УДК 512.552.12

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COADEQUATE ELEMENTS OF A COMMUTATIVE BEZOUT DOMAIN

T. Kysil'. *Coadequate elements of a commutative Bezout domain*, Matematychni Studii, **22** (2004) 94–96.

In this paper the theory of coadequate elements of a commutative Bezout domains is developed by investigating the structure of maximal noncoadequate ideals.

Т. Кысиль. *Коадекватные елементы коммутативной области Безу* // Математичні Студії. – 2004. – Т.22, №1. – C.94–96.

В статье развита теория коадекватных елементов коммутативной области Безу посредством исследования структуры максимальных некоадекватных идеалов.

Adequate domains appeared in the ring theory as elementary divisor rings without limitations on descending chains [1–6].

Henriksen constructed an example of an elementary divisor ring which is not adequate. In fact, he presented an example of a general adequate ring, which naturally generalizes adequate rings [7]. The notion of a coadequaete element, which is dual to that of adequate element, is considered in [7] and was introduced in [3], where it was shown that the idempotents of a ring are such elements.

Recall that a commutative ring R with $1 \neq 0$ is a Bezout ring, if every finitely generated ideal of R is principal.

In what follows, R is a commutative Bezout domain with $1 \neq 0$.

We say that a nonzero element $a \in R$ is *adequate*, if for every $b \in R$ there exist $r, s \in R$ such that

(i) $a = rs$,

(ii) $rR + bR = R$,

(iii) the ideal $s'R + bR$ is proper for all $s' \in R$ such that $sR \subset s'R \neq R$.

A ring whose elements are all adequate, is called an adequate ring.

Definition 1. An element $a \in R$ is said to be *coadequate* if for any nonzero $b \in R$ there exist $r, s \in R$ such that

(i) $b = rs$,

(ii) $rR + aR = R$,

(iii) $s'R + aR \neq R$ for any noninvertible divisor s' of s.

2000 *Mathematics Subject Classification:* 16S10.

Denote by S the set of all adequate elements of R. It is clear that $S \neq \varnothing$ because S contains all invertible elements.

Proposition 1. *If R is a commutative Bezout domain, then the set S is multiplicatively closed.*

Proof. Let $a, b \in S$ and $c \in R \setminus \{0\}$. By the definition of coadequate elements we have the decomposition

$$c = r_a s_a = r_b s_b. \tag{1}$$

Let $rR = r_a R + r_b R$, then $r_a = r r_a^0$ and $r_b = r r_b^0$. Therefore, $c = r r_a^0 s_a = r r_b^0 s_b$. We are going to show that $rR + s_a R = R$. Let $rR + s_a R = hR \neq R$. Since $rR \subset hR$, according to (1) we have $hR + aR = R$. On the other hand, $s_a R \subset hR \neq R$, then by the definition of a coadequate element and by (1) we have $hR + aR \neq R$. Thus, $rR + s_a R = R$. Since $c = r r_a^0 s_a = r r_b^0 s_b$ and R is a domain, we obtain

$$r_a^0 s_a = r_b^0 s_b. \tag{2}$$

Using the equality $r_a^0 R + r_b^0 R = R$ and (2), we obtain $s_b R \subset r_a^0 R$. Then $c = r r_a^0 s_a$ is the required decomposition. Indeed, $rR + abR = R$ and if $r_a^0 R \subset s'R \neq R$, then $s'R + abR \neq R$. □

Definition 2. An ideal N of R is called *noncoadequate* if R does not contain coadequate elements.

The following proposition demonstrates that the set of all noncoadequate ideals contains a maximal element, i.e. an ideal N such that any ideal M with $N \subsetneq M$ contains a noncoadequate element. An ideal having this property is called maximal noncoadequate.

Proposition 2. *Every noncoadequate ideal is contained in a maximal noncoadequate ideal.*

Proof. Denote by H the set of all noncoadequte ideals of the ring R. Since $(0) \in H$, $H \neq \varnothing$. Let $\{I_\alpha\}_{\alpha \in \Omega}$ be a chain in the set H with respect to inclusion. Consider the ideal $I = \bigcup_{\alpha \in \Omega} I_\alpha$. If $I \in H$, then there exists $a \in I \cap S$, and hence there exists $\beta \in \Omega$ such that $a \in I_\beta$. This is a contradiction to $I_\beta \cap S = \varnothing$. Therefore $I \in H$. Thus the set H is inductive, and by Zorn's lemma there exists at least one maximal element in H which is a maximal noncoadequate ideal. □

Proposition 3. *Every maximal noncoadequate ideal P of R is prime.*

Proof. Suppose to the contrary that there exist $a, b \in R \setminus P$ such that $ab \in P$. Consider the ideal $P + aR$. Since $a \notin P$, by the definition of P we have $(P + aR) \cap S \neq \varnothing$, that is there exists an element $c \in S$ such that $c \in (P + aR)$. Now consider the ideal $J = \{x \mid cx \in P\}$. Obviously, $P \subset J$, but $P \neq J$ because $b \in J$ and $b \notin P$. Then there exists an element $d \in S \cap J$ for which $cd \in P$. Note that according to Proposition 1 $cd \in S$, which is impossible. □

Theorem 1. *Let R be a commutative Bezout domain, in which every maximal noncoadequate ideal is not a maximal ideal. Then every nonzero prime ideal of R is contained in a maximal noncoadequate ideal.*

Proof. Suppose that for some nonzero prime ideal $P \in R$ we have $P \subset M \cap M'$, where M and M' are two distinct maximal noncoadequate ideals. Due to the conditions on R there exists a coadequate element $m \in R$ such that $M \subset mR \neq R$. Since $M \neq M'$, we see that $M + M' = R$, and then $mR + M' = R$. Thus $mR + m'R = R$ for some $m' \in M'$. For any $p \in P \setminus \{0\}$ we have $p = rs$, where $rR + mR = R$ and $s'R + mR \neq R$ for every noninertible divisor s' of the element s. Since $P \subset M \cap M'$, $p \subset mR$ and by $rR + mR = R$ we obtain that $p = rs \in P$ implies $s \in P$. Let $sR + m'R = tR$. Since $s \in P \subset M'$, t is a noninvertible element of R. But $tR + mR \supset m'R + mR = R$, and therefore t is an invertible element of R; this is impossible. □

As a corollary we obtain the following result.

Theorem 2. *Let R be a commutative Bezout domain and let P be a prime ideal of R that contains at least one coadequate element. Then P is contained in exactly one maximal ideal.*

Proof. If P is a maximal ideal then we have nothing to prove. Let $P \subseteq M \cap M'$, where M and M' are distinct maximal ideals and $a \in P \cap S$. From the maximality of M and M' it follows that there exist $m \in M$ and $m' \in M'$ such that $mR + m'R = R$. If $a \in S$, then m can be decomposed as $m = rs$, where $rR + aR = R$ and $s'R + aR \neq R$ for every noninvertible divisor s' of the element s. Since P is prime and $P \subseteq M \cap M'$, $s \in P$. Let $dR = sR + m'R$. Then d is not an invertible element of R because $P \subset M'$. But $dR + nR \supseteq m'R + mR = R$, and this is impossible. Thus P is contained in exactly one maximal ideal. □

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Received 04.12.2003
Revised 25.05.2004