T}\int_0^T u(t)\,dt$ exists, then*

$$\frac{1}{T}\int_0^T \log^+ u(t)\,dt \le \max(1,\log\mathrm{I}).$$

Lemma HL ([1]).

$$\int_2^T |\zeta(\sigma+it)|^2\,dt \le AVT, \quad 2\le T, \quad 1/2\le\sigma\le 2,$$

where $A=\text{const}$, $V=\min\{\log T, (\sigma-1/2)^{-1}\}$.

To formulate the following lemma we introduce some notions.

Let φ be a holomorphic function in the closure of the rectangle $R_\alpha=\{s=\sigma+it : \alpha<\sigma<\beta,\ 0<t<T\}$ that does not have zeroes on ∂R_α. Denote by $\{\rho_j\}$ the set of its zeroes in R_α, $\rho_j=\sigma_j+it_j$.

Let $\log\varphi(\beta)$ be determined. Put

$$\log\varphi(s)-\log\varphi(\beta)=\int_\beta^s \frac{\varphi'(\xi)}{\varphi(\xi)}\,d\xi,$$

where the integral is taken along a path in $\overline{R_\alpha}$ with the slits $\left\{\tau\sigma_j+i\,t_j : \frac{\alpha}{\sigma_j}\le\tau\le 1\right\}$, whose endpoints are β and s.

Further, we denote

$$l_k(\sigma,T)=\frac{1}{T}\int_0^T e^{-iKt}\log\varphi(\sigma+it)\,dt, \quad k\in\mathbb{Z},$$

$$N_k(\sigma,T)=\sum_{\rho_j\in R_\sigma} e^{-iKt_j}, \quad k\in\mathbb{Z}, \quad t_j=\mathrm{Im}\rho_j,$$

$$M_k(\sigma,T)=\frac{2\pi}{T}\int_\sigma^\beta N_k(\eta,T)\,d\eta, \quad k\in\mathbb{Z}.$$

Denote also $N_0(\sigma,T)=N(\sigma,T)$. The function $N(\sigma,T)$ gives the number of zeroes of φ in R_σ.

Lemma 2. *The following relations hold:*

$$l_k(\alpha,T)=\frac{2\pi}{T}e^{K\alpha}\int_\alpha^\beta e^{-K\sigma}N_k(\sigma,T)\,d\sigma+$$

$$+\frac{i}{T}e^{K\alpha}\int_\alpha^\beta e^{-K\sigma}\left(\log\varphi(\sigma+iT)-\log\varphi(\sigma)\right)\,d\sigma+e^{K(\alpha-\beta)}l_k(\beta,T), \quad k\in\mathbb{Z}, \tag{5}$$

$$M_k(\alpha,T)=l_k(\alpha,T)+K\int_\alpha^\beta l_k(\sigma,T)\,d\sigma+$$

$$+\frac{i}{T}\int_\alpha^\beta\left(\log\varphi(\sigma)-\log\varphi(\sigma+iT)\right)\,d\sigma-l_k(\beta,T), \quad k\in\mathbb{Z}. \tag{6}$$

The proof is a routine calculation based on the known idea of integrating the Argument Principle and consists in frequent integration by parts and elementary transformations.

Theorem 1 is Lemma 2 formulated for $\zeta(s)$.

Remark. If $\sigma > 1$ then neither zeroes nor poles of $\zeta(s)$ lie in R. So, $N_k(\sigma, T) = 0$, $\eta \geq \sigma$, $\log \zeta(s)$ is a holomorphic function on $\overline{R}$ [2] and relation (3) takes the form

$$l_k(\sigma, T) = \frac{i}{T} e^{K\sigma} \int\limits_{\sigma}^{\beta} e^{-K\eta} \left(\log \zeta(\eta + iT) - \log \zeta(\eta)\right) d\eta + e^{K(\sigma-\beta)} l_k(\beta, T), \quad k \in \mathbb{Z}. \tag{7}$$

This is Cauchy's theorem for the holomorphic function $e^{-Ks} \log \zeta(s)$ on $\overline{R}$.

Integrating (7) over σ from α to β we obtain

$$l_k(\alpha, T) + K \int\limits_{\alpha}^{\beta} l_k(\sigma, T)\, d\sigma + \frac{i}{T} \int\limits_{\alpha}^{\beta} \left(\log \zeta(\sigma) - \log \zeta(\sigma + iT)\right) d\sigma - l_k(\beta, T) = 0, \quad k \in \mathbb{Z},$$

after routine calculations. This is (4) for $\sigma = \alpha$.

2⁰. Proof of Theorem 2. Let $k = 0$ then relation (3) written for $\zeta(s)$ implies

$$l_0(\sigma, T) = \frac{2\pi}{T} \int\limits_{\sigma}^{\beta} N_0(\eta, T)\, d\eta + \frac{i}{T} \int\limits_{\sigma}^{\beta} \left(\log \zeta(\eta + iT) - \log \zeta(\eta)\right) d\eta + l_0(\beta, T). \tag{8}$$

If in (8) we make $\beta \to +\infty$ then the last term disappears, and the first integral of the right side is equal to $\int_\sigma^1 N_0(\eta, T)\, d\eta$. Taking into account that $\zeta(s)$ has the simple pole at $s = 1$ we obtain

$$l_0(\sigma, T) = \frac{2\pi}{T} \int\limits_{\sigma}^{1} N_0(\eta, T)\, d\eta - \frac{i}{T} \int\limits_{\sigma}^{+\infty} \left(\log \zeta(\eta) - \log \zeta(\eta + iT)\right) d\eta, \tag{9}$$

where

$$N_0(\sigma, T) = \begin{cases} N(\sigma, T) - \frac{1}{2}, & \sigma < 1, \\ 0, & \sigma > 1. \end{cases}$$

It was proved in [1] that

$$\int\limits_{\sigma}^{+\infty} \left(\log \zeta(\eta) - \log \zeta(\eta + iT)\right) d\eta = O(\log T), \quad T \to +\infty, \quad 1/2 \leq \sigma \leq 2, \tag{10}$$

So, (9) gives

$$l_0(\sigma, T) = \frac{2\pi}{T} \int\limits_{\sigma}^{1} \left(N(\eta, T) - \frac{1}{2}\right) d\eta + O\left(\frac{\log T}{T}\right), \quad T \to +\infty.$$

Using Selberg's result (see, for example, [2, p. 240])

$$\int\limits_{1/2}^{1} N(\sigma, T)\, d\sigma = O(T), \quad T \to +\infty, \tag{11}$$

we obtain

$$l_0(\sigma, T) = O(1), \quad T \to +\infty, \quad 1/2 \le \sigma.$$

Let $k \in \mathbb{N}$. If in (3) we make $\beta \to +\infty$ then

$$\frac{2\pi}{T} \int\limits_{\sigma}^{+\infty} e^{-K\eta} N_k(\eta T)\, d\eta = e^{-K\sigma} l_k(\sigma, T) + \frac{i}{T} \int\limits_{\sigma}^{+\infty} e^{-K\eta} \left(\log \zeta(\eta) - \log \zeta(\eta + iT)\right) d\eta, \quad k \in \mathbb{Z},$$

where $N_k(\sigma, T) = \sum_{\rho_j \in R_\sigma} e^{-iK\gamma_j} - \frac{1}{2}$.

Consequently,

$$|l_k(\sigma, T)| \le \frac{2\pi}{T} \int\limits_{\sigma}^{+\infty} e^{-K(\eta-\sigma)} N_k(\eta, T)\, d\eta +$$

$$+ \frac{1}{T} \left| \int\limits_{\sigma}^{+\infty} e^{-K(\eta-\sigma)} \left(\log \zeta(\eta) - \log \zeta(\eta + iT)\right) d\eta \right|. \tag{12}$$

We will estimate the integral on the right side of inequality (12). Integration by parts in the second integral of the right side of relation (12) gives

$$\left| \int\limits_{\sigma}^{+\infty} e^{-K(\eta-\sigma)} \left(\log \zeta(\eta) - \log \zeta(\eta + iT)\right) d\eta \right| =$$

$$= \left| \int\limits_{\sigma}^{+\infty} e^{-K(\eta-\sigma)}\, d \int\limits_{\eta}^{+\infty} \left(\log \zeta(\omega) - \log \zeta(\omega + iT)\right) d\omega \right| =$$

$$= \left| - \int\limits_{\sigma}^{+\infty} \left(\log \zeta(\eta) - \log \zeta(\eta + iT)\right) d\eta \right. +$$

$$\left. + K \int\limits_{\sigma}^{+\infty} e^{-K(\eta-\sigma)} \int\limits_{\eta}^{+\infty} \left(\log \zeta(\omega) - \log \zeta(\omega + iT)\right) d\omega\, d\eta \right| \le \tag{13}$$

$$\le \left| \int\limits_{\sigma}^{+\infty} \left(\log \zeta(\eta) - \log \zeta(\eta + iT)\right) d\eta \right| +$$

$$+ K \left| \int\limits_{\sigma}^{+\infty} e^{-K(\eta-\sigma)} \int\limits_{\eta}^{+\infty} \left(\log \zeta(\omega) - \log \zeta(\omega + iT)\right) d\omega\, d\eta \right|.$$

Applying (10) to both last integral in (13) we have

$$\left|\int\limits_{\sigma}^{+\infty} e^{-K(\eta-\sigma)}\left(\log\zeta(\eta)-\log\zeta(\eta+iT)\right)d\eta\right| = O(\log T), \quad T\to+\infty. \tag{14}$$

Using once more Selberg's result (11) and the inequality $|N_k(\sigma,T)|\le|N(\sigma,T)|$ we obtain from (12) and (14)

$$|l_k(\sigma,T)| = O(1), \quad T\to+\infty, \quad k\in\mathbb{N}.$$

Now let $1/2<\sigma_0\le\sigma\le 2$, $1\le T$.

Taking the imaginary parts in (9) and using some properties of $\zeta(s)$ J. Littlewood obtained [1]:

$$2\pi\int\limits_{1/2}^{1} N(\sigma,T)\,d\sigma = \int\limits_{0}^{T}\log\left|\zeta\left(\frac{1}{2}+it\right)\right|dt + O(\log T), \quad T\to+\infty.$$

Since the left side of this equality is nonnegative, we have

$$\int\limits_{0}^{T}\log^{-}|\zeta(\sigma+it)|\,dt \le \int\limits_{0}^{T}\log^{+}|\zeta(\sigma+it)|\,dt + O(\log T), \quad T\to+\infty.$$

Consequently,

$$\int\limits_{0}^{T}\big|\log|\zeta(\sigma+it)|\big|\,dt \le 2\int\limits_{0}^{T}\log^{+}|\zeta(\sigma+it)|\,dt + O(\log T), \quad T\to+\infty.$$

On the other hand, using Lemma 1 and Lemma HL for sufficiently large T we obtain

$$\frac{2}{T-2}\int\limits_{2}^{T}\log^{+}|\zeta(\sigma+it)|\,dt \le \max\left(1,\log\left(\frac{1}{T-2}\int\limits_{2}^{T}|\zeta(\sigma+it)|^2\,dt\right)\right)\le$$
$$\le\max\left(1,\log\left(\frac{AT}{T-2}\min\left(\log T,\frac{1}{\sigma_0-\frac{1}{2}}\right)\right)\right)\le\log\frac{AT}{T-2}+\log\frac{1}{\sigma_0-\frac{1}{2}}=$$
$$=\log\frac{1}{\sigma_0-\frac{1}{2}}+O(1), \quad T\to+\infty.$$

So,

$$\frac{1}{T}\int\limits_{0}^{T}\big|\log|\zeta(\sigma+it)|\big|\,dt \le \frac{2}{T}\int\limits_{0}^{T}\log^{+}|\zeta(\sigma+it)|\,dt + O(1) = \tag{15}$$
$$=\log\frac{1}{\sigma_0-\frac{1}{2}}+O(1), \quad T\to+\infty.$$

Denote the Fourier coefficients of $\log|\zeta(s)|$ and $\arg\zeta(s)$ by $C_k(\sigma,T)$ and $A_k(\sigma,T)$ respectively.

It follows from (15) that the Fourier coefficients $C_k(\sigma,T)$, $k\in\mathbb{Z}$, are bounded for $\sigma_0\le\sigma\le 2$, $1\le T$.

Since $l_k(\sigma,T)=C_k(\sigma,T)+i\,A_k(\sigma,T)$, $A_{-k}(\sigma,T)=\overline{A_k(\sigma,T)}$, and

$$|A_k(\sigma,T)|\le|l_k(\sigma,T)|+|C_k(\sigma,T)|=\log\frac{1}{\sigma_0-\frac12}+O(1),\quad T\to+\infty,\quad k\in\mathbb{N},$$

we have

$$|l_{-k}(\sigma,T)|\le|C_k(\sigma,T)|+|A_k(\sigma,T)|=\log\frac{1}{\sigma_0-\frac12}+O(1),\quad T\to+\infty,\quad k\in\mathbb{N},$$

where the constant in $O(1)$ is absolute. This completes the proof.

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