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INVARIANT SUBSPACES OF LORENTZ TYPE FOR UNBOUNDED OPERATORS ON INTERPOLATION SPACES

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Unbounded operators on intermediate spaces of Banach spaces generated by the method of real interpolation are investigated. Invariant subspaces of Lorentz type for such unbounded operators are described.

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Изучаются неограниченные операторы на промежуточных пространствах банаховых пространств, построенные методом действительной интерполяции. Для таких неограниченных операторов описаны инвариантные подпространства типа Лоренца.

1. Introduction. In the paper we investigate certain intermediate spaces having property of invariancy between the given Banach space and the domain of an arbitrary unbounded operator. For this purpose we introduce a new definition of Lorentz type invariant subspaces for an unbounded operator in Banach space. We study some properties of such subspaces and also properties of the given operator on these subspaces. The obtained results are illustrated for the operator of differentiation. The technique of research is based on interpolation properties of invariant subspaces of unbounded operators advanced in [3], [4]. For interpolation it the known Lions-Peetre method is used. We use the standard notations and definitions from [1], [6].

2. Invariant subspaces of Lorentz type. In this section, for arbitrary closed linear operator A on a Banach space one special class of its invariant subspaces $l^{\nu}_{q,p}(\mathcal{C}^m, \mathcal{C}^{m+r})_{\theta,g}$ is determined. As will be noticed later, the determined subspaces are interpolation Lorentz spaces. Therefore we call them as *A-invariant subspaces of Lorentz type*.

Let in a complex Banach space $(\mathfrak{X}, \|\cdot\|)$ a closed unbounded linear operator A with the dense domain $\mathfrak{X}^1$ is given. Denote by $\mathfrak{X}^m = \{x \in \mathfrak{X}^{m-1} : \ Ax \in \mathfrak{X}^{m-1}\}$, $(m \in \mathbb{N})$ the domain of the operator power A^m with the norm

$$\|x\|_{\mathfrak{X}^m} = \sum_{k=0}^{m} \|A^k x\|, \qquad x \in \mathcal{C}^m,$$

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and set $\mathfrak{X}^\infty := \bigcap_{m=0}^{\infty} \mathfrak{X}^m$. Clearly, $\mathfrak{X}^0 = \mathfrak{X}$ for $A^0 = I$. Suppose that the condition $\rho(A) \neq \varnothing$ for the resolvent set is satisfied. Then the conditions of density $\overline{\mathfrak{X}^\infty} = \mathfrak{X}$, $\overline{\mathfrak{X}^{m+1}} = \mathfrak{X}^m$ hold and the operator powers A^m are closed in $\mathfrak{X}$ (see [2, Theorem VII.9.7]). From closeness of the operators A^m completeness of the spaces $\mathfrak{X}^m$ follows.

Using the Lions-Peetre method [1] we shall construct a scale of intermediate spaces between $\{\mathfrak{X}^m : m \in \mathbb{Z}_+\}$. Just on the sum of spaces $\mathfrak{X}^m + \mathfrak{X}^{m+1}$ we define the norm $K_m(t,x) = \inf_{x=x_0+x_1} \left(\|x_0\|_{\mathfrak{X}^m} + t\|x_1\|_{\mathfrak{X}^{m+1}}\right)$, where $x_0 \in \mathfrak{X}^m$, $x_1 \in \mathfrak{X}^{m+1}$ and $t > 0$. On the intersection of the spaces $\mathfrak{X}^m \cap \mathfrak{X}^{m+1}$ we define the norm $J_m(t,x) = \max\left(\|x\|_{\mathfrak{X}^m}, t\|x\|_{\mathfrak{X}^{m+1}}\right)$, $x \in \mathfrak{X}^m \cap \mathfrak{X}^{m+1}$. For numbers $0 < \theta < 1$ and $1 \le g \le \infty$ we defined the intermediate subspaces

$$\left(\mathfrak{X}^m, \mathfrak{X}^{m+1}\right)_{\theta,g} = \left\{x \in \mathfrak{X}^m + \mathfrak{X}^{m+1} : \|x\|_{\theta,g,m} < \infty\right\}$$

with the norm

$$\|x\|_{\theta,g,m} = \begin{cases} \left[\displaystyle\int_0^\infty [t^{-\theta}\Phi_m(t,x)]^g \frac{dt}{t}\right]^{\frac{1}{g}} : & 1 \le g < \infty, \\ \displaystyle\sup_{0<t<\infty} t^{-\theta}\Phi_m(t,x) : & g = \infty, \end{cases}$$

where $\Phi_m(t,x) = K_m(t,x)$ or $J_m(t,x)$. In view of [1, Theorem 3.3.1] the norms defined by the functions $K_m(t,x)$ and $J_m(t,x)$ on the space $(\mathfrak{X}^m, \mathfrak{X}^{m+1})_{\theta,g}$ are equivalent.

For any numbers $0 < \nu < \infty$, $1 \le p, q \le \infty$ we define the space

$$l_{q,p}^{\nu}\left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g} = \left\{x \in \mathfrak{X}^\infty : \|x\|_{l_{q,p}^{\nu}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} < \infty\right\}$$

with the norm

$$\|x\|_{l_{q,p}^{\nu}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} = \begin{cases} \left[\displaystyle\sum_{m=0}^{\infty} (m+1)^{\frac{p}{q}-1} \left(\sup_{n\ge m} \frac{\|A^n x\|_{\theta,g,0}}{\nu^n}\right)^p\right]^{\frac{1}{p}} : & p < \infty, \\ \displaystyle\sup_{m\in\mathbb{Z}_+} \left[(m+1)^{\frac{1}{q}} \left(\sup_{n\ge m} \frac{\|A^n x\|_{\theta,g,0}}{\nu^n}\right)\right] : & p = \infty. \end{cases}$$

In the following statement, basic spectral and interpolations properties of the specified spaces are established.

Theorem 2.1. *The restriction $A\,|_{(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}$ of an unbounded operator A onto the intermediate subspace $(\mathfrak{X}, \mathfrak{X}^1)_{\theta,g}$ is a closed operator with the dense domain $(\mathfrak{X}^1, \mathfrak{X}^2)_{\theta,g}$. For arbitrary $\nu > 0$ the normed spaces $l_{q,p}^{\nu}(\mathfrak{X}, \mathfrak{X}^1)_{\theta,g}$ is complete and is an invariant subspace of A. The restriction of the operator A onto $l_{q,p}^{\nu}(\mathfrak{X}, \mathfrak{X}^1)_{\theta,g}$ is a bounded operator and its norm satisfies the inequality*

$$\left\|A\,|_{l_{q,p}^{\nu}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}\right\| \le \nu. \tag{2.1}$$

The inclusions

$$l_{q,p}^{\nu}\left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g} \hookrightarrow \left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g} \hookrightarrow \mathfrak{X}, \qquad l_{q,p}^{\nu}\left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g} \hookrightarrow l_{q,p}^{\nu+\varepsilon}\left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g}$$

for all $\varepsilon > 0$ are continuous. The spectra of the operator A and the spectra of the restrictions $A\,|_{(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}$, $A\,|_{l_{q,p}^{\nu}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}$ satisfy the inclusions

$$\sigma\left(A\,|_{l_{q,p}^{\nu}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}\right) \subset \sigma\left(A\,|_{(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}\right) \subset \sigma(A).$$

Proof. As it was noted, the inclusions $\mathfrak{X}^{m+r} \hookrightarrow \mathfrak{X}^m \hookrightarrow \mathfrak{X}$, $(m, r \in \mathbb{Z}_+)$ are continuous and dense. For $x = x_0 + x_1 \in \mathfrak{X}^m$ we have $\|x\|_{\mathfrak{X}^m} = \|x_0 + x_1\|_{\mathfrak{X}^m} \le \|x_0\|_{\mathfrak{X}^m} + \|x_1\|_{\mathfrak{X}^m} \le \|x_0\|_{\mathfrak{X}^m} + \alpha_m \|x_1\|_{\mathfrak{X}^{m+1}}$, where α_m is the norm of embedding $\mathfrak{X}^{m+1} \hookrightarrow \mathfrak{X}^m$. Hence $\|x\|_{\mathfrak{X}^m} \le K_m(\alpha_m, x)$. As the result we obtain the following isomorphism of Banach spaces, $\mathfrak{X}^m + \mathfrak{X}^{m+1} = \mathfrak{X}^m$. On the other hand, $\|x\|_{\mathfrak{X}^{m+1}} \le J_m(1, x)$. Thus we obtain the isomorphism $\mathfrak{X}^m \cap \mathfrak{X}^{m+1} = \mathfrak{X}^{m+1}$.

Since $\|A^m x\|_{\mathfrak{X}^r} \le \|x\|_{\mathfrak{X}^{r+m}}$, the operator A^m is continuous from $\mathfrak{X}^{r+m}$ to $\mathfrak{X}^r$ for every $r \in \mathbb{Z}_+$ and the conditions of [1, Theorem 3.1.2] are fulfilled. Therefore the operator A^m is continuous from $(\mathfrak{X}^m, \mathfrak{X}^{m+1})_{\theta,g}$ to $(\mathfrak{X}, \mathfrak{X}^1)_{\theta,g}$. Moreover, there exists a constant β_m such that

$$K_m(\alpha_m, x) \le \beta_m \alpha_m^\theta \|x\|_{\theta,g,0}, \qquad x \in \left(\mathfrak{X}^m, \mathfrak{X}^{m+1}\right)_{\theta,g},$$

and the inclusions $\left(\mathfrak{X}^m, \mathfrak{X}^{m+1}\right)_{\theta,g} \hookrightarrow \mathfrak{X}^m \hookrightarrow \mathfrak{X}$ are continuous. Let a sequence $\{x_n\} \subset \left(\mathfrak{X}^m, \mathfrak{X}^{m+1}\right)_{\theta,g}$ be such that $x_n \to x$ and $A^m x_n \to y$ in $\left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g}$, then $x_n \to x$ and $A^m x_n \to y$ by the norm of $\mathfrak{X}$. From the closeness of A^m on the space $\mathfrak{X}$ it follows that $A^m x = y$. Therefore the operator A^m on the space $\left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g}$ is closed.

By [1, Theorem 3.2.2] there exists a constant γ_m such that

$$\|x\|_{\theta,g,m} \le \gamma_m J_m(1, x), \qquad x \in \mathfrak{X}^{m+1},$$

and the inclusions $\mathfrak{X}^{m+2} \hookrightarrow \mathfrak{X}^{m+1} \hookrightarrow \left(\mathfrak{X}^m, \mathfrak{X}^{m+1}\right)_{\theta,g}$ are continuous. Let a non-trivial continuous linear functional $f \in \left(\mathfrak{X}^m, \mathfrak{X}^{m+1}\right)'_{\theta,g}$ be such that $f \perp \mathfrak{X}^{m+2}$. Then f is continuous on the space $\mathfrak{X}^{m+2}$. In view of density and continuity of the inclusion $\mathfrak{X}^{m+2} \hookrightarrow \mathfrak{X}^{m+1}$ for dual spaces we have the embedding $(\mathfrak{X}^{m+1})' \subset (\mathfrak{X}^{m+2})'$. Thus $f \in (\mathfrak{X}^{m+1})'$ and $f \perp \mathfrak{X}^{m+2}$. By density of the inclusion $\mathfrak{X}^{m+2} \hookrightarrow \mathfrak{X}^{m+1}$ we get $f = 0$, i.e. the embedding $\mathfrak{X}^{m+2} \subset \left(\mathfrak{X}^m, \mathfrak{X}^{m+1}\right)_{\theta,g}$ is dense. Since $\mathfrak{X}^2 \subset \left(\mathfrak{X}^1, \mathfrak{X}^2\right)_{\theta,g} \subset \left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g}$, the space $\left(\mathfrak{X}^1, \mathfrak{X}^2\right)_{\theta,g}$ is dense in $\left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g}$.

Now we establish the inequality

$$\|A^m x\|_{\theta,g,0} \le \begin{cases} \nu^m \|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} : & q \le p, \\ \nu^m (m+1)^{\frac{1}{p}-\frac{1}{q}} \|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} : & p < q\,. \end{cases} \tag{2.2}$$

Let $x \in l^\nu_{q,p}\left(\mathfrak{X}, \mathfrak{X}^1\right)_{\theta,g}$ and $p = \infty$. Since $1 \le (m+1)^{\frac{1}{q}}$, $(m \in \mathbb{Z}_+)$, we obtain

$$\frac{\|A^m x\|_{\theta,g,0}}{\nu^m} \le \sup_{n \ge m} \frac{\|A^n x\|_{\theta,g,0}}{\nu^n} \le (m+1)^{\frac{1}{q}} \sup_{n \ge m} \frac{\|A^n x\|_{\theta,g,0}}{\nu^n} \le \|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}.$$

Let $1 \le p < \infty$. For all $m \in \mathbb{Z}_+$ we have

$$(m+1)^{\frac{p}{q}-1} \sup_{n \ge m} \frac{\|A^n x\|^p_{\theta,g,0}}{\nu^{np}} \le \|x\|^p_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}. \tag{2.3}$$

If $p \ge q$, then $1 \le \dfrac{p}{q}$ and $1 \le (m+1)^{\frac{p}{q}-1}$. Therefore, for all $m \in \mathbb{Z}_+$

$$\sup_{n \ge m} \frac{\|A^n x\|_{\theta,g,0}}{\nu^n} \le \|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} \quad \text{and} \quad \frac{\|A^m x\|_{\theta,g,0}}{\nu^m} \le \|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}.$$

If $p < q$, then for all $m \in \mathbb{Z}_+$ and $n \ge m$ we have

$$(n+1)^{\frac{p}{q}-1} \sup_{n \ge m} \frac{\|A^n x\|^p_{\theta,g,0}}{\nu^{np}} \le (m+1)^{\frac{p}{q}-1} \sup_{n \ge m} \frac{\|A^n x\|^p_{\theta,g,0}}{\nu^{np}}.$$

From here and (2.3) for all $n \geq m$ it follows

$$(n+1)^{\frac{p}{q}-1} \frac{\|A^m x\|^p_{\theta,g,0}}{\nu^{mp}} \leq (n+1)^{\frac{p}{q}-1} \sup_{n\geq m} \frac{\|A^n x\|^p_{\theta,g,0}}{\nu^{np}} \leq \|x\|^p_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}.$$

In particular, at $n = m$ we obtain

$$\frac{\|A^m x\|_{\theta,g,0}}{\nu^m} \leq (m+1)^{\frac{1}{p}-\frac{1}{q}} \|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} = (m+1)^{\left(1-\frac{p}{q}\right)\frac{1}{p}} \|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}.$$

Inequality (2.2) is proved. From inequality (2.2) it immediately follows: if $\{x_n\}$ is a Cauchy sequence in $l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$, then $\{A^m x_n\}$ for every $m \in \mathbb{Z}_+$ and $\{x_n\}$ are also Cauchy sequences in $(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$. By completeness of $(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$ there exist $x, y \in (\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$ such that $x_n \to x$ and $A^m x_n \to y$. The graph of A^m is a closed subspace in $(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$, therefore $y = A^m x$ and $x \in (\mathfrak{X}^m,\mathfrak{X}^{m+1})_{\theta,g}$. Since this is valid for every $m \in \mathbb{Z}_+$, we get $x \in \bigcap_m (\mathfrak{X}^m,\mathfrak{X}^{m+1})_{\theta,g} = \mathfrak{X}^\infty$. At last, for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\|x_n\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} \leq \|x_{n_\varepsilon}\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} + \|x_n - x_{n_\varepsilon}\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} \leq \|x_{n_{n_\varepsilon}}\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} + \varepsilon$ for all $n \geq n_\varepsilon$. Passing to the limit at $n \to \infty$, we obtain $\|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} \leq \|x_{n_\epsilon}\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} + \varepsilon < \infty$. Thus $x \in l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$. Similarly, passing to the limit in $\|x_n - x_k\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} < \varepsilon$ we obtain $\|x - x_k\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} \leq \varepsilon$ for all $k \geq n_\varepsilon$, i.e. the space $l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$ is complete.

The property of A-invariancy for the space $l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$ and inequality (2.1) follows from the calculations:

$$\begin{aligned}
\|Ax\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} &= \begin{cases} \left[\sum\limits_{m=0}^{\infty}(m+1)^{\frac{p}{q}-1}\left(\sup\limits_{n\geq m}\dfrac{\nu\|A^{n+1}x\|_{\theta,g,0}}{\nu^{n+1}}\right)^p\right]^{\frac{1}{p}} : 1\leq p<\infty, \\ \sup\limits_{m\in\mathbb{Z}_+}\left[(m+1)^{\frac{1}{q}}\left(\sup\limits_{n\geq m}\dfrac{\nu\|A^{n+1}x\|_{\theta,g,0}}{\nu^{n+1}}\right)\right] : \quad p=\infty, \end{cases} \leq \\
&\leq \begin{cases} \nu\left[\sum\limits_{m=0}^{\infty}(m+1)^{\frac{p}{q}-1}\left(\sup\limits_{n\geq m}\dfrac{\|A^{n}x\|_{\theta,g,0}}{\nu^{n}}\right)^p\right]^{\frac{1}{p}} : 1\leq p<\infty, \\ \nu\sup\limits_{m\in\mathbb{Z}_+}\left[(m+1)^{\frac{1}{q}}\left(\sup\limits_{n\geq m}\dfrac{\|A^{n}x\|_{\theta,g,0}}{\nu^{n}}\right)\right] : \quad p=\infty. \end{cases} = \\
&= \nu\|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}.
\end{aligned}$$

From the definition of the norm the inequalities $\|x\|_{l^{\nu+\epsilon}_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} \leq \|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}$ and $\|x\|_{\theta,g,0} \leq \|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}$ for all $x \in l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$ follow immediately. Hence, the corresponding inclusions are continuous.

Let $\lambda \in \rho(A)$, where $\rho(A)$ is the resolvent set of A on the space $\mathfrak{X}$. Since the resolvent $(\lambda - A)^{-1}$ is continuous on the spaces $\mathfrak{X}$ and $\mathfrak{X}^1$ simultaneously, according to [1, Theorem 3.1.2] it is continuous on the space $(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$ i.e., for the resolvent set of the restriction $A\,|_{(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}$ we obtain $\lambda \in \rho\big(A\,|_{(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}\big)$. Hence the inclusion $\sigma\big(A\,|_{(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}\big) \subset \sigma(A)$ is proved. From the inequality $K_1\big[t, (\lambda - A)^{-1}x\big] \leq \|(\lambda - A)^{-1}\|\, K_1(t,x)$ it follows that $\big\|(\lambda - A)^{-1}x\big\|_{\theta,g,0} \leq \|(\lambda - A)^{-1}\|\,\|x\|_{\theta,g,0}$ for each $x \in (\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$. Using the property of A-invariancy for the space $l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}$ from here we obtain

$$\|(\lambda - A)^{-1}x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}} \leq \|(\lambda - A)^{-1}\|\,\|x\|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}, \quad x \in l^\nu_{q,p}\left(\mathfrak{X},\mathfrak{X}^1\right)_{\theta,g}$$

and for the resolvent set of $A\,|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}$ we get $\lambda \in \rho\big(A\,|_{l^\nu_{q,p}(\mathfrak{X},\mathfrak{X}^1)_{\theta,g}}\big)$. □

3. Case of differentiation operator. Set $\mathfrak{X} = L_\rho$, $(1 \le \rho \le \infty)$. Then $\mathfrak{X}^1 = W_\rho$ is the complex Sobolev space of the functions $\mathbb{R} \ni t \longrightarrow \varphi(t)$ with the norm

$$\|\varphi\|_{W_\rho} = \begin{cases} \left(\int_{\mathbb{R}} |\varphi(t)|^\rho \, dt\right)^{1/\rho} + \left(\int_{\mathbb{R}} |\varphi'(t)|^\rho \, dt\right)^{1/\rho} & : \ 1 \le \rho < \infty, \\ \operatorname*{ess\,sup}_{t\in\mathbb{R}} |\varphi(t)| + \operatorname*{ess\,sup}_{t\in\mathbb{R}} |\varphi'(t)| & : \ \rho = \infty. \end{cases}$$

Note that the norm $\|\cdot\|_{L_\rho}$ of the space L_ρ consists only of the first addends of the previous formula. Set $A = d/dt$ and note that the operator d/dt is the generator of the isometric group of shifts $T_s \colon \varphi(t) \longrightarrow \varphi(t+s)$, $(t, s \in \mathbb{R})$ in the space L_ρ is closed. For the Besov space $B^\theta_{\rho,g}$ of complex functions on $\mathbb{R}$ the isomorphism of Banach space $B^\theta_{\rho,g} = (L_\rho, W_\rho)_{\theta,g}$, $0 < \theta < 1$ is known [1, Theorem 6.7.4]. In the cases $g = 1, \infty$ the last equality can be extended onto $0 \le \theta \le 1$ by the relations $B^0_{\rho,\infty} = L_\rho$ and $B^1_{\rho,1} = W_\rho$ [1, Theorem 3.5.2].

For numbers $\nu > 0$ and $1 \le p, q \le \infty$ we define the space of Lorentz type $l^\nu_{q,p}(B^\theta_{\rho,g}) := l^\nu_{q,p}(L_\rho, W_\rho)_{\theta,g}$. As the operator d/dt is closed, the space $l^\nu_{q,p}(B^\theta_{\rho,g})$ is complete. Moreover it is invariant with respect to the differentiation d/dt.

Let M_ν be the space of entire analytic functions $\Phi \colon \mathbb{C} \ni t+i\tau \longrightarrow \Phi(t+i\tau) \in \mathbb{C}$ satisfying the inequality

$$|\Phi(t+i\tau)| \le C_\Phi \exp\big(\nu|\tau|\big), \qquad t + i\tau \in \mathbb{C}$$

for some constant C_Φ. It is known [5, 3.1] that the space M_ν is invariant with respect to the differentiation and consists of entire functions of the exponential type ν. Denote

$$M^\theta_{\rho,g,\nu} := \big\{\Phi(t+i\tau) \in M_\nu : \ \varphi(t) := \Phi(t+i0) \in B^\theta_{\rho,g}\big\}.$$

Proposition 3.1. *For $1 < \rho < \infty$ the restriction*

$$M^\theta_{\rho,g,\nu} \ni \Phi(t+i\tau) \longrightarrow \varphi(t) \in l^\nu_{\infty,\infty}\left(B^\theta_{\rho,g}\right)$$

realizes a linear isomorphism.

Proof. Let $\Phi \in M^\theta_{\rho,g,\nu}$. Since $B^\theta_{\rho,g} \subset L_\rho$, the restriction φ of the function $\Phi(t+i\tau)$ on the real axis $\mathbb{R}$ satisfies the Bernstein inequality [5, 3.2.2]

$$\|\varphi^{(k)}\|_{L_\rho} \le \nu^k \, \|\varphi\|_{L_\rho}, \qquad k \in \mathbb{Z}_+. \tag{3.1}$$

In particular, from here it follows $\varphi^{(k)} \in L_\rho$. As also $\varphi^{(k)} \in M_\nu$, from (3.1) we get $\|\varphi^{(k+1)}\|_{L_\rho} \le \nu \, \|\varphi^{(k)}\|_{L_\rho}$. Therefore

$$\begin{aligned} \|\varphi^{(k+1)}\|_{W_\rho} = \|\varphi^{(k+1)}\|_{L_\rho} + \|\varphi^{(k+2)}\|_{L_\rho} \le \nu\big(\|\varphi^{(k)}\|_{L_\rho} + \|\varphi^{(k+1)}\|_{L_\rho}\big) = \\ = \nu \, \|\varphi^{(k)}\|_{W_\rho} \le \nu^{k+1} \|\varphi\|_{W_\rho} \end{aligned}$$

and the inequality $K_1\big(t, \varphi^{(k)}\big) \le \nu^k K_1(t, \varphi)$ is valid for all $t > 0$. Integrating the inequality we get $\|\varphi^{(k)}\|_{\theta,g,0} \le \nu^k \|\varphi\|_{\theta,g,0}$, $\Phi \in M^\theta_{\rho,g,\nu}$. From here, it follows

$$\|\varphi\|_{l^\nu_{\infty,\infty}(B^\theta_{\rho,g})} = \sup_{k\in\mathbb{Z}_+} \frac{\|\varphi^{(k)}\|_{\theta,g,0}}{\nu^k} \le \|\varphi\|_{\theta,g,0},$$

hence $\varphi \in l^\nu_{\infty,\infty}\left(B^\theta_{\rho,g}\right)$.

Vice versa, let $\varphi \in l^{\nu}_{\infty,\infty}\left(B^{\theta}_{\rho,g}\right)$. Then, in particular $\varphi \in W_\rho$. For the function φ we put in conformity the series

$$\Phi(t+i\tau) := \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(t)}{k!}(i\tau)^k .$$

From the definition of the norm in the space $l^{\nu}_{\infty,\infty}\left(B^{\theta}_{\rho,g}\right)$ it follows

$$\|\varphi^{(k)}\|_{\theta,g,0} \le \nu^k \|\varphi\|_{l^{\nu}_{\infty,\infty}(B^{\theta}_{\rho,g})}, \qquad k \in \mathbb{Z}_+ . \tag{3.2}$$

Calculating the norm of the function $\Phi(t+i\tau)$ by the variable t we get

$$\|\Phi(\cdot+i\tau)\|_{\theta,g,0} \le \sum_{k=0}^{\infty} \frac{|\tau^k|\,\|\varphi^{(k)}\|_{\theta,g,0}}{k!} \le \exp\left(\nu|\tau|\right)\|\varphi\|_{l^{\nu}_{\infty,\infty}(B^{\theta}_{\rho,g})}.$$

Hence $\Phi(\cdot+i\tau) \in B^{\theta}_{\rho,g}$ for all $\tau \in \mathbb{R}$. Now it is necessary to prove that $\Phi \in M_\nu$. Let C_b be the space of continuous and bounded functions on $\mathbb{R}$ with the norm $\|\cdot\|_{L_\infty}$. By the Sobolev Theorem [6, 2.8.1] the following continuous embedding $W_\rho \hookrightarrow C_b$ holds. Thus there exists a constant $\alpha > 0$ such that

$$\|\varphi^{(k)}\|_{L_\infty} \le \alpha\|\varphi^{(k)}\|_{W_\rho} \le \alpha\nu^k\|\varphi\|_{l^{\nu}_{\infty,\infty}(B^{\theta}_{\rho,g})}, \qquad k \in \mathbb{Z}_+ .$$

From here the inequalities

$$\|\Phi(\cdot+i\tau)\|_{L_\infty} \le \sum_{k=0}^{\infty} \frac{|\tau^k|\,\|\varphi^{(k)}\|_{L_\infty}}{k!} \le \alpha\exp\left(\nu|\tau|\right)\|\varphi\|_{l^{\nu}_{\infty,\infty}(B^{\theta}_{\rho,g})}$$

follow for all $\tau \in \mathbb{R}$. Hence $\Phi \in M_\nu$. □

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